PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 11: Rigid bodies – Chap. 5 (F &W)

1. More about moment of inertia tensor
2. Torque free motion
3. Digression on matrix diagonalization

George Holzwarth Memorial Biophysics Lecture

- Thursday - September 19, 2024

Something's wrong in the cellular neighborhood: the biophysics of how cells respond to nearby wounds

When a layer of epithelial cells is injured, surrounding cells respond in a distance-dependent manner to reseal the wound. So, how do the surrounding cells "know" that there is a wound nearby, i.e., that something is wrong in the cellular neighborhood? The earliest response is conserved across a wide range of organisms, including both plants and animals, and that response is a dramatic increase in cytosolic calcium concentrations. We have investigated this process in fruit flies using fast and reproducible laser wounds. Our results show that this increase occurs quickly – calcium floods into damaged cells within tens of milliseconds, moves into adjacent cells over ~20 s, and appears in a much larger set of surrounding cells via a delayed second expansion over 40-300 s – but calcium is nonetheless a reporter: cells must detect wounds even earlier. We will discuss how measurements of laser-tissue interactions can be combined with quantitative image analysis and the genetic tools available n terms of its biophysical and biochemical mechanisms. We will discuss the experimental evidence and develop a corresponding computational model that matches experimental observations, tests the plausibility of hypothesized mechanisms, and makes experimentally testable predictions. We will then discuss how these early signals relate to subsequent cell behaviors such as cell-cell fusion, cell migration into the wounded area, and the re-establishment of epithelial tension. This work supported by NIH Grant 1R01GM130130.

Professor M. Shane Hutson

Vanderbilt University

Reception 3:30
Olin Lobby

Colloquium 4:00
Olin 101

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/26/2024		Introduction and overview	#1
2	Wed, 8/28/2024	Chap. 3(17)	Calculus of variation	#2
3	Fri, 8/30/2024	Chap. 3(17)	Calculus of variation	#3
4	Mon, 9/02/2024	Chap. 3	Lagrangian equations of motion	#4
5	Wed, 9/04/2024	Chap. 3 & 6	Lagrangian equations of motion	#5
6	Fri, 9/06/2024	Chap. 3 & 6	Lagrangian equations of motion	#6
7	Mon, 9/09/2024	Chap. 3 & 6	Lagrangian to Hamiltonian formalism	#7
8	Wed, 9/11/2024	Chap. 3 & 6	Phase space	#8
9	Fri, 9/13/2024	Chap. 3 & 6	Canonical Transformations	
10	Mon, 9/16/2024	Chap. 5	Dynamics of rigid bodies	#9
11	Wed, 9/18/2024	Chap. 5	Dynamics of rigid bodies	#10
12	Fri, 9/20/2024	Chap. 5	Dynamics of rigid bodies	

PHY 711 – Assignment #10

Assigned: 09/18/2024 Due: 09/23/2024

The material for this exercise is covered in the lecture notes and in Chapters 5 of Fetter and Walecka.

1. Consider a matrix representing the moment of inertia of a system having the form

$$I = \begin{pmatrix} A & B & 0 \\ B & C & 0 \\ 0 & 0 & D \end{pmatrix},$$

where A, B, C, D are all real values. A related moment of inertia matrix I' can be found from a transformation matrix R, where

$$I' = RIR^{-1}.$$

Suppose the transformation matrix R has the form

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where θ is a real angle.

(a) Find the form of the matrix I' as function of the angle θ.

(b) For what value of θ is matrix I' diagonal?

(c) For the value of θ found in part (b), determine the eigenvalues and eigenvectors of the moment of inertia matrix for this system

Moment of inertia tensor:

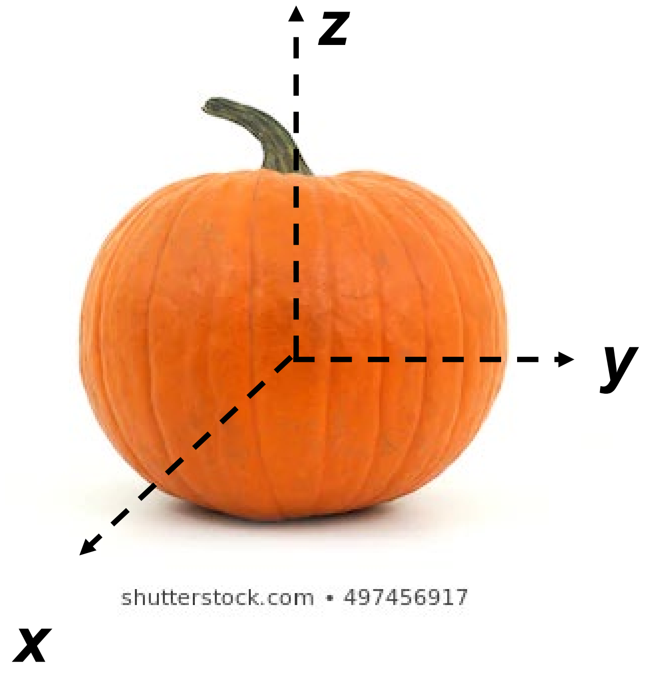
$$\overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right) \qquad \text{(dyad notation)}$$

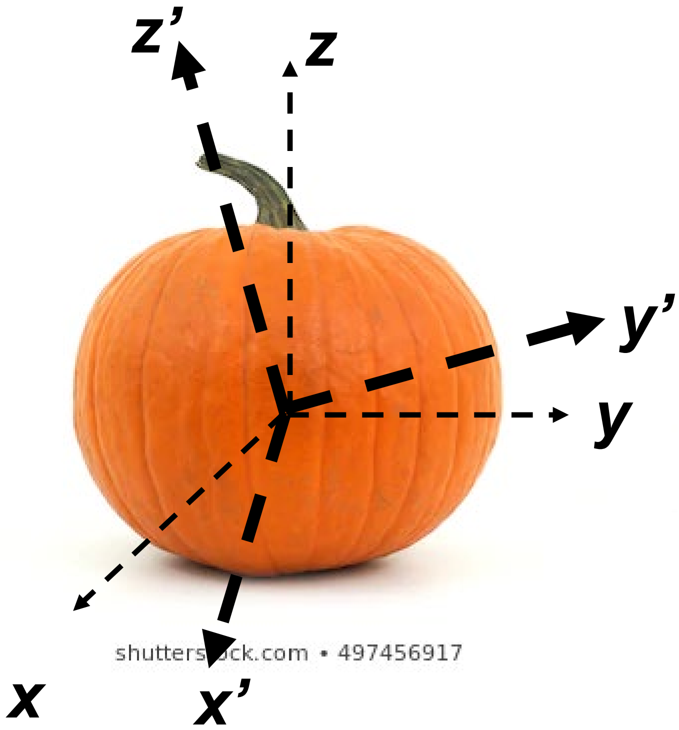
Note: For a given object and a given coordinate system, one can find the moment of inertia matrix

Matrix notation :

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$





Moment of inertia in original coordinates

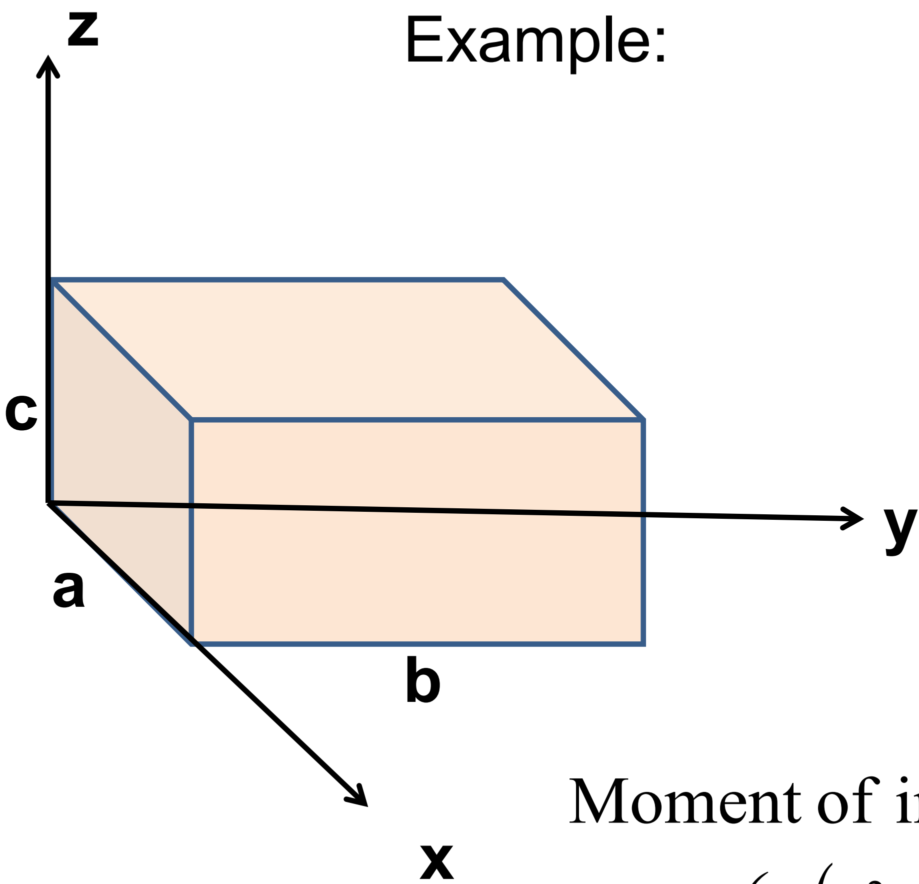
$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

Moment of inertia in principal axes *(x',y',z')*

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Example:



Moment of inertia tensor :

$$\overleftrightarrow{\mathbf{I}} = M\begin{pmatrix} \frac{1}{3}\left(b^2+c^2\right) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}\left(a^2+c^2\right) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}\left(a^2+b^2\right) \end{pmatrix}$$

Properties of moment of inertia tensor:

- Symmetric matrix → real eigenvalues I_1, I_2, I_3
- → orthogonal eigenvectors

$$\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad i = 1,2,3$$

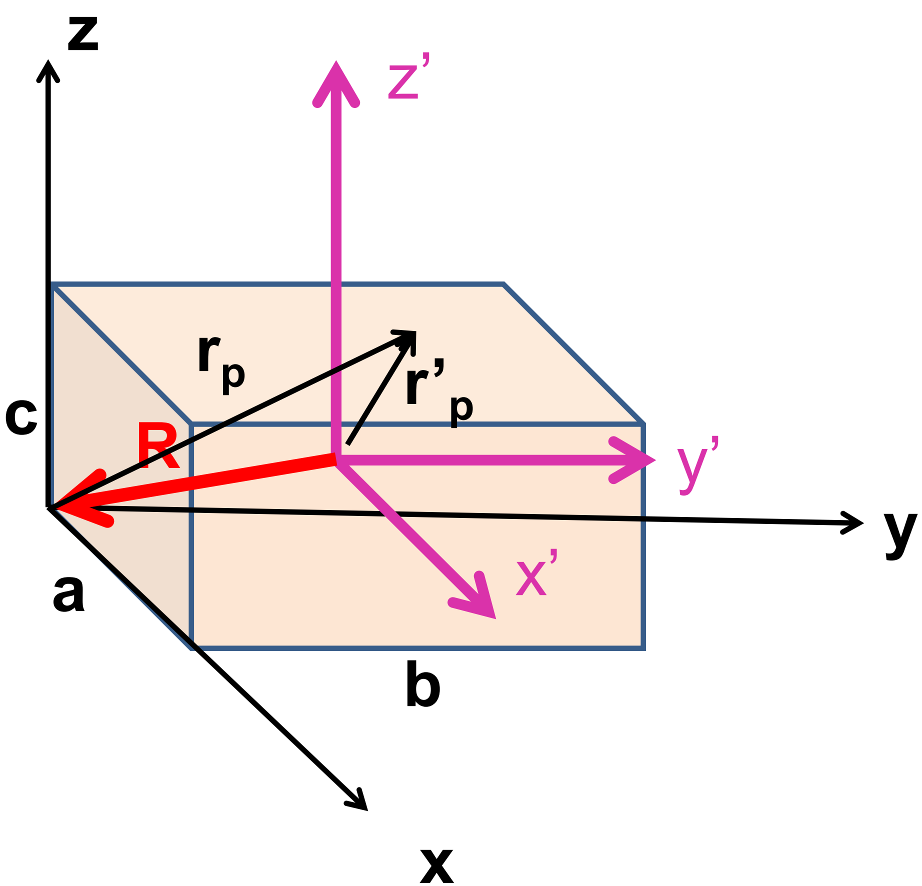
Moment of inertia tensor :

$$\overleftrightarrow{\mathbf{I}} = M \begin{pmatrix} \frac{1}{3}\left(b^2 + c^2\right) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}\left(a^2 + c^2\right) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}\left(a^2 + b^2\right) \end{pmatrix}$$

For $\quad a = b = c :$

$$I_1 = \frac{1}{6} Ma^2 \quad I_2 = \frac{11}{12} Ma^2 \quad I_3 = \frac{11}{12} Ma^2$$

Changing origin of rotation



$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

$$I'_{ij} \equiv \sum_p m_p \left(\delta_{ij} r'^2_p - r'_{pi} \, r'_{pj} \right)$$

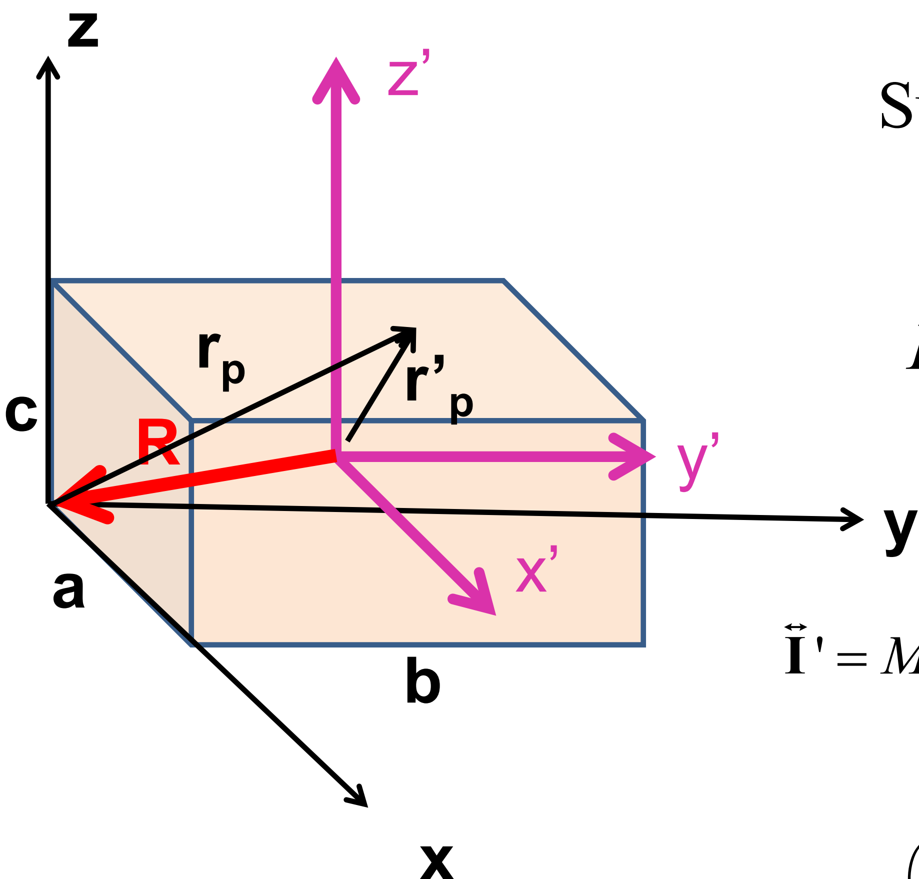
$$\mathbf{r}'_p = \mathbf{r}_p + \mathbf{R}$$

Define the center of mass:

$$\mathbf{r}_{CM} = \frac{\sum_p m_p \mathbf{r}_p}{\sum_p m_p} \equiv \frac{\sum_p m_p \mathbf{r}_p}{M}$$

$$I'_{ij} = I_{ij} + M\left(R^2 \delta_{ij} - R_i R_j\right) + M\left(2\mathbf{r}_{CM} \cdot \mathbf{R} \delta_{ij} - r_{CMi} R_j - R_i r_{CMj}\right)$$

$$I'_{ij} = I_{ij} + M\left(R^2\delta_{ij} - R_iR_j\right) + M\left(2\mathbf{r}_{CM}\cdot\mathbf{R}\delta_{ij} - r_{CMi}R_j - R_ir_{CMj}\right)$$



Suppose that $\mathbf{R} = -\frac{a}{2}\hat{\mathbf{x}} - \frac{b}{2}\hat{\mathbf{y}} - \frac{c}{2}\hat{\mathbf{z}}$

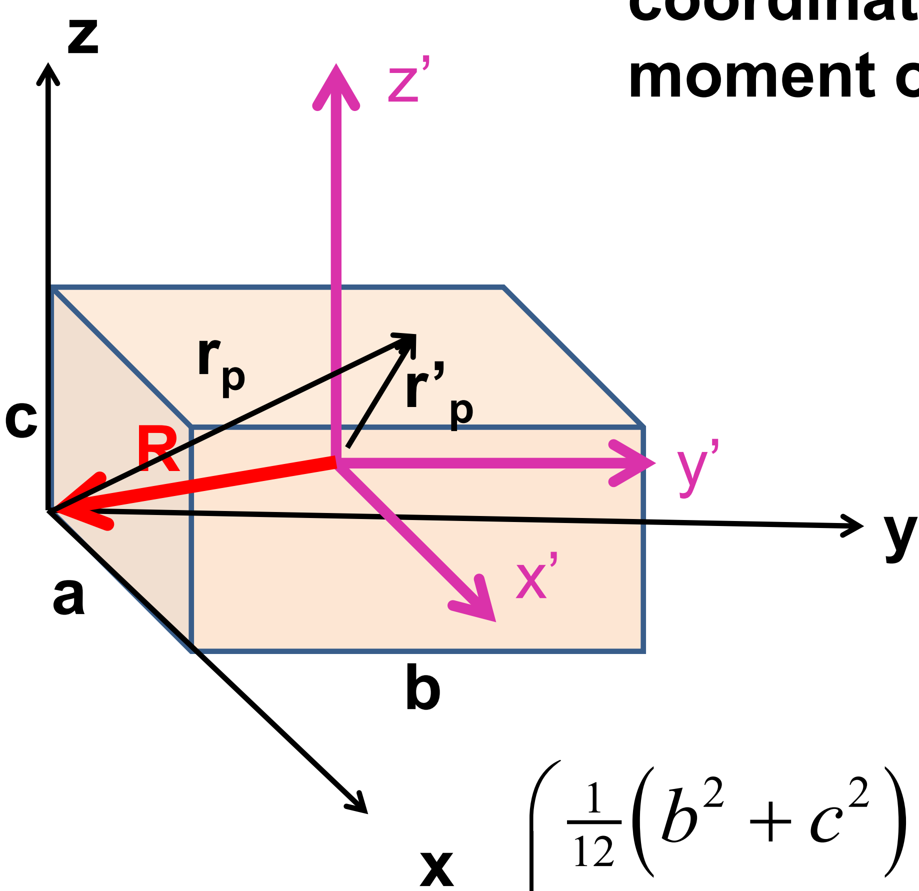
and $\mathbf{r}_{\mathrm{CM}} = -\mathbf{R}$

$$I'_{ij} = I_{ij} - M\left(R^2\delta_{ij} - R_iR_j\right)$$

$$\overleftrightarrow{\mathbf{I}}' = M\begin{pmatrix} \frac{1}{3}\left(b^2+c^2\right) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}\left(a^2+c^2\right) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}\left(a^2+b^2\right) \end{pmatrix}$$

$$-\ M\begin{pmatrix} \frac{1}{4}\left(b^2+c^2\right) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{4}\left(a^2+c^2\right) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{4}\left(a^2+b^2\right) \end{pmatrix}$$

Note that changing origin of coordinate system changes moment of inertia tensor.



Note: This is a special case; changing the center of rotation does not necessarily result in a diagonal **I'**

$$\overleftrightarrow{\mathbf{I}}' = M\begin{pmatrix} \frac{1}{12}\left(b^2+c^2\right) & 0 & 0 \\ 0 & \frac{1}{12}\left(a^2+c^2\right) & 0 \\ 0 & 0 & \frac{1}{12}\left(a^2+b^2\right) \end{pmatrix}$$

Descriptions of rotation about a given origin

For general coordinate system

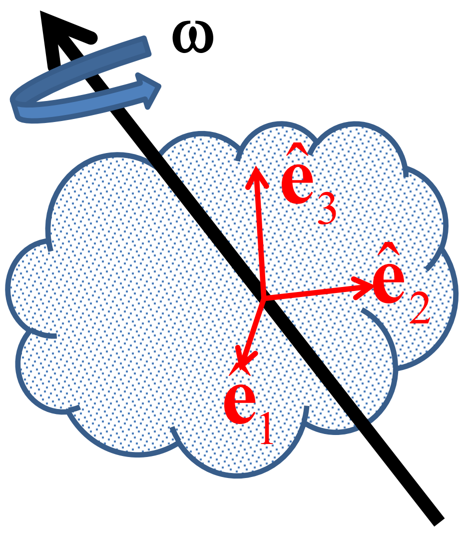
$$T = \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j$$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor :

$$\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad i = 1,2,3$$

$$\boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\Rightarrow T = \frac{1}{2}\sum_i I_i \tilde{\omega}_i^2$$



Descriptions of rotation about a given origin -- continued

Time rate of change of angular momentum

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L}$$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor:

$$\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad \boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\frac{d\mathbf{L}}{dt} = I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 + \tilde{\omega}_2 \tilde{\omega}_3 \left(I_3 - I_2\right) \hat{\mathbf{e}}_1 + \tilde{\omega}_3 \tilde{\omega}_1 \left(I_1 - I_3\right) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 \left(I_2 - I_1\right) \hat{\mathbf{e}}_3$$

Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

For $\boldsymbol{\tau} = 0$ we can solve the Euler equations:

$$\frac{d\mathbf{L}}{dt} = I_1\dot{\tilde{\omega}}_1\hat{\mathbf{e}}_1 + I_2\dot{\tilde{\omega}}_2\hat{\mathbf{e}}_2 + I_3\dot{\tilde{\omega}}_3\hat{\mathbf{e}}_3 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right)\hat{\mathbf{e}}_1 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right)\hat{\mathbf{e}}_2 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right)\hat{\mathbf{e}}_3 = 0$$

Torqueless Euler equations for rotation in body fixed frame:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

➔ Solution for symmetric object with $I_2 = I_1$:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_1\right) = 0$$

$$I_1\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 = 0 \qquad \Rightarrow \tilde{\omega}_3 = (\text{constant})$$

Define: $\Omega \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_1}$

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2\Omega$$

$$\dot{\tilde{\omega}}_2 = \tilde{\omega}_1\Omega$$

Solution of Euler equations for symmetric object continued

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2 \Omega \qquad \dot{\tilde{\omega}}_2 = \tilde{\omega}_1 \Omega$$

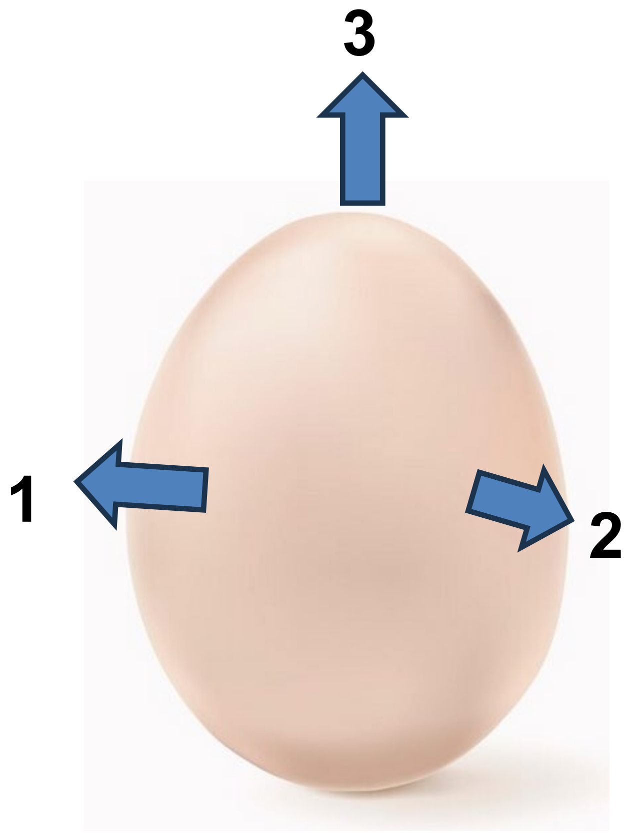
$$\text{where } \Omega \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_1}$$

Solution:
$$\tilde{\omega}_1(t) = A\cos(\Omega t + \phi)$$
$$\tilde{\omega}_2(t) = A\sin(\Omega t + \phi)$$
$$\tilde{\omega}_3(t) = \tilde{\omega}_3 \quad \text{(constant)}$$

$$T = \frac{1}{2}\sum_i I_i \tilde{\omega}_i^2 = \frac{1}{2} I_1 A^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$
$$= I_1 A\left(\cos(\Omega t + \varphi)\hat{\mathbf{e}}_1 + \sin(\Omega t + \varphi)\hat{\mathbf{e}}_2\right) + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

Example symmetric top --



Torqueless Euler equations for rotation in body fixed frame:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

→ Solution for asymmetric object: $I_3 \neq I_2 \neq I_1$:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

Suppose: $\dot{\tilde{\omega}}_3 \approx 0$

Define: $\Omega_1 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_2}{I_1}$

Define: $\Omega_2 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_2}$

Euler equations for rotation in body fixed frame:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

Solution for asymmetric object $I_3 \neq I_2 \neq I_1$:

Approximate solution --

Suppose: $\dot{\tilde{\omega}}_3 \approx 0$ Define: $\Omega_1 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_2}{I_1}$

Define: $\Omega_2 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_2}$

Euler equations for asymmetric object continued

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

If $\dot{\tilde{\omega}}_3 \approx 0$, Define: $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$ $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

$$\dot{\tilde{\omega}}_1 = -\Omega_1\tilde{\omega}_2 \qquad \dot{\tilde{\omega}}_2 = \Omega_2\tilde{\omega}_1$$

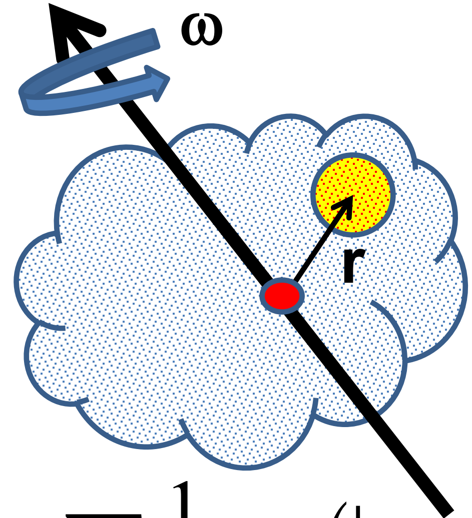
If Ω_1 and Ω_2 are both positive or both negative:

$$\tilde{\omega}_1(t) \approx A\cos\left(\sqrt{\Omega_1\Omega_2}\,t + \varphi\right)$$

$$\tilde{\omega}_2(t) \approx A\sqrt{\frac{\Omega_2}{\Omega_1}}\sin\left(\sqrt{\Omega_1\Omega_2}\,t + \varphi\right)$$

$\Rightarrow$ If Ω_1 and Ω_2 have opposite signs, solution is unstable.

Summary of previous results describing rigid bodies rotating about a fixed origin ●



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

Kinetic energy: $T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\left|\boldsymbol{\omega} \times \mathbf{r}_p\right|\right)^2$

$$= \sum_p \frac{1}{2} m_p \left(\boldsymbol{\omega} \times \mathbf{r}_p\right) \cdot \left(\boldsymbol{\omega} \times \mathbf{r}_p\right)$$

$$= \sum_p \frac{1}{2} m_p \left[\left(\boldsymbol{\omega} \cdot \boldsymbol{\omega}\right)\left(\mathbf{r}_p \cdot \mathbf{r}_p\right) - \left(\mathbf{r}_p \cdot \boldsymbol{\omega}\right)^2\right]$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega} \qquad \overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p\right)$$

Moment of inertia tensor

Matrix notation:

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \qquad I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

For general coordinate system: $T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor: $\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad i = 1,2,3$

$$\boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3 \qquad \Rightarrow T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2$$

Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

In principle,

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} = I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 +$$

$$\tilde{\omega}_2 \tilde{\omega}_3 \left(I_3 - I_2\right) \hat{\mathbf{e}}_1 + \tilde{\omega}_3 \tilde{\omega}_1 \left(I_1 - I_3\right) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 \left(I_2 - I_1\right) \hat{\mathbf{e}}_3$$

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 \left(I_3 - I_2\right) = \tau_1$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 \left(I_1 - I_3\right) = \tau_2$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 \left(I_2 - I_1\right) = \tau_3$$

Only useful if we can express $\boldsymbol{\tau}$ in the body fixed coordinate frame. Next time, Euler angles will come to the rescue.

Digression on matrix diagonalization

Moment of inertia tensor in original coordinates

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

Note that $I_{ij} = I_{ji}$

Moment of inertia tensor in principal axis system

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

$$\hat{\mathbf{e}}_1 \qquad \hat{\mathbf{e}}_2 \qquad \hat{\mathbf{e}}_3$$

➔Eigenvalues of a symmetric matrix are real and eigenvectors are orthogonal.

Finding eigenvalues/eigenvectors by hand for a general matrix $\mathbf{M}$ --

$$\mathbf{M}\mathbf{y}^{\alpha} = \lambda^{\alpha}\mathbf{y}^{\alpha}$$

$$\left(\mathbf{M} - \lambda^{\alpha}\mathbf{I}\right)\mathbf{y}^{\alpha} = 0$$

$$\left|\mathbf{M} - \lambda^{\alpha}\mathbf{I}\right| \equiv \det\left(\mathbf{M} - \lambda^{\alpha}\mathbf{I}\right) = 0 \quad \Rightarrow \text{polynomial for solutions } \lambda^{\alpha}$$

For each α and λ^{α} solve for the eigenvector coefficients $\mathbf{y}^{\alpha}$

Example

$$\mathbf{M} = \begin{pmatrix} A & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A \end{pmatrix}$$

$$\left|\mathbf{M} - \lambda^{\alpha}\mathbf{I}\right| = \begin{vmatrix} A-\lambda^{\alpha} & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B-\lambda^{\alpha} & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A-\lambda^{\alpha} \end{vmatrix} = \lambda^{\alpha}\left(\lambda^{\alpha} - A\right)\left(\lambda^{\alpha} - (A+2B)\right) = 0$$

Example -- continued

$$\left|\mathbf{M}-\lambda^{\alpha}\mathbf{I}\right| = \begin{vmatrix} A-\lambda^{\alpha} & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B-\lambda^{\alpha} & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A-\lambda^{\alpha} \end{vmatrix} = \lambda^{\alpha}\left(\lambda^{\alpha}-A\right)\left(\lambda^{\alpha}-(A+2B)\right)$$

Solving for eigenvector corresponding to $\lambda^{\alpha} \equiv \lambda^{1} = 0$

$$\begin{pmatrix} A & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A \end{pmatrix}\begin{pmatrix} y_x^1 \\ y_y^1 \\ y_z^1 \end{pmatrix} = 0 \qquad \Rightarrow \frac{y_x^1}{y_y^1} = \frac{y_z^1}{y_y^1} = \sqrt{\frac{B}{A}}$$

Normalized eigenvector: $\hat{\mathbf{e}}_1 = \dfrac{1}{\sqrt{1+2B/A}}\begin{pmatrix} \sqrt{B/A} \\ 1 \\ \sqrt{B/A} \end{pmatrix}$

Digression on matrices -- continued

Eigenvalues of a matrix are “invariant” under a similarity transformation

Eigenvalue properties of matrix: $\mathbf{M}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$

Transformed matrix: $\mathbf{M}'\mathbf{y}'_\alpha = \lambda'_\alpha \mathbf{y}'_\alpha$

If $\mathbf{M}' = \mathbf{S}\mathbf{M}\mathbf{S}^{-1}$ then $\lambda'_\alpha = \lambda_\alpha$ and $\mathbf{S}^{-1}\mathbf{y}'_\alpha = \mathbf{y}_\alpha$

Proof $\mathbf{S}\mathbf{M}\mathbf{S}^{-1}\mathbf{y}'_\alpha = \lambda'_\alpha \mathbf{y}'_\alpha$

$$\mathbf{M}\left(\mathbf{S}^{-1}\mathbf{y}'_\alpha\right) = \lambda'_\alpha \left(\mathbf{S}^{-1}\mathbf{y}'_\alpha\right)$$

This means that if a matrix is “similar” to a Hermitian matrix, it has the same eigenvalues. The corresponding eigenvectors of **M** and **M’** are not the same but $\mathbf{y}_\alpha = \mathbf{S}^{-1}\mathbf{y}'_\alpha$

Note, here we have defined **S** as a transformation matrix (often called a similarity transformation matrix)

Sometimes, the similarity transformation is also unitary so that

$$\mathbf{U}^{-1} = \mathbf{U}^{H}$$

Example for 2x2 case --

$$\mathbf{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \mathbf{U}^{-1} = \mathbf{U}^{H} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

How can you find a unitary transformation that also diagonalizes a matrix?

Example -- $\mathbf{M} = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \qquad \mathbf{M}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

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$\mathbf{M}' = \mathbf{U}\mathbf{M}\mathbf{U}^H$ $\quad$ for $\mathbf{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$\mathbf{M}' = \begin{pmatrix} A\cos^2\theta + C\sin^2\theta + B\sin 2\theta & -B\cos 2\theta - \frac{1}{2}(C-A)\sin 2\theta \\ -B\cos 2\theta - \frac{1}{2}(C-A)\sin 2\theta & A\sin^2\theta + C\cos^2\theta - B\sin 2\theta \end{pmatrix}$$

$$\Rightarrow \text{choose} \quad \theta = \frac{1}{2}\tan^{-1}\left(\frac{-2B}{C-A}\right)$$

$$\Rightarrow \lambda_1 = A\cos^2\theta + C\sin^2\theta + B\sin 2\theta$$

$$\Rightarrow \lambda_2 = A\sin^2\theta + C\cos^2\theta - B\sin 2\theta$$

Note that this “trick” is special for 2x2 matrices, but numerical extensions based on the trick are possible.

Note that transformations using unitary matrices are often convenient and they can be easily constructed from the eigenvalues of a matrix.

Suppose you have an $N \times N$ matrix $\mathbf{M}$ and find all N eigenvalues/vectors:

$\mathbf{M}\mathbf{y}^{\alpha} = \lambda^{\alpha}\mathbf{y}^{\alpha}$ orthonormalized so that $\left\langle \mathbf{y}^{\alpha} \middle| \mathbf{y}^{\beta} \right\rangle = \delta_{\alpha\beta}$

Now construct an $N \times N$ matrix $\mathbf{U}$ by listing the eigenvector columns:

$$\mathbf{U} \equiv \begin{pmatrix} y_1^1 & y_1^2 & \cdots & y_1^N \\ y_2^1 & y_2^2 & \cdots & y_2^N \\ \vdots & \vdots & \cdots & \vdots \\ y_N^1 & y_N^2 & \cdots & y_N^N \end{pmatrix} \qquad \mathbf{U}^{-1} \equiv \begin{pmatrix} y_1^{1*} & y_2^{1*} & \cdots & y_N^{1*} \\ y_1^{2*} & y_2^{2*} & \cdots & y_N^{N*} \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{N*} & y_2^{N*} & \cdots & y_N^{N*} \end{pmatrix} \qquad \Rightarrow \text{ by construction } \mathbf{U}^{-1}\mathbf{U} = \mathbf{I}$$

$$\text{Also by construction} \quad \mathbf{U}^{-1}\mathbf{M}\mathbf{U} = \begin{pmatrix} \lambda^1 & 0 & \cdots & 0 \\ 0 & \lambda^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda^N \end{pmatrix}$$