PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 10: Rigid bodies – Chap. 5 (F &W)

1. Rigid body motion
2. Notion of the center of mass
3. Moment of inertia tensor
4. Torque free motion

Course schedule

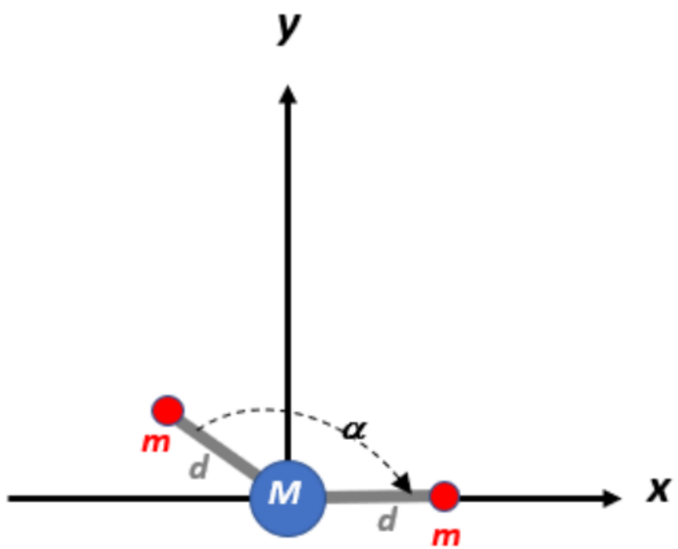
(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/26/2024		Introduction and overview	#1
2	Wed, 8/28/2024	Chap. 3(17)	Calculus of variation	#2
3	Fri, 8/30/2024	Chap. 3(17)	Calculus of variation	#3
4	Mon, 9/02/2024	Chap. 3	Lagrangian equations of motion	#4
5	Wed, 9/04/2024	Chap. 3 & 6	Lagrangian equations of motion	#5
6	Fri, 9/06/2024	Chap. 3 & 6	Lagrangian equations of motion	#6
7	Mon, 9/09/2024	Chap. 3 & 6	Lagrangian to Hamiltonian formalism	#7
8	Wed, 9/11/2024	Chap. 3 & 6	Phase space	#8
9	Fri, 9/13/2024	Chap. 3 & 6	Canonical Transformations	
10	Mon, 9/16/2024	Chap. 5	Dynamics of rigid bodies	#9
11	Wed, 9/18/2024	Chap. 5	Dynamics of rigid bodies	
12	Fri, 9/20/2024	Chap. 5	Dynamics of rigid bodies	

PHY 711 -- Assignment #9

Assigned: 9/16/2024 Due: 9/23/2024

Start reading Chapter 5 in **Fetter & Walecka**.



1. The figure above shows a rigid 3 atom molecule placed in the *x*-*y* plane as shown. Assume that the rigid bonds are massless.
   a. Find the moment of inertia tensor in the given coordinate system placed of mass *M* in terms of the atom masses, bond lengths *d*, and angle α.
   b. Find the principal moments moments of inertia I_1, I_2,I_3 and the corresponding principal axes.
   c. (Extra credit.) Find the principal moments and axes for a coordinate system with its origin placed at the center of mass of the molecule.

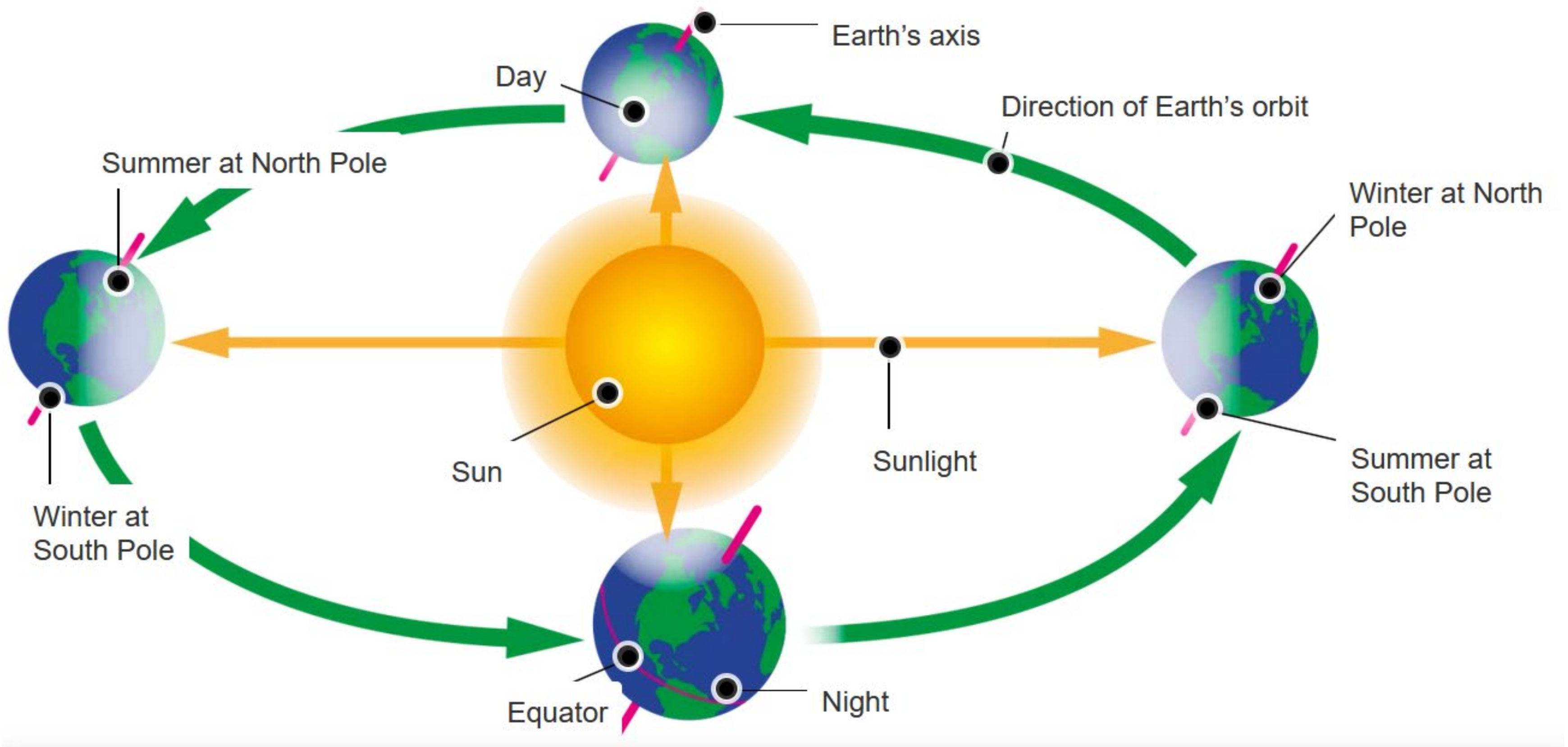
Up to now, we have considered the motions of idealized point particles of mass *m*, moving along a trajectory with generalized coordinates $q_\sigma(t)$ according to Newton's laws and the Lagrangian and Hamiltonian equations of motion. In this case, the kinetic energy of the particle depends only on the squared velocity of the particle scaled by its mass *m*.

For example, the kinetic energy of point mass m expressed in Cartesian coordinates is

$$K = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

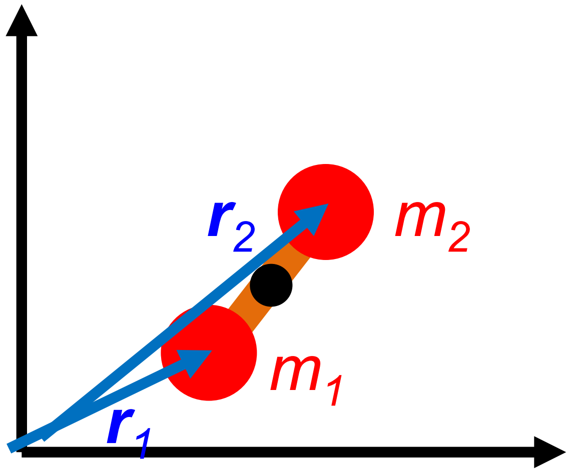
In studying rigid body motion, we consider a system with distributed mass in which the motion is more complicated.

https://www.dkfindout.com/us/space/solar-system/earths-orbit/



Knowing that the laws of physics are most conveniently applied with in an inertial frame of reference, we will focus on how to analyze rotations of a rigid body.

Example of a rigid body system consisting of two masses:



Center of mass:

$$\mathbf{R}_{CM} \equiv \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i}$$

With rigid bodies, we should consider motion of the body, both relative to an inertial frame of reference and also internal motion of the body. For rigid body motion, it is assumed that no deformations or vibrations occur. It turns out that the details of the shape of the rigid body can be characterized by the "moment of inertia tensor" to describe the internal motion, while the overall motion will also be important.

The physics of rigid body motion; body fixed frame vs inertial frame; (using notation from Chapter 2 of F & W)

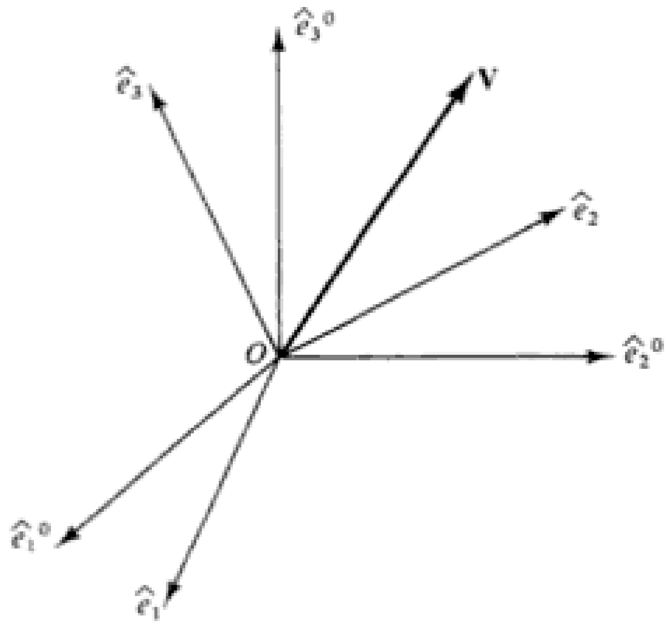


Figure 6.1 Transformation to a rotating coordinate system.

Let **V** be a general vector, e.g., the position of a particle. This vector can be characterized by its components with respect to either orthonormal triad. Thus we can write

$$\mathbf{V} = \sum_{i=1}^{3} V_i^0 \hat{e}_i^0 \tag{6.1a}$$

$$\mathbf{V} = \sum_{i=1}^{3} V_i \hat{e}_i \tag{6.1b}$$

Comparison of analysis in "inertial frame" versus "non-inertial frame"

Denote by $\hat{e}_i^0$ a fixed coordinate system

Denote by $\hat{e}_i$ a moving coordinate system

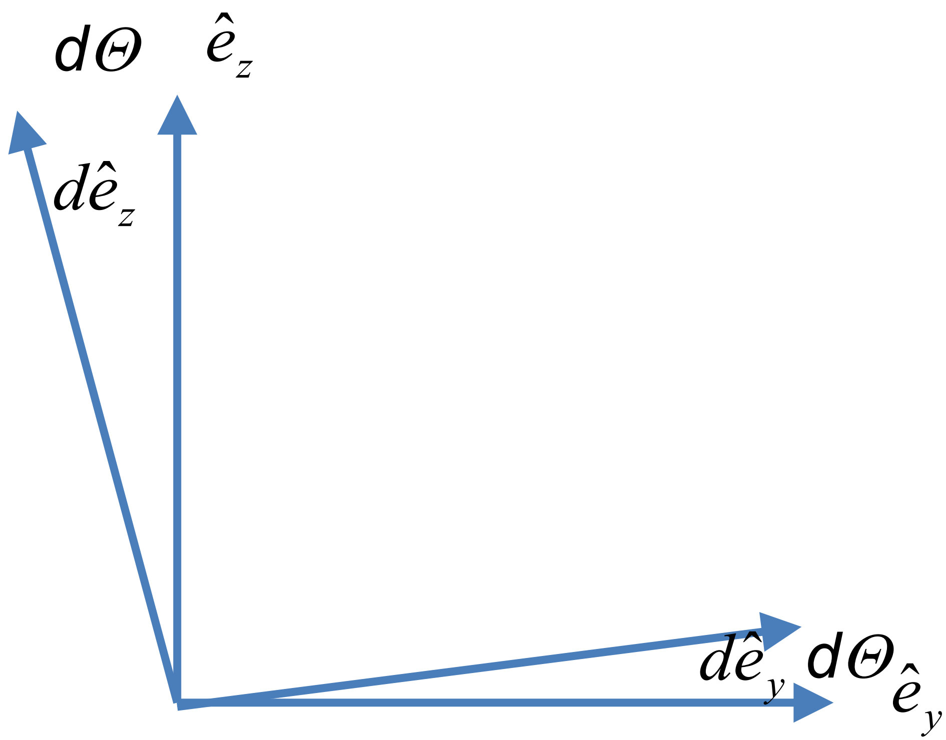
For an arbitrary vector **V**: $\mathbf{V} = \sum_{i=1}^{3} V_i^0 \hat{e}_i^0 = \sum_{i=1}^{3} V_i \hat{e}_i$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \sum_{i=1}^{3} \frac{dV_i^0}{dt}\hat{e}_i^0 = \sum_{i=1}^{3} \frac{dV_i}{dt}\hat{e}_i + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

Define: $\left(\frac{d\mathbf{V}}{dt}\right)_{body} \equiv \sum_{i=1}^{3} \frac{dV_i}{dt}\hat{e}_i$

$$\Rightarrow \left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

Properties of the frame motion (rotation):



$$d\hat{e}_y = d\Theta \hat{e}_z$$

$$d\hat{e}_z = -d\Theta \hat{e}_y$$

$$\Rightarrow d\hat{\mathbf{e}} = d\Theta \times \hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \frac{d\Theta}{dt} \times \hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

$$\begin{pmatrix} d\hat{e}_y \\ d\hat{e}_z \end{pmatrix} = \begin{pmatrix} \cos(d\Theta) & \sin(d\Theta) \\ -\sin(d\Theta) & \cos(d\Theta) \end{pmatrix} \begin{pmatrix} \hat{e}_y \\ \hat{e}_z \end{pmatrix} - \begin{pmatrix} \hat{e}_y \\ \hat{e}_z \end{pmatrix} \approx \begin{pmatrix} 0 & d\Theta \\ -d\Theta & 0 \end{pmatrix} \begin{pmatrix} \hat{e}_y \\ \hat{e}_z \end{pmatrix}$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

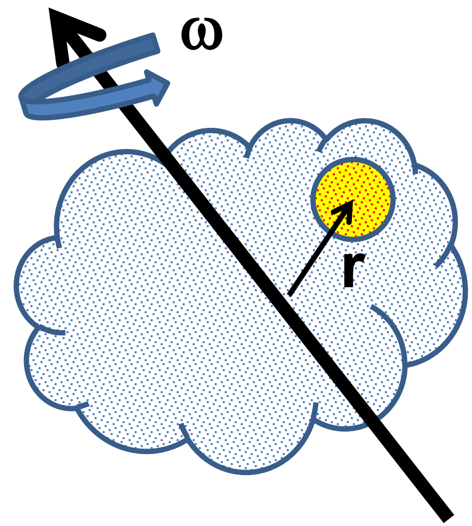
$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{V}$$

Effects on acceleration:

$$\left(\frac{d}{dt}\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega} \times\right)\left\{\left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{V}\right\}$$

$$\left(\frac{d^2\mathbf{V}}{dt^2}\right)_{inertial} = \left(\frac{d^2\mathbf{V}}{dt^2}\right)_{body} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{V} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{V}$$

Kinetic energy of rigid body,
rotating at angular velocity $\boldsymbol{\omega}$



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{r}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{r}$$

=0 for rigid body

$$\rightarrow \left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega}\times\mathbf{r}$$

$$T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\left|\boldsymbol{\omega}\times\mathbf{r}_p\right|\right)^2$$

$$= \sum_p \frac{1}{2} m_p \left(\boldsymbol{\omega}\times\mathbf{r}_p\right)\cdot\left(\boldsymbol{\omega}\times\mathbf{r}_p\right)$$

$$= \sum_p \frac{1}{2} m_p \left[(\boldsymbol{\omega}\cdot\boldsymbol{\omega})(\mathbf{r}_p\cdot\mathbf{r}_p) - (\mathbf{r}_p\cdot\boldsymbol{\omega})^2\right]$$

$$T = \sum_p \frac{1}{2} m_p \left[(\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r}_p \cdot \mathbf{r}_p) - (\mathbf{r}_p \cdot \boldsymbol{\omega})^2 \right]$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega}$$

Moment of inertia tensor:

$$\overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right) \qquad \text{(dyad notation)}$$

Matrix notation :

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

Moment of inertia tensor:

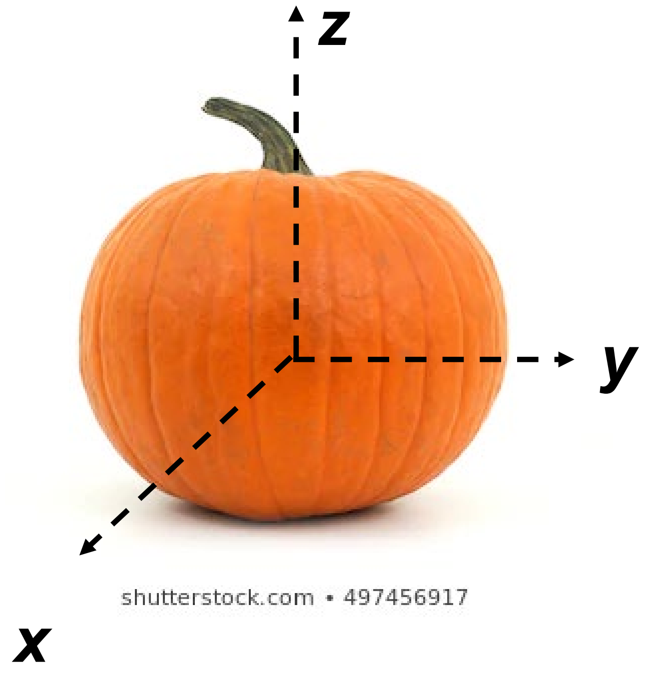
$$\overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right) \qquad \text{(dyad notation)}$$

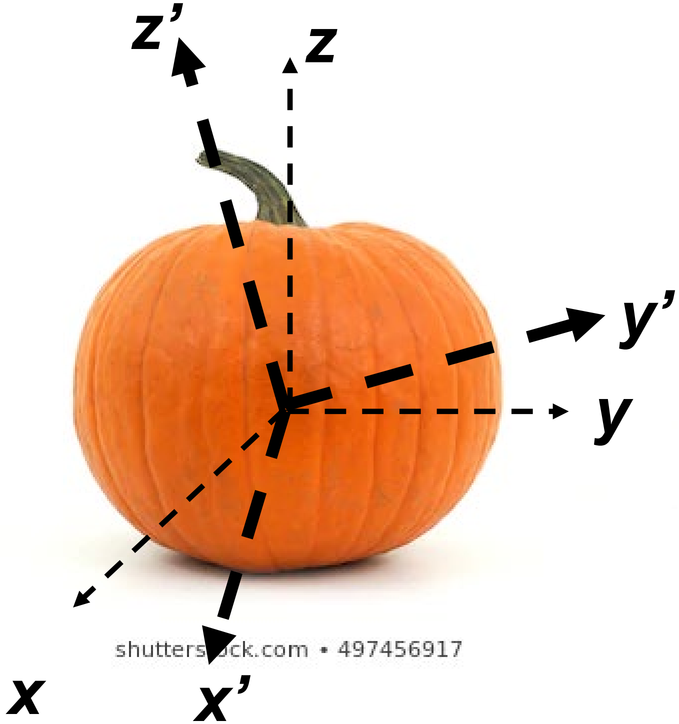
Note: For a given object and a given coordinate system, one can find the moment of inertia matrix

Matrix notation :

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$





Moment of inertia in original coordinates

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

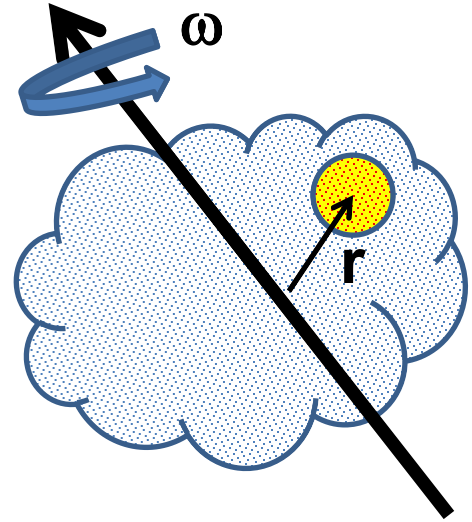
Moment of inertia in principal axes ***(x',y',z')***

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Angular momentum of rigid body:

$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{r}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{r}$$

=0 for rigid body



$$\rightarrow \left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \mathbf{v}_{inertial} = \boldsymbol{\omega}\times\mathbf{r}$$

$$\mathbf{r}\rightarrow\mathbf{r}_p \qquad \mathbf{v}\rightarrow\mathbf{v}_p$$

$$\mathbf{L} = \sum_p \mathbf{r}_p \times \left(m_p \mathbf{v}_p\right) = \sum_p m_p \mathbf{r}_p \times \left(\boldsymbol{\omega}\times\mathbf{r}_p\right)$$

$$= \sum_p m_p \left(\boldsymbol{\omega}\, r_p^2 - \mathbf{r}_p\left(\boldsymbol{\omega}\cdot\mathbf{r}_p\right)\right) = \overleftrightarrow{\mathbf{I}}\cdot\boldsymbol{\omega}$$

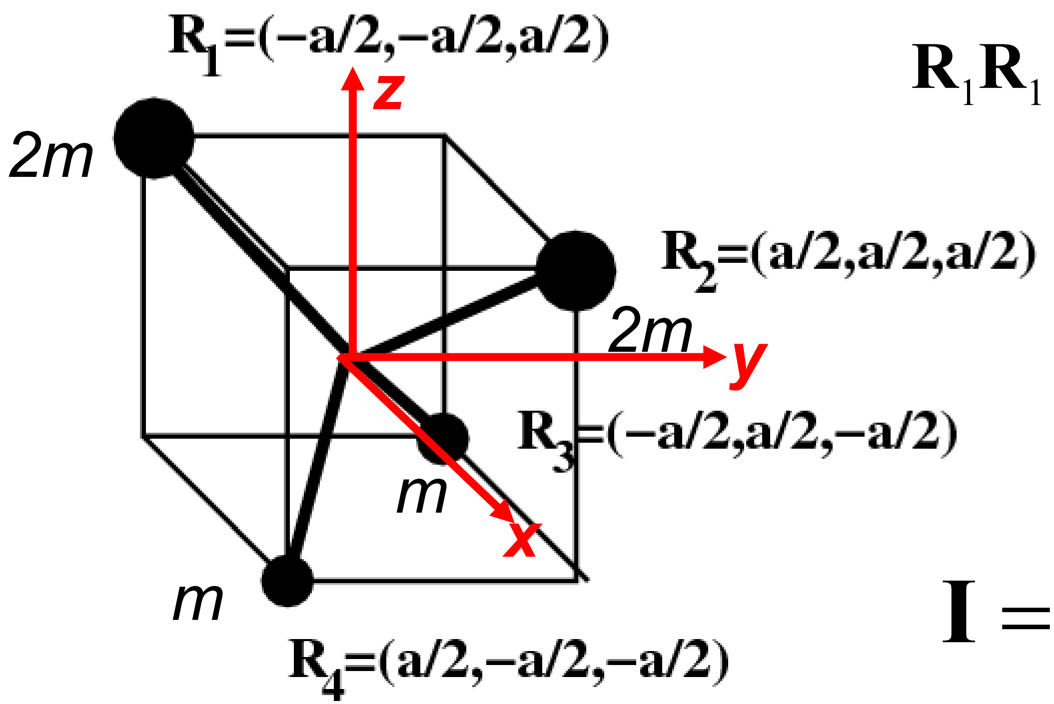
where $\overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p\right)$

An example with 4 point masses and massless rigid bonds

$$\overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right)$$

$$R_1^2 = R_2^2 = R_3^2 = R_4^2 = \frac{3a^2}{4}$$

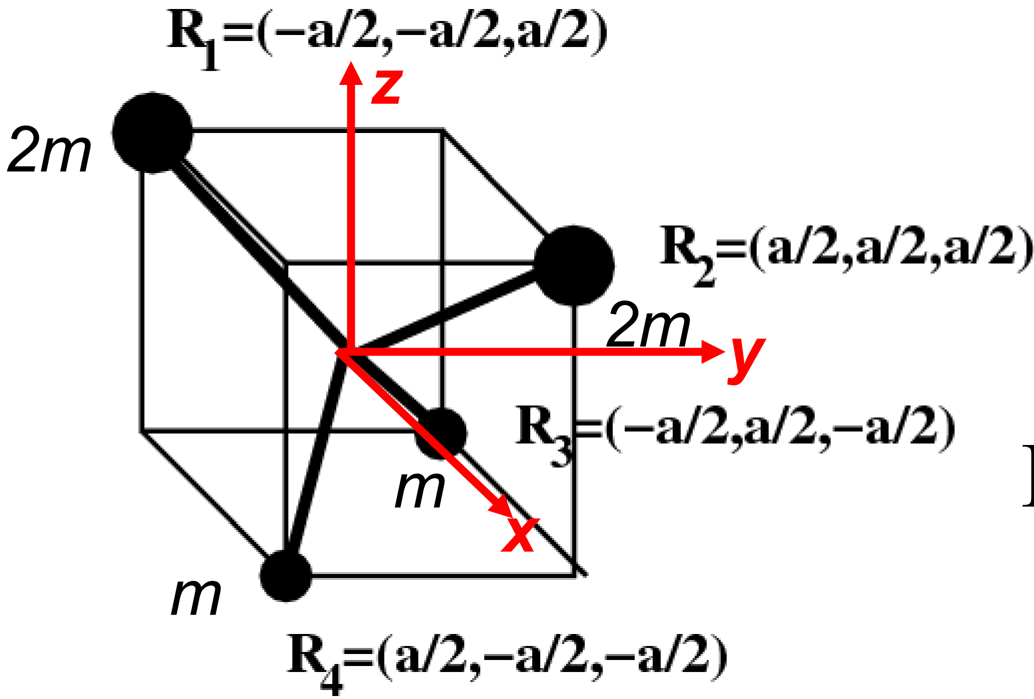
$$\mathbf{R}_1\mathbf{R}_1 = \frac{a^2}{4}\left(-\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}\right)\left(-\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}\right)$$



$$\mathbf{I} = ma^2 \begin{pmatrix} 3 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Example continued --

$$\overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right)$$



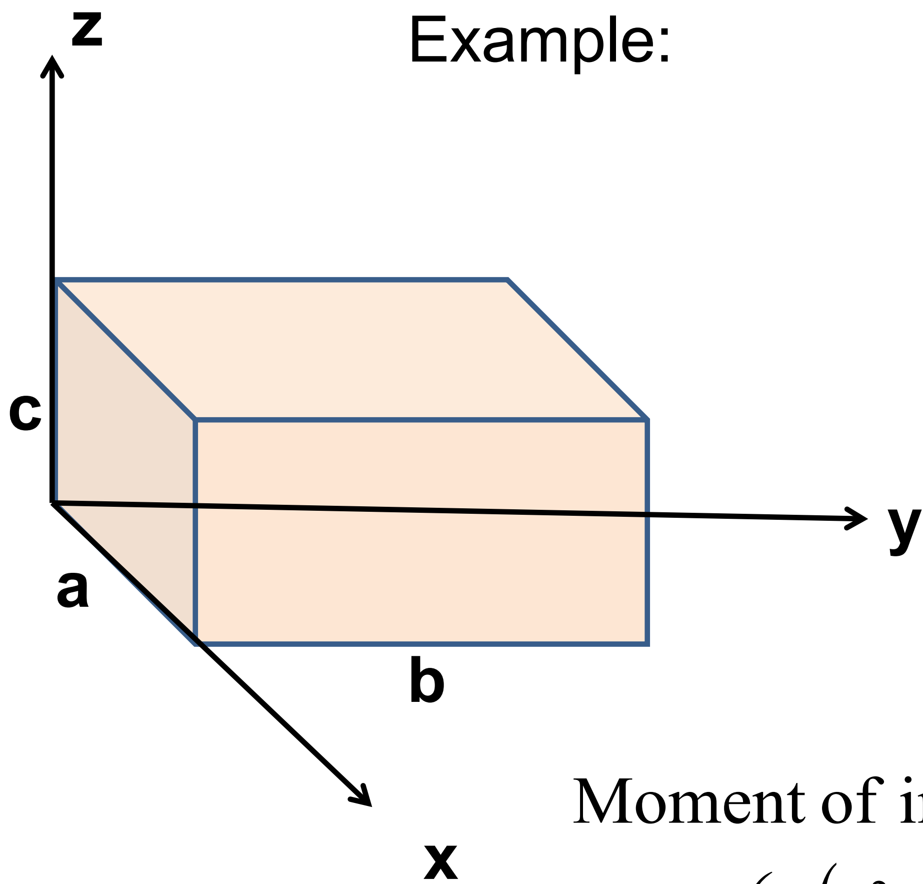
$$\mathbf{I} = ma^2 \begin{pmatrix} 3 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Principal moments:

$$I_1 = \frac{7}{2} ma^2 \quad \mathbf{v}_1 = \sqrt{\frac{1}{2}} \left(\hat{\mathbf{x}} - \hat{\mathbf{y}} \right)$$

$$I_2 = \frac{5}{2} ma^2 \quad \mathbf{v}_2 = \sqrt{\frac{1}{2}} \left(\hat{\mathbf{x}} + \hat{\mathbf{y}} \right)$$

$$I_3 = 3ma^2 \quad \mathbf{v}_3 = \hat{\mathbf{z}}$$



Example:

Moment of inertia tensor :

$$\overleftrightarrow{\mathbf{I}} = M\begin{pmatrix} \frac{1}{3}\left(b^2+c^2\right) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}\left(a^2+c^2\right) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}\left(a^2+b^2\right) \end{pmatrix}$$

Properties of moment of inertia tensor:

- Symmetric matrix → real eigenvalues I_1, I_2, I_3
- → orthogonal eigenvectors

$$\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad i = 1,2,3$$

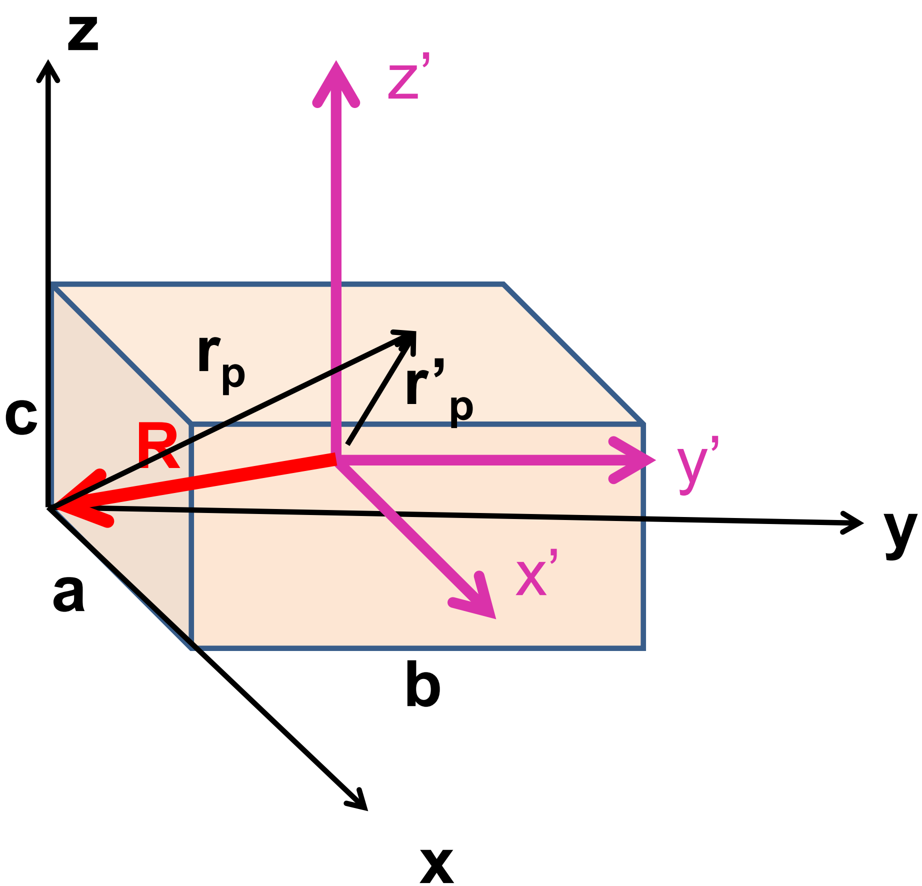
Moment of inertia tensor :

$$\overleftrightarrow{\mathbf{I}} = M \begin{pmatrix} \frac{1}{3}\left(b^2 + c^2\right) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}\left(a^2 + c^2\right) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}\left(a^2 + b^2\right) \end{pmatrix}$$

For $\quad a = b = c :$

$$I_1 = \frac{1}{6}Ma^2 \quad I_2 = \frac{11}{12}Ma^2 \quad I_3 = \frac{11}{12}Ma^2$$

Changing origin of rotation



$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

$$I'_{ij} \equiv \sum_p m_p \left(\delta_{ij} r'^2_p - r'_{pi}\, r'_{pj} \right)$$

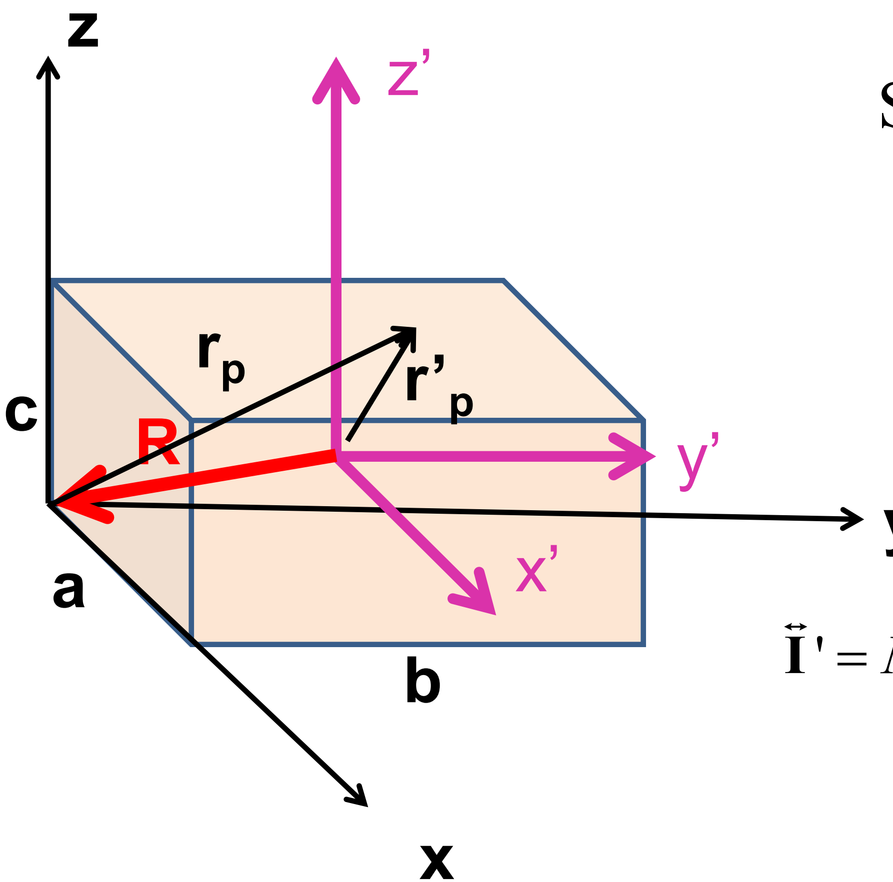
$$\mathbf{r}'_p = \mathbf{r}_p + \mathbf{R}$$

Define the center of mass :

$$\mathbf{r}_{CM} = \frac{\sum_p m_p \mathbf{r}_p}{\sum_p m_p} \equiv \frac{\sum_p m_p \mathbf{r}_p}{M}$$

$$I'_{ij} = I_{ij} + M\left(R^2 \delta_{ij} - R_i R_j\right) + M\left(2\mathbf{r}_{CM} \cdot \mathbf{R}\delta_{ij} - r_{CMi} R_j - R_i r_{CMj}\right)$$

$$I'_{ij} = I_{ij} + M\left(R^2\delta_{ij} - R_iR_j\right) + M\left(2\mathbf{r}_{CM}\cdot\mathbf{R}\delta_{ij} - r_{CMi}R_j - R_ir_{CMj}\right)$$



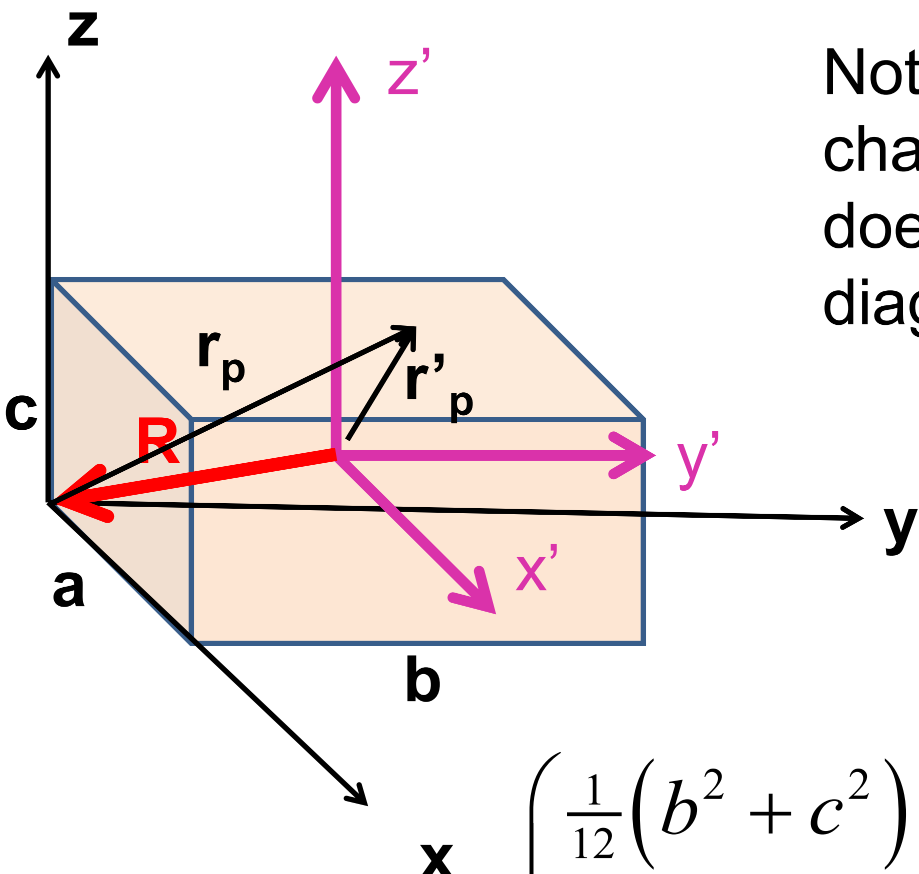
Suppose that $\mathbf{R} = -\frac{a}{2}\hat{\mathbf{x}} - \frac{b}{2}\hat{\mathbf{y}} - \frac{c}{2}\hat{\mathbf{z}}$

and $\mathbf{r}_{\mathrm{CM}} = -\mathbf{R}$

$$I'_{ij} = I_{ij} - M\left(R^2\delta_{ij} - R_iR_j\right)$$

$$\overleftrightarrow{\mathbf{I}}' = M\begin{pmatrix} \frac{1}{3}\left(b^2+c^2\right) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}\left(a^2+c^2\right) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}\left(a^2+b^2\right) \end{pmatrix}$$

$$-\;M\begin{pmatrix} \frac{1}{4}\left(b^2+c^2\right) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{4}\left(a^2+c^2\right) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{4}\left(a^2+b^2\right) \end{pmatrix}$$



Note: This is a special case; changing the center of rotation does not necessarily result in a diagonal **I'**

$$\overleftrightarrow{\mathbf{I}}' = M \begin{pmatrix} \frac{1}{12}\left(b^2 + c^2\right) & 0 & 0 \\ 0 & \frac{1}{12}\left(a^2 + c^2\right) & 0 \\ 0 & 0 & \frac{1}{12}\left(a^2 + b^2\right) \end{pmatrix}$$

Descriptions of rotation about a given origin

For general coordinate system

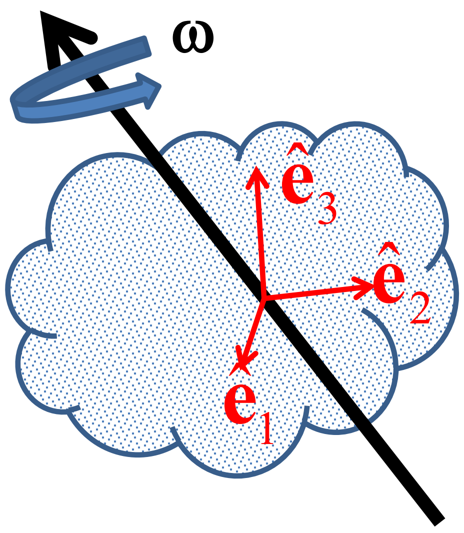
$$T = \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j$$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor :

$$\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad\qquad i = 1,2,3$$

$$\boldsymbol{\omega} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\Rightarrow T = \frac{1}{2}\sum_i I_i \tilde{\omega}_i^2$$



Descriptions of rotation about a given origin -- continued

Time rate of change of angular momentum

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L}$$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor:

$$\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad \boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\frac{d\mathbf{L}}{dt} = I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 + \tilde{\omega}_2 \tilde{\omega}_3 \left(I_3 - I_2\right) \hat{\mathbf{e}}_1$$

$$+ \tilde{\omega}_3 \tilde{\omega}_1 \left(I_1 - I_3\right) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 \left(I_2 - I_1\right) \hat{\mathbf{e}}_3$$

Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

For $\boldsymbol{\tau} = 0$ we can solve the Euler equations:

$$\frac{d\mathbf{L}}{dt} = I_1\dot{\tilde{\omega}}_1\hat{\mathbf{e}}_1 + I_2\dot{\tilde{\omega}}_2\hat{\mathbf{e}}_2 + I_3\dot{\tilde{\omega}}_3\hat{\mathbf{e}}_3 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right)\hat{\mathbf{e}}_1$$

$$+ \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right)\hat{\mathbf{e}}_2 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right)\hat{\mathbf{e}}_3 = 0$$

Torqueless Euler equations for rotation in body fixed frame:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

➔ Solution for symmetric object with $I_2 = I_1$:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_1\right) = 0$$

$$I_1\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 = 0 \qquad \Rightarrow \tilde{\omega}_3 = \text{(constant)}$$

Define: $\Omega \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_1}$

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2\Omega$$

$$\dot{\tilde{\omega}}_2 = \tilde{\omega}_1\Omega$$

Solution of Euler equations for symmetric object continued

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2 \Omega \qquad \dot{\tilde{\omega}}_2 = \tilde{\omega}_1 \Omega$$

where $\Omega \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_1}$

Solution: $\tilde{\omega}_1(t) = A\cos(\Omega t + \phi)$

$$\tilde{\omega}_2(t) = A\sin(\Omega t + \phi)$$

$$\tilde{\omega}_3(t) = \tilde{\omega}_3 \quad \text{(constant)}$$

$$T = \frac{1}{2}\sum_i I_i \tilde{\omega}_i^2 = \frac{1}{2} I_1 A^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$
$$= I_1 A\left(\cos(\Omega t + \varphi)\hat{\mathbf{e}}_1 + \sin(\Omega t + \varphi)\hat{\mathbf{e}}_2\right) + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

Torqueless Euler equations for rotation in body fixed frame:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

→ Solution for asymmetric object: $I_3 \neq I_2 \neq I_1$:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

Suppose: $\dot{\tilde{\omega}}_3 \approx 0$

Define: $\Omega_1 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_2}{I_1}$

Define: $\Omega_2 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_2}$

Euler equations for rotation in body fixed frame:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

Solution for asymmetric object $I_3 \neq I_2 \neq I_1$:

Approximate solution --

Suppose: $\dot{\tilde{\omega}}_3 \approx 0$ Define: $\Omega_1 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_2}{I_1}$

Define: $\Omega_2 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_2}$

Euler equations for asymmetric object continued

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

If $\dot{\tilde{\omega}}_3 \approx 0,$ Define: $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$ $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

$$\dot{\tilde{\omega}}_1 = -\Omega_1\tilde{\omega}_2 \qquad \dot{\tilde{\omega}}_2 = \Omega_2\tilde{\omega}_1$$

If Ω_1 and Ω_2 are both positive or both negative:

$$\tilde{\omega}_1(t) \approx A\cos\left(\sqrt{\Omega_1\Omega_2}\,t + \varphi\right)$$

$$\tilde{\omega}_2(t) \approx A\sqrt{\frac{\Omega_2}{\Omega_1}}\sin\left(\sqrt{\Omega_1\Omega_2}\,t + \varphi\right)$$

$\Rightarrow$ If Ω_1 and Ω_2 have opposite signs, solution is unstable.