PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 9 -- Chap. 6 (F & W)
Extensions of Hamiltonian formalism

1. Virial theorem

2. Canonical transformations

3. Hamilton-Jacobi formalism

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/28/2023		Introduction and overview	#1
2	Wed, 8/30/2023	Chap. 3(17)	Calculus of variation	#2
3	Fri, 9/01/2023	Chap. 3(17)	Calculus of variation	#3
4	Mon, 9/04/2023	Chap. 3	Lagrangian equations of motion	#4
5	Wed, 9/06/2023	Chap. 3 & 6	Lagrangian equations of motion	#5
6	Fri, 9/08/2023	Chap. 3 & 6	Lagrangian equations of motion	#6
7	Mon, 9/11/2023	Chap. 3 & 6	Lagrangian to Hamiltonian formalism	#7
8	Wed, 9/13/2023	Chap. 3 & 6	Phase space	
9	Fri, 9/15/2023	Chap. 3 & 6	Canonical Transformations	#8
10	Mon, 9/18/2023			
11	Wed, 9/20/2023			
12	Fri, 9/22/2023			
13	Mon, 9/25/2023			
14	Wed, 9/27/2023			

PHY 711 -- Assignment #8

Assigned: 9/15/2023 Due: 9/25/2023

Finish reading Chapters 3 and 6 in **Fetter & Walecka**.

1. Consider the example of the mass m initially at height h falling from rest with constant acceleration g to verify the analysis of the action S at time $t>0$ discussed in the lecture notes.

Virial theorem (Rudolf Clausius ~ 1870)

$$2\langle T\rangle = -\left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma \right\rangle$$

Proof:

Define: $A \equiv \sum_\sigma \mathbf{p}_\sigma \cdot \mathbf{r}_\sigma$

$$\frac{dA}{dt} = \sum_\sigma \left(\dot{\mathbf{p}}_\sigma \cdot \mathbf{r}_\sigma + \mathbf{p}_\sigma \cdot \dot{\mathbf{r}}_\sigma \right) = \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma + 2T$$

Because $\dot{\mathbf{p}}_\sigma = \mathbf{F}_\sigma$

$$\left\langle \frac{dA}{dt} \right\rangle = \left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma \right\rangle + 2\langle T\rangle$$

$$\left\langle \frac{dA}{dt} \right\rangle = \frac{1}{\tau}\int_0^\tau \frac{dA(t)}{dt}dt = \frac{A(\tau) - A(0)}{\tau} \Rightarrow 0$$

Note that this implies that the motion is periodic or bounded (not for all systems).

When it is true -- $\Rightarrow \left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma \right\rangle + 2\langle T\rangle = 0$

Examples of the Virial Theorem

$$2\langle T\rangle = -\left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma \right\rangle$$

Harmonic oscillator:

$$\mathbf{F} = -kx\hat{\mathbf{x}} \qquad T = \frac{1}{2}m\dot{x}^2 \qquad \langle m\dot{x}^2\rangle = \langle kx^2\rangle$$

Check: for $x(t) = X\sin\left(\sqrt{\frac{k}{m}}t + \alpha\right)$

$$\langle 2T\rangle = \langle m\dot{x}^2\rangle = kX^2\left\langle \cos^2\left(\sqrt{\frac{k}{m}}t + \alpha\right)\right\rangle = \frac{1}{2}kX^2$$

$$-\left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma \right\rangle = \langle kx^2\rangle = kX^2\left\langle \sin^2\left(\sqrt{\frac{k}{m}}t + \alpha\right)\right\rangle = \frac{1}{2}kX^2$$

Premise true because of periodicity.

Examples of the Virial Theorem

$$2\langle T\rangle = -\left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma \right\rangle$$

Circular orbit due to gravitational field of massive object:

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \qquad T = \frac{1}{2}mv^2$$

$$\left\langle mv^2 \right\rangle = \left\langle \frac{GMm}{r} \right\rangle$$

Check: for $\dfrac{v^2}{r} = \dfrac{GM}{r^2}$

$$\Rightarrow \left\langle mv^2 \right\rangle = \left\langle \frac{GMm}{r} \right\rangle$$

centripetal acceleration

gravitational force

Premise true because of periodicity.

Hamiltonian formalism and the canonical equations of motion:

$$H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$$

Canonical equations of motion

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma}$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

In the next slides we will consider finding different coordinates and momenta that can also describe the system. Why?

a. Because we can
b. Because it might be useful

Notion of "Canonical" generalized coordinate transformations

$$q_\sigma = q_\sigma\left(\{Q_1 \cdots Q_n\},\{P_1 \cdots P_n\},t\right) \qquad \text{for each } \sigma$$

$$p_\sigma = p_\sigma\left(\{Q_1 \cdots Q_n\},\{P_1 \cdots P_n\},t\right) \qquad \text{for each } \sigma$$

Note that because of the way we set up the problem we can always add such a term.

For some $\tilde{H}$ and F, using Legendre transformations

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H\left(\{q_\sigma\},\{p_\sigma\},t\right) = \sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) + \frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right)$$

Apply Hamilton's principle:

$$\delta \int_{t_i}^{t_f} \left[\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) + \frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right)\right] dt = 0$$

$$\delta \int_{t_i}^{t_f} \left[\frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right)\right] dt = \int_{t_i}^{t_f} \left[\frac{d}{dt}\delta F\left(\{q_\sigma\},\{Q_\sigma\},t\right)\right] dt$$

$$= \delta F\left(t_f\right) - \delta F\left(t_i\right) = 0 \qquad \text{and} \qquad \dot{Q}_\sigma = \frac{\partial \tilde{H}}{\partial P_\sigma} \qquad\qquad \dot{P}_\sigma = -\frac{\partial \tilde{H}}{\partial Q_\sigma}$$

Some details --

$$q_\sigma = q_\sigma\left(\{Q_1 \cdots Q_n\},\{P_1 \cdots P_n\},t\right) \qquad \text{for each } \sigma$$

$$p_\sigma = p_\sigma\left(\{Q_1 \cdots Q_n\},\{P_1 \cdots P_n\},t\right) \qquad \text{for each } \sigma$$

For some $\tilde{H}$ and F, using Legendre transformations

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H\left(\{q_\sigma\},\{p_\sigma\},t\right) = \sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) + \frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right)$$

Action integral:

$$S = \int_{t_i}^{t_f} dt \left(\sum_\sigma p_\sigma \dot{q}_\sigma - H\left(\{q_\sigma\},\{p_\sigma\},t\right) \right)$$

$$\delta S = \int_{t_i}^{t_f} dt \left(\sum_\sigma \left(\delta p_\sigma \dot{q}_\sigma + p_\sigma \delta \dot{q}_\sigma\right) - \delta H\left(\{q_\sigma\},\{p_\sigma\},t\right) \right)$$

Note that $\delta \int_{t_i}^{t_f} dt \left(\frac{dF(t)}{dt} \right) = \int_{t_i}^{t_f} dt \left(\frac{d\delta F(t)}{dt} \right) = 0$

Some relations between old and new variables:

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t)$$

$$\frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t) = \sum_\sigma\left(\left(\frac{\partial F}{\partial q_\sigma}\right)\dot{q}_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\dot{Q}_\sigma\right) + \frac{\partial F}{\partial t}$$

$$\Rightarrow \sum_\sigma\left(p_\sigma - \left(\frac{\partial F}{\partial q_\sigma}\right)\right)\dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma\left(P_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\right)\dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{\partial F}{\partial t}$$

$$\sum_\sigma \left(p_\sigma - \left(\frac{\partial F}{\partial q_\sigma} \right) \right) \dot{q}_\sigma - H(\{q_\sigma\}, \{p_\sigma\}, t) =$$

$$\sum_\sigma \left(P_\sigma + \left(\frac{\partial F}{\partial Q_\sigma} \right) \right) \dot{Q}_\sigma - \widetilde{H}(\{Q_\sigma\}, \{P_\sigma\}, t) + \frac{\partial F}{\partial t}$$

$$\Rightarrow p_\sigma = \left(\frac{\partial F}{\partial q_\sigma} \right) \qquad P_\sigma = -\left(\frac{\partial F}{\partial Q_\sigma} \right)$$

$$\Rightarrow \widetilde{H}(\{Q_\sigma\}, \{P_\sigma\}, t) = H(\{q_\sigma\}, \{p_\sigma\}, t) + \frac{\partial F}{\partial t}$$

Note that it is conceivable that if we were extraordinarily clever, we could find all of the constants of the motion!

$$\dot{Q}_\sigma = \frac{\partial \widetilde{H}}{\partial P_\sigma} \qquad \dot{P}_\sigma = -\frac{\partial \widetilde{H}}{\partial Q_\sigma}$$

$$\text{Suppose:} \quad \dot{Q}_\sigma = \frac{\partial \widetilde{H}}{\partial P_\sigma} = 0 \quad \text{and} \quad \dot{P}_\sigma = -\frac{\partial \widetilde{H}}{\partial Q_\sigma} = 0$$

$$\Rightarrow Q_\sigma, P_\sigma \quad \text{are constants of the motion}$$

Possible solution – Hamilton-Jacobi theory:

$$\text{Suppose:} \quad F(\{q_\sigma\},\{Q_\sigma\},t) \Rightarrow -\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)$$

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}\left(-\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

$$= -\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) - \sum_\sigma \dot{P}_\sigma Q_\sigma + \sum_\sigma \left(\frac{\partial S}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial S}{\partial P_\sigma}\dot{P}_\sigma\right) + \frac{\partial S}{\partial t}$$

Solution :

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \qquad\qquad Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$

$$\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) = H(\{q_\sigma\},\{p_\sigma\},t) + \frac{\partial S}{\partial t}$$

When the dust clears:

Assume $\{Q_\sigma\}, \{P_\sigma\}, \tilde{H}$ are constants; choose $\tilde{H} = 0$

Need to find $S(\{q_\sigma\}, \{P_\sigma\}, t)$

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \qquad Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$

$$\Rightarrow H\left(\{q_\sigma\}, \left\{\frac{\partial S}{\partial q_\sigma}\right\}, t\right) + \frac{\partial S}{\partial t} = 0$$

Note: S is the "action":

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\}, \{p_\sigma\}, t) =$$

$$\underbrace{\sum_\sigma P_\sigma \dot{Q}_\sigma}_{0} - \underbrace{\tilde{H}(\{Q_\sigma\}, \{P_\sigma\}, t)}_{0} + \frac{d}{dt}\left(-\underbrace{\sum_\sigma P_\sigma Q_\sigma}_{0} + S(\{q_\sigma\}, \{P_\sigma\}, t)\right)$$

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \widetilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}\left(-\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

0 0 0

$$\int_{t_i}^{t_f}\left(\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t)\right)dt = \int_{t_i}^{t_f}\left(\frac{d}{dt}\left(S(\{q_\sigma\},\{P_\sigma\},t)\right)\right)dt$$

$$= S(\{q_\sigma\},\{P_\sigma\},t)\Big|_{t_i}^{t_f}$$

Differential equation for ***S***:

$$H\left(\{q_\sigma\},\left\{\frac{\partial S}{\partial q_\sigma}\right\},t\right)+\frac{\partial S}{\partial t}=0$$

$$\text{Example:}\qquad H(\{q\},\{p\},t)=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2$$

$$\text{Hamilton - Jacobi Eq:}\quad H\left(\{q\},\left\{\frac{\partial S}{\partial q}\right\},t\right)+\frac{\partial S}{\partial t}=0$$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+\frac{1}{2}m\omega^2q^2+\frac{\partial S}{\partial t}=0$$

Does this look familiar?

$$\text{Assume:}\qquad S(q,t)\equiv W(q)-Et \qquad (E \ \ \text{constant})$$

Continued:

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 + \frac{\partial S}{\partial t} = 0$$

$$\text{Assume:} \qquad S(q,t) \equiv W(q) - Et \qquad\qquad (E \quad \text{constant})$$

$$\frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + \frac{1}{2}m\omega^2 q^2 = E$$

$$\frac{dW}{dq} = \sqrt{2mE - (m\omega)^2 q^2}$$

$$W(q) = \int \sqrt{2mE - (m\omega)^2 q^2}\, dq$$

Continued:

$$W(q) = \int \sqrt{2mE - (m\omega)^2 q^2}\, dq$$

$$= \frac{1}{2} q \sqrt{2mE - (m\omega)^2 q^2} + \frac{E}{\omega} \sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}} \right) + C$$

$$S(q, E, t) = \frac{1}{2} q \sqrt{2mE - (m\omega)^2 q^2} + \frac{E}{\omega} \sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}} \right) - Et$$

$$\frac{\partial S}{\partial E} = Q = \frac{1}{\omega} \sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}} \right) - t$$

$$\Rightarrow q(t) = \frac{\sqrt{2mE}}{m\omega} \sin\left(\omega\left(t + Q\right)\right)$$

Another example of Hamilton Jacobi equations

Example: $$H\left(\{y\},\{p\},t\right)=\frac{p^2}{2m}+mgy$$

Assume $y(0)=h; \qquad p(0)=0$

Hamilton-Jacobi Eq: $$H\left(\{y\},\left\{\frac{\partial S}{\partial y}\right\},t\right)+\frac{\partial S}{\partial t}=0$$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2+mgy+\frac{\partial S}{\partial t}=0$$

Assume: $S(y,t)\equiv W(y)-Et \qquad (E \quad \text{constant})$

Example: $$H\left(\{y\},\{p\},t\right)=\frac{p^2}{2m}+mgy$$

Assume $y(0)=h; \qquad p(0)=0$

$$\frac{1}{2m}\left(\frac{dW}{dy}\right)^2+mgy=E\equiv mgh$$

$$W(y)=m\int_y^h\sqrt{2g\left(h-y'\right)}dy'=\frac{2}{3}m\sqrt{2g}\left(h-y\right)^{3/2}$$

$$S(y,t)=W(y)-Et=\frac{2}{3}m\sqrt{2g}\left(h-y\right)^{3/2}-mght$$

Check action:

For this case: $y(t) = h - \frac{1}{2}gt^2$

$$S = \int_0^t \left(\frac{1}{2}m\dot{y}^2 - mgy \right) dt' = \frac{1}{3}mg^2t^3 - mght$$

$$S(y,t) = W(y) - Et = \frac{2}{3}m\sqrt{2g}\left(h-y\right)^{3/2} - mght$$

Agrees with Hamilton-Jacobi analysis.

Alternatively, keeping E notation:

$$W(y) = \int_y^h \sqrt{2mE - 2m^2 g y'} dy'$$

$$= \sqrt{\frac{2}{m}} \frac{1}{g} \left(E - mgy\right)^{3/2}$$

$$S(y,t) = W(y) - Et = \sqrt{\frac{2}{m}} \frac{2}{3g} \left(E - mgy\right)^{3/2} - Et$$

$$\frac{\partial S}{\partial E} = Q = \sqrt{\frac{2}{m}} \frac{1}{g} \left(E - mgy\right)^{1/2} - t$$

$$\Rightarrow y(t) = \frac{E}{mg} - \frac{1}{2} g \left(t + Q\right)^2$$

In our case, $Q = 0$

$E = mgh$

What do you think of Hamilton-Jacobi method

a. Historically important
b. Hysterical
c. Painful
d. Might be useful

The next 3 slides contain important equations that you will hopefully remember for this material contained in Chapters 3 & 6 of Fetter and Walecka. On Monday we will start with Chapter 5 and discuss one of the many applications of these ideas – the case of rigid body motion.

Recap --

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Hamiltonian picture

$$H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

$\Rightarrow$ Coupled first order differential equations for

$q_\sigma(t)$ and $p_\sigma(t)$

General treatment of particle of mass m and charge q moving in 3 dimensions in an potential $U(\mathbf{r})$ as well as electromagnetic scalar and vector potentials $\Phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$:

Lagrangian: $$L(\mathbf{r},\dot{\mathbf{r}},t)=\frac{1}{2}m\dot{\mathbf{r}}^2-U(\mathbf{r})-q\Phi(\mathbf{r},t)+\frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

Hamiltonian: $$\mathbf{p}=\frac{\partial L}{\partial \dot{\mathbf{r}}}=m\dot{\mathbf{r}}+\frac{q}{c}\mathbf{A}(\mathbf{r},t)$$

$$H(\mathbf{r},\mathbf{p},t)=\mathbf{p}\cdot\dot{\mathbf{r}}-L(\mathbf{r},\dot{\mathbf{r}},t)$$

$$=\frac{1}{2m}\left(\mathbf{p}-\frac{q}{c}\mathbf{A}(\mathbf{r},t)\right)^2+U(\mathbf{r})+q\Phi(\mathbf{r},t)$$

Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function: $L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$
2. Compute generalized momenta: $p_\sigma \equiv \dfrac{\partial L}{\partial \dot{q}_\sigma}$
3. Construct Hamiltonian expression: $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$
4. Form Hamiltonian function: $H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$
5. Analyze canonical equations of motion:

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$