PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 12 – Chap. 5 (F &W)

Rotational motion of rigid bodies

1. Review of rigid body motion
2. Euler angles
3. Symmetric top motion

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/26/2024		Introduction and overview	#1
2	Wed, 8/28/2024	Chap. 3(17)	Calculus of variation	#2
3	Fri, 8/30/2024	Chap. 3(17)	Calculus of variation	#3
4	Mon, 9/02/2024	Chap. 3	Lagrangian equations of motion	#4
5	Wed, 9/04/2024	Chap. 3 & 6	Lagrangian equations of motion	#5
6	Fri, 9/06/2024	Chap. 3 & 6	Lagrangian equations of motion	#6
7	Mon, 9/09/2024	Chap. 3 & 6	Lagrangian to Hamiltonian formalism	#7
8	Wed, 9/11/2024	Chap. 3 & 6	Phase space	#8
9	Fri, 9/13/2024	Chap. 3 & 6	Canonical Transformations	
10	Mon, 9/16/2024	Chap. 5	Dynamics of rigid bodies	#9
11	Wed, 9/18/2024	Chap. 5	Dynamics of rigid bodies	#10
12	Fri, 9/20/2024	Chap. 5	Dynamics of rigid bodies	#11
13	Mon, 9/23/2024	Chap. 1	Scattering analysis	

PHY 711 -- Assignment #11

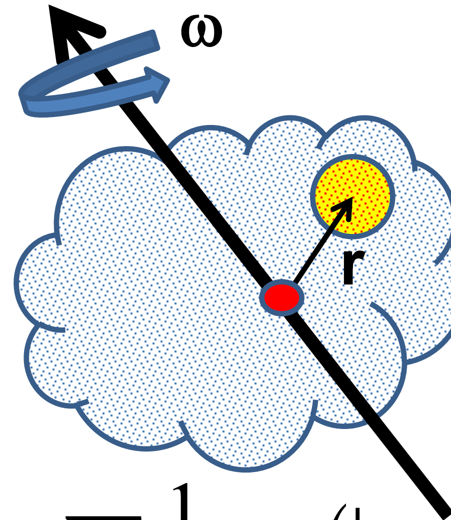
Assigned: 9/20/2024 Due: 9/23/2024

Finish reading Chapter 5 in **Fetter & Walecka**.

Consider problem 5.10 at the end of chapter 5 in **Fetter & Walecka**

a. Work part (a).
b. (Extra credit) Work part (b).

Summary of previous results describing rigid bodies rotating about a fixed origin



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

Kinetic energy: $T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\left|\boldsymbol{\omega} \times \mathbf{r}_p\right|\right)^2$

$$= \sum_p \frac{1}{2} m_p \left(\boldsymbol{\omega} \times \mathbf{r}_p\right) \cdot \left(\boldsymbol{\omega} \times \mathbf{r}_p\right)$$

$$= \sum_p \frac{1}{2} m_p \left[\left(\boldsymbol{\omega} \cdot \boldsymbol{\omega}\right)\left(\mathbf{r}_p \cdot \mathbf{r}_p\right) - \left(\mathbf{r}_p \cdot \boldsymbol{\omega}\right)^2\right]$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega} \qquad \overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p\right)$$

Moment of inertia tensor

Matrix notation:

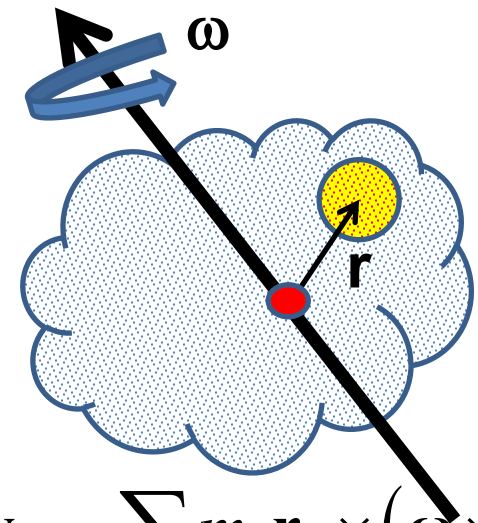
$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \qquad I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

For general coordinate system: $T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor: $\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad i = 1,2,3$

$$\boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3 \qquad \Rightarrow T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2$$

Continued -- summary of previous results describing rigid bodies rotating about a fixed origin



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

Angular momentum: $\mathbf{L} = \sum_p m_p \mathbf{r}_p \times \mathbf{v}_p = \sum_p m_p \mathbf{r}_p \times \left(\boldsymbol{\omega} \times \mathbf{r}_p\right)$

$$\mathbf{L} = \sum_p m_p \left[\boldsymbol{\omega}\left(\mathbf{r}_p \cdot \mathbf{r}_p\right) - \mathbf{r}_p\left(\mathbf{r}_p \cdot \boldsymbol{\omega}\right)\right]$$

$$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega} \qquad \overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p\right)$$

Descriptions of rotation about a given origin -- continued

For (body fixed) coordinate system that diagonalizes moment of inertia tensor:

$$\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad \boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

Time derivative: $\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L}$

$$\frac{d\mathbf{L}}{dt} = I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 +$$

$$\tilde{\omega}_2 \tilde{\omega}_3 \left(I_3 - I_2\right) \hat{\mathbf{e}}_1 + \tilde{\omega}_3 \tilde{\omega}_1 \left(I_1 - I_3\right) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 \left(I_2 - I_1\right) \hat{\mathbf{e}}_3$$

Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

In principle,

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} = I_1\dot{\tilde{\omega}}_1\hat{\mathbf{e}}_1 + I_2\dot{\tilde{\omega}}_2\hat{\mathbf{e}}_2 + I_3\dot{\tilde{\omega}}_3\hat{\mathbf{e}}_3 +$$

$$\tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right)\hat{\mathbf{e}}_1 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right)\hat{\mathbf{e}}_2 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right)\hat{\mathbf{e}}_3$$

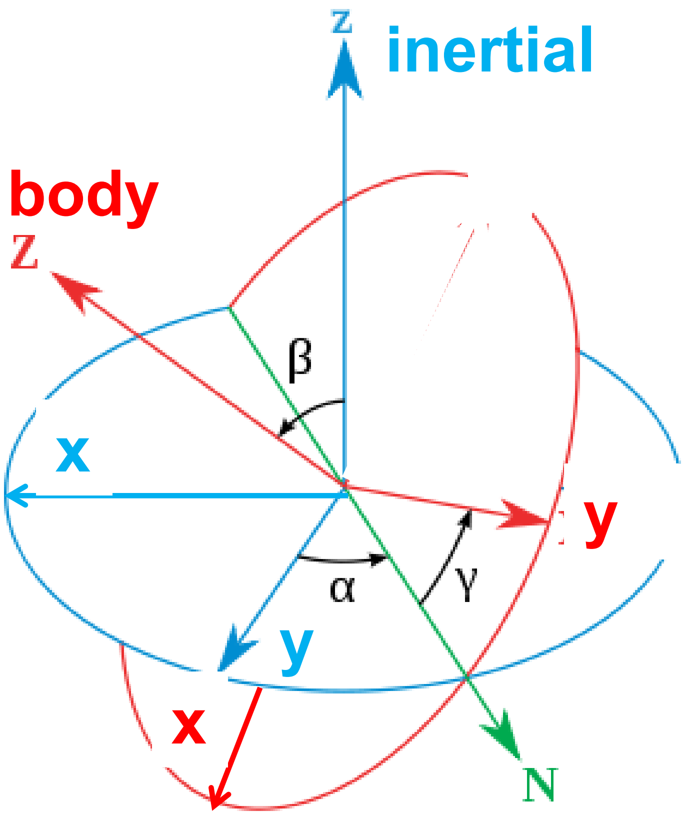
$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = \tau_1$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = \tau_2$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = \tau_3$$

Only useful if we can express $\boldsymbol{\tau}$ in the body fixed coordinate frame. Euler angles will come to the rescue.

Transformation between body-fixed and inertial coordinate systems – Euler angles

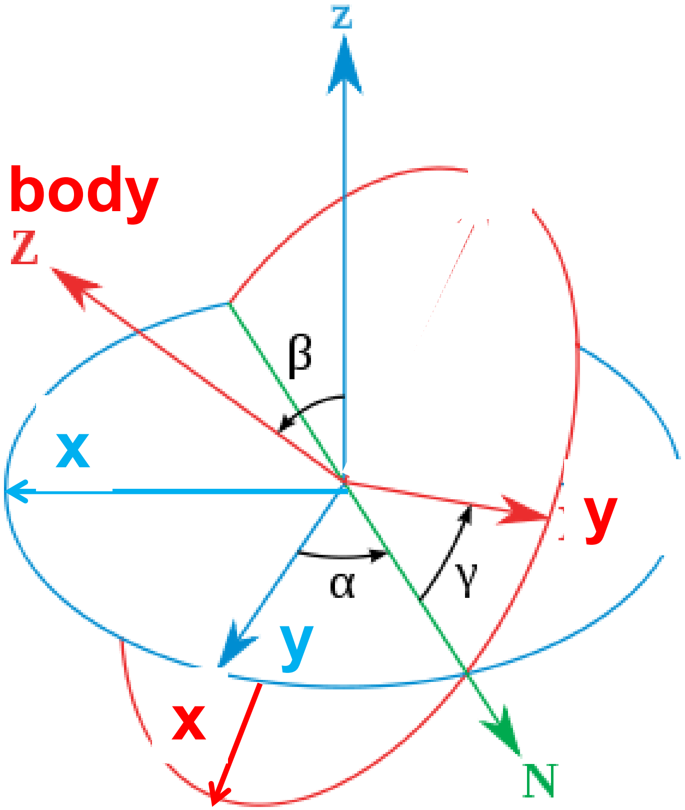


Comment – Since this is an old and intriguing subject, there are a lot of terminologies and conventions, not all of which are compatible. We are following the convention found in most quantum mechanics texts and NOT the convention found in most classical mechanics texts. Euler's main point is that any rotation can be described by 3 successive rotations about 3 different (not necessarily orthogonal) axes. In this case, one is along the inertial z axis and another is along the body fixed Z axis. The middle rotation is along an intermediate N axis.

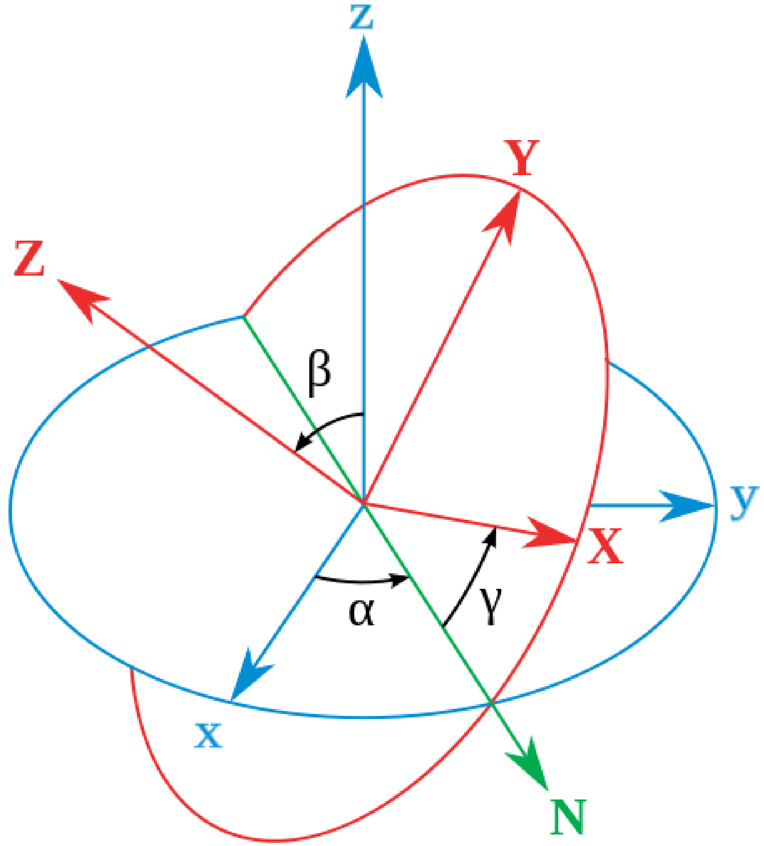
http://en.wikipedia.org/wiki/Euler_angles

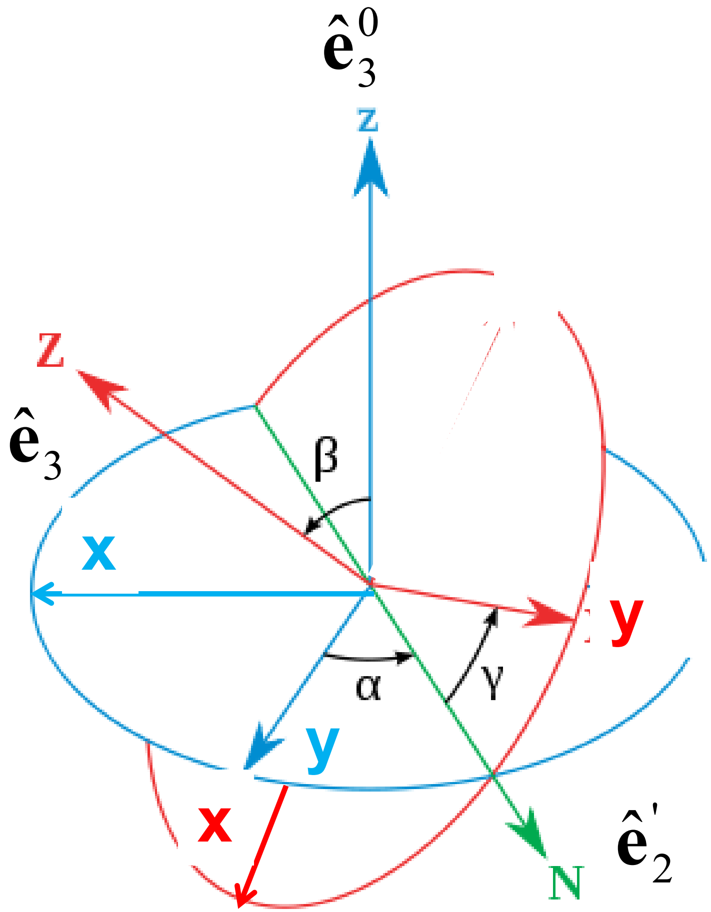
Comment on conventions

Our diagram



On web (for CM)





$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

Need to express all components in body-fixed frame:

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

When the dust clears --

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix} + \dot{\beta}\begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix} + \dot{\gamma}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

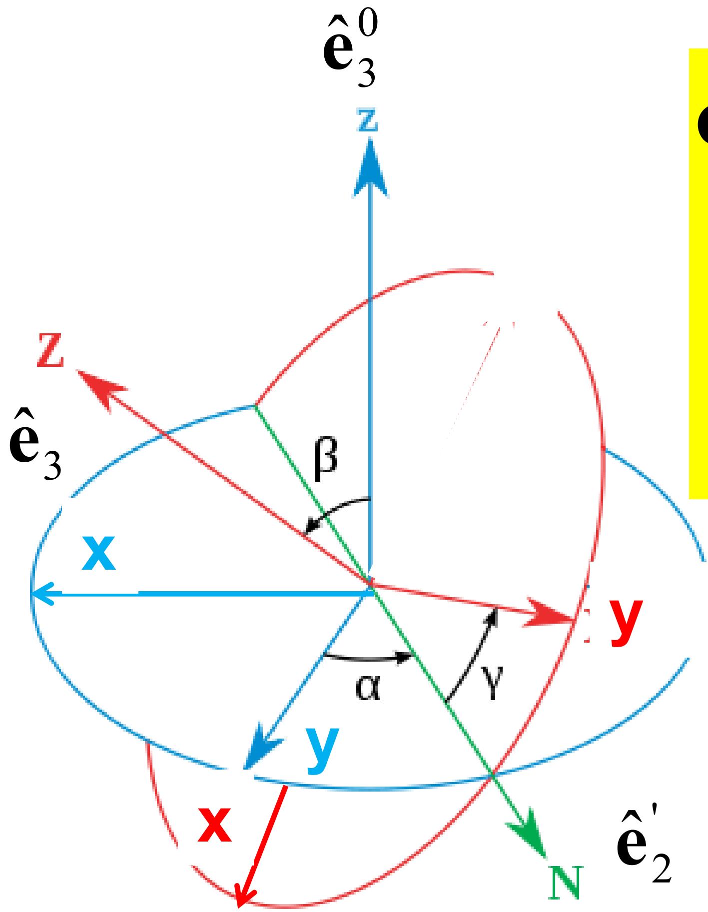
$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix} + \dot{\beta}\begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix} + \dot{\gamma}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\omega}_1 = \dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma$$

$$\tilde{\omega}_2 = \dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma$$

$$\tilde{\omega}_3 = \dot{\alpha}\cos\beta + \dot{\gamma}$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2' + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



$$\begin{aligned}\tilde{\boldsymbol{\omega}} &= \left[\dot{\alpha}\left(-\sin\beta\cos\gamma\right) + \dot{\beta}\sin\gamma\right]\hat{\mathbf{e}}_1 \\ &+ \left[\dot{\alpha}\left(\sin\beta\sin\gamma\right) + \dot{\beta}\cos\gamma\right]\hat{\mathbf{e}}_2 \\ &+ \left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]\hat{\mathbf{e}}_3\end{aligned}$$

Rotational kinetic energy

$$T(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1\tilde{\omega}_1^2 + \frac{1}{2}I_2\tilde{\omega}_2^2 + \frac{1}{2}I_3\tilde{\omega}_3^2$$

$$= \frac{1}{2}I_1\left[\dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma\right]^2$$

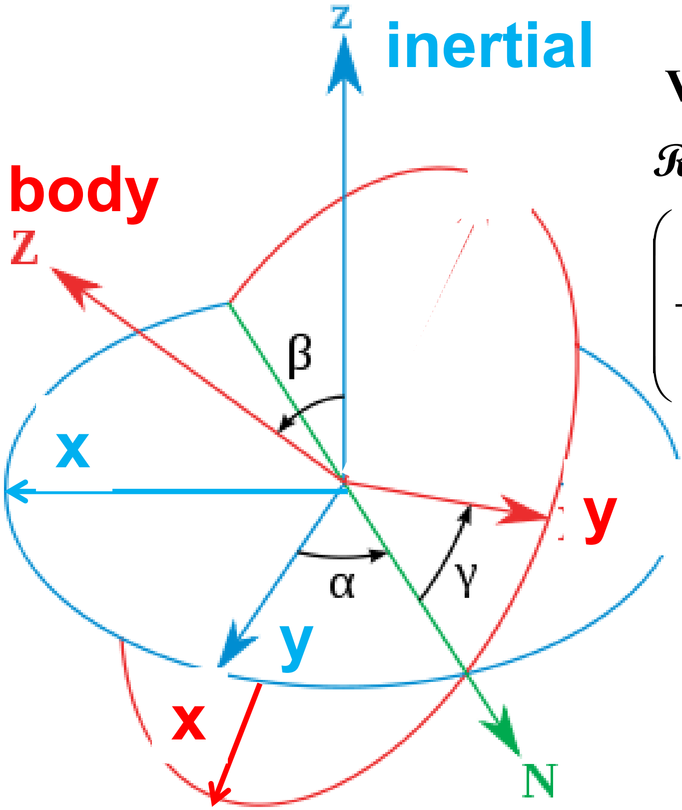
$$+ \frac{1}{2}I_2\left[\dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma\right]^2$$

$$+ \frac{1}{2}I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2$$

If $I_1 = I_2$:

$$T(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) + \frac{1}{2}I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2$$

General transformation between rotated coordinates – Euler angles



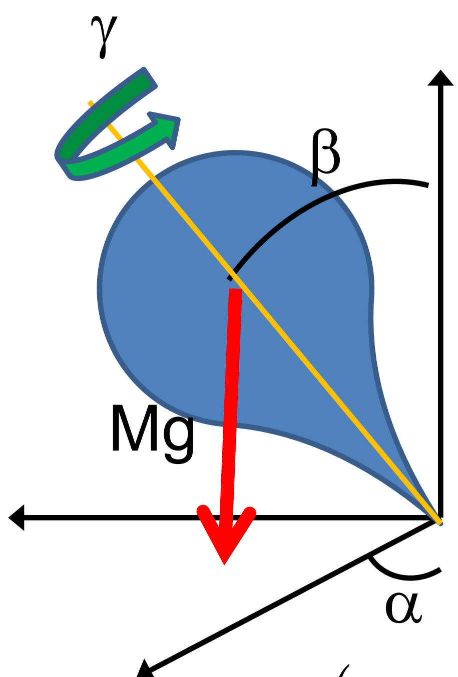
$$\mathbf{V}' = \mathcal{R}\mathbf{V} = \mathcal{R}_\alpha \mathcal{R}_\beta \mathcal{R}_\gamma \mathbf{V}$$

$$\mathcal{R} =$$

$$\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

http://en.wikipedia.org/wiki/Euler_angles

Motion of a symmetric top under the influence of the torque of gravity:



$$L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma})=\frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2\right)+$$

$$\frac{1}{2}I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]^2-Mgl\cos\beta$$

$$L\left(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}\right)=\frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2\right)+$$

$$\frac{1}{2}I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]^2-Mgl\cos\beta$$

Constants of the motion :

$$p_\alpha=\frac{\partial L}{\partial\dot{\alpha}}=I_1\dot{\alpha}\sin^2\beta+I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]\cos\beta$$

$$p_\gamma=\frac{\partial L}{\partial\dot{\gamma}}=I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]$$

$$E=\frac{1}{2}I_1\dot{\beta}^2+\frac{p_\gamma^2}{2I_3}+V_{eff}(\beta)$$

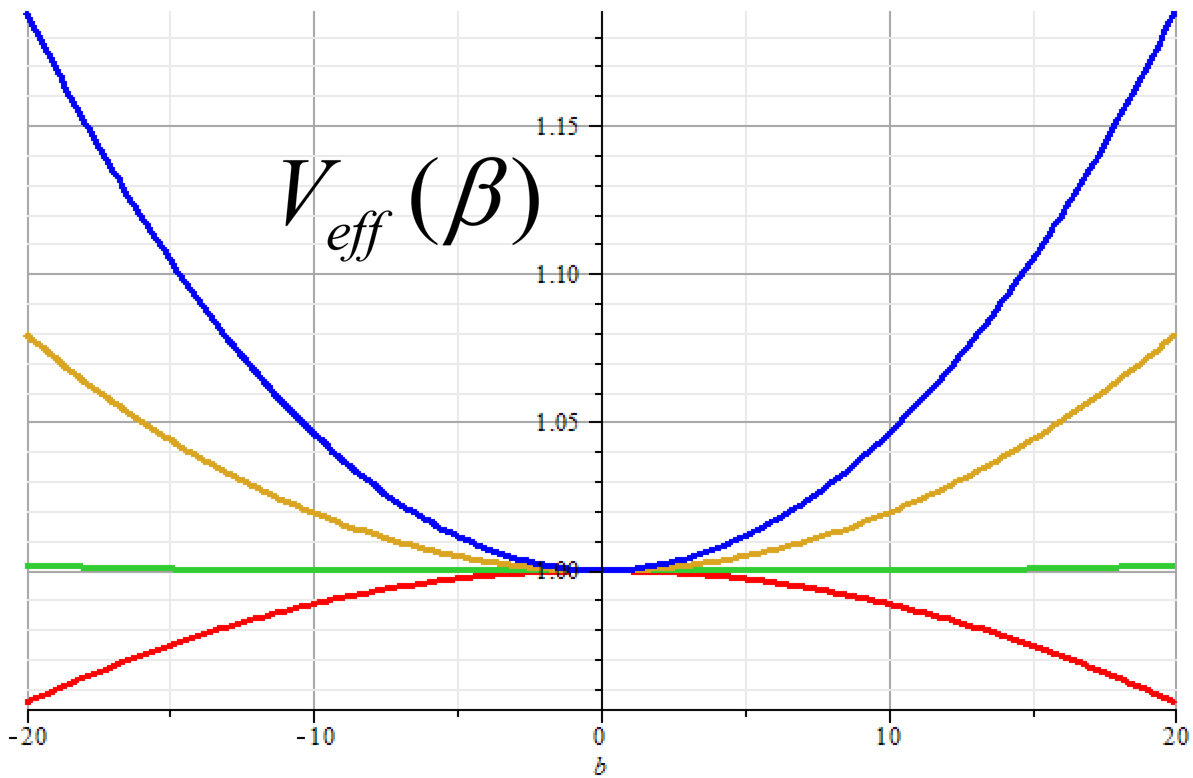
$$L\left(\beta,\dot{\beta}\right)=\frac{1}{2}I_1\dot{\beta}^2+\frac{\left(p_\alpha-p_\gamma\cos\beta\right)^2}{2I_1\sin^2\beta}+\frac{p_\gamma^2}{2I_3}-Mgl\cos\beta$$

$$V_{eff}(\beta)=\frac{\left(p_\alpha-p_\gamma\cos\beta\right)^2}{2I_1\sin^2\beta}+Mgl\cos\beta$$

$$E=\frac{1}{2}I_1\dot{\beta}^2+\frac{p_\gamma^2}{2I_3}+\frac{\left(p_\alpha-p_\gamma\cos\beta\right)^2}{2I_1\sin^2\beta}+Mgl\cos\beta$$

$$E'=E-\frac{p_\gamma^2}{2I_3}=\frac{1}{2}I_1\dot{\beta}^2+\frac{\left(p_\alpha-p_\gamma\cos\beta\right)^2}{2I_1\sin^2\beta}+Mgl\cos\beta$$

Stable/unstable solutions near β=0



What happens when β→0?

Constants of the motion:

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1 \dot{\alpha} \sin^2 \beta + I_3 \left[\dot{\alpha} \cos \beta + \dot{\gamma} \right] \cos \beta$$

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3 \left[\dot{\alpha} \cos \beta + \dot{\gamma} \right]$$

$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + V_{eff}\left(\beta\right)$$

$$L\left(\beta, \dot{\beta}\right) = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{\left(p_\alpha - p_\gamma \cos \beta\right)^2}{2I_1 \sin^2 \beta} + \frac{p_\gamma^2}{2I_3} - Mgl \cos \beta$$

$$V_{eff}\left(\beta\right) = \frac{\left(p_\alpha - p_\gamma \cos \beta\right)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + V_{eff}(\beta)$$

$$V_{eff}(\beta) = \frac{\left(p_\alpha - p_\gamma \cos\beta\right)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

Note that for $\beta \approx 0$, $\cos\beta \approx 1 - \frac{1}{2}\beta^2 + \ldots$

$$\sin\beta \approx \beta$$

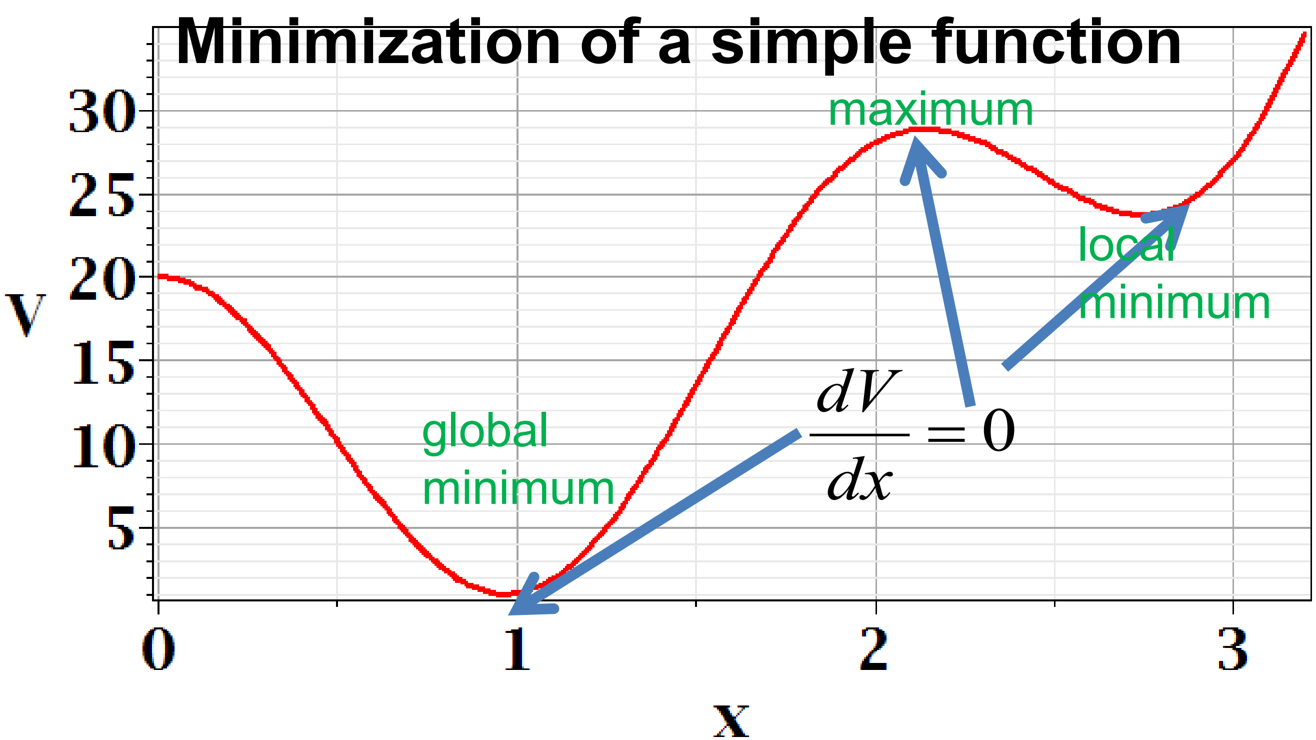
If $p_\alpha = p_\gamma$: $\quad V_{eff}(\beta) \approx \frac{p_\gamma^2}{8I_1}\beta^2 + Mgl\left(1 - \frac{1}{2}\beta^2\right) + \ldots$

$$E - \frac{p_\gamma^2}{2I_3} \approx \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{8I_1}\beta^2 + Mgl\left(1 - \frac{1}{2}\beta^2\right) + \ldots$$

$$E - \frac{p_\gamma^2}{2I_3} \approx \frac{1}{2} I_1 \dot{\beta} + Mgl + \left(\frac{p_\gamma^2}{8I_1} - \frac{Mgl}{2}\right)\beta^2 + \ldots$$

Question: How do we decide stable/unstable solutions for the symmetric top motion?

Comment – When we discussed one dimensional motion, we discussed stable and unstable equilibrium points. At equilibrium dV/dx=0, but only when V(x) has a minimum at that point, is the system stable in the sense that for small displacements from equilibrium, there are restoring forces to move the system back to the equilibrium point.



Suppose $p_\alpha = p_\gamma$ and $\beta \approx 0$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

$$E' \approx \frac{1}{2} I_1 \dot{\beta}^2 + \frac{{p_\gamma}^2}{2I_1} \frac{\left(1 - 1 + \frac{1}{2}\beta^2\right)^2}{\beta^2} + Mgl\left(1 - \frac{1}{2}\beta^2\right)$$

$$\approx \frac{1}{2} I_1 \dot{\beta}^2 + \left(\frac{{p_\gamma}^2}{8I_1} - \frac{Mgl}{2}\right)\beta^2 + Mgl$$

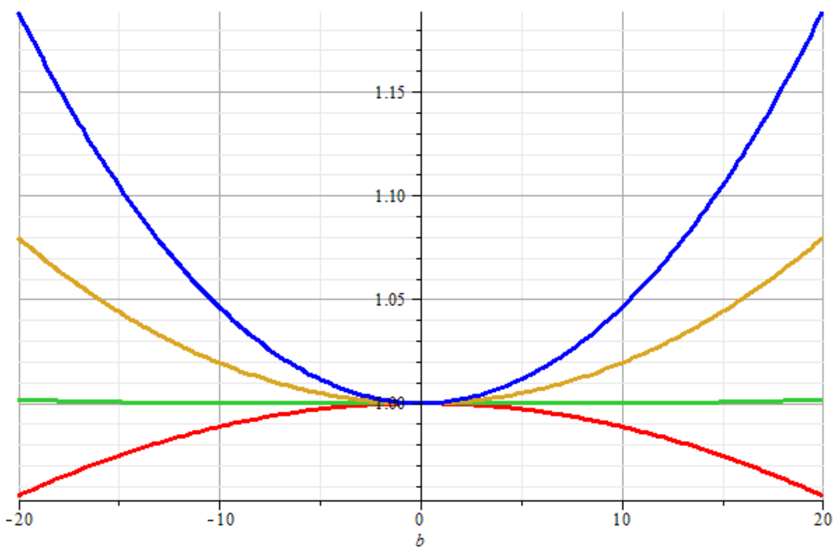
$\Rightarrow$ Stable solution if

$$p_\gamma \geq \sqrt{4MglI_1}$$

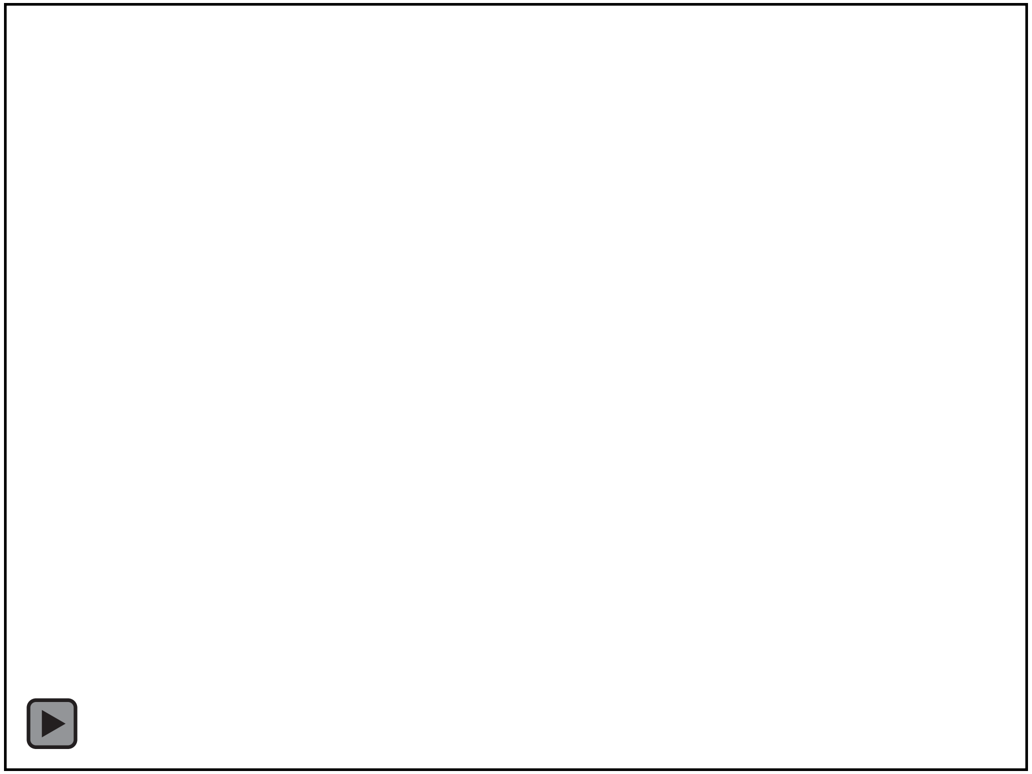
Note that

$p_\gamma = I_3\omega_3$

$\Rightarrow \omega_3$ must be sufficiently large for the top to maintain vertical orientation $(\beta \approx 0)$.



http://www.physics.usyd.edu.au/~cross/SPINNING%20TOPS.htm



See also --

Home > American Journal of Physics > Volume 81, Issue 4 > 10.1119/1.4776195

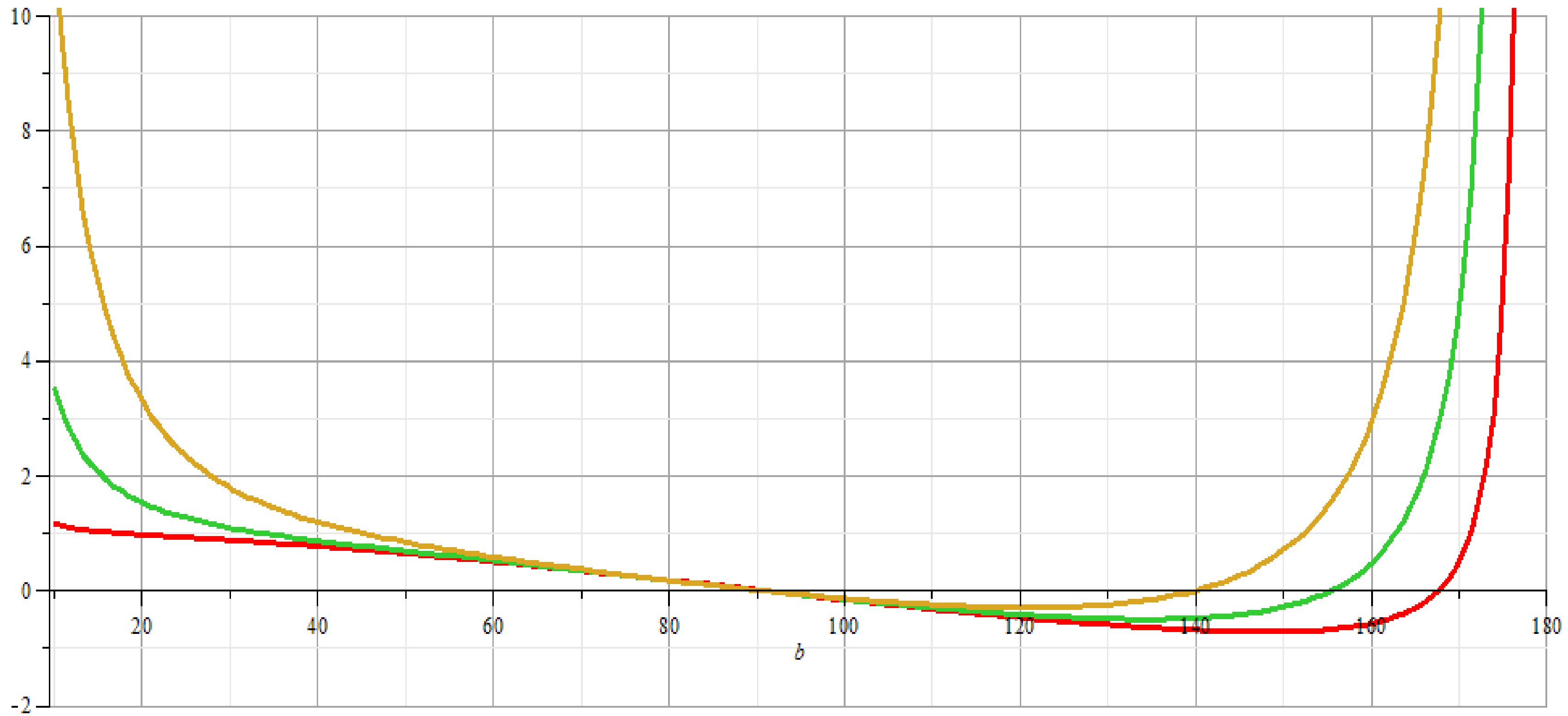
Full . Published Online: 18 March 2013 Accepted: December 2012

The rise and fall of spinning tops

American Journal of Physics **81**, 280 (2013); https://doi.org/10.1119/1.4776195

More general case:

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$



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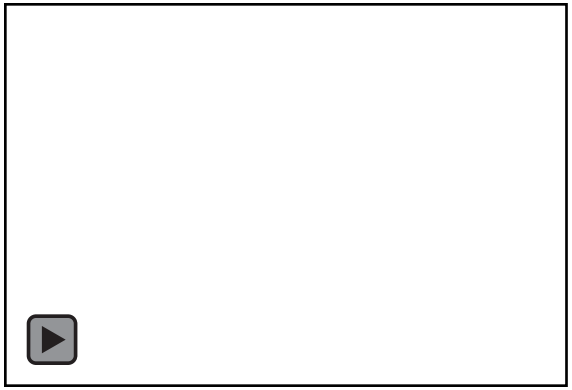

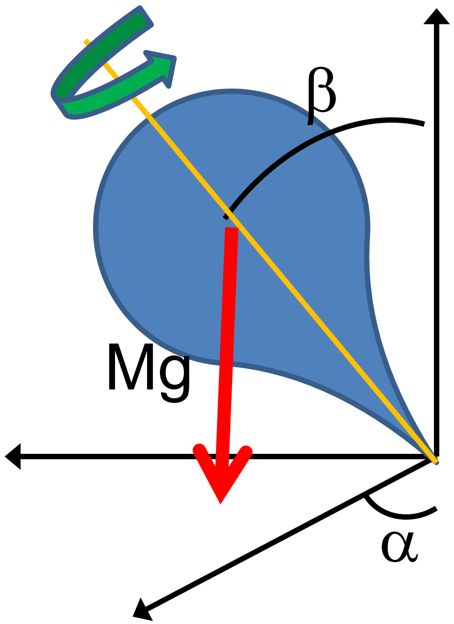

Constants of the motion :

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]$$

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1\dot{\alpha}\sin^2\beta + I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]\cos\beta$$

$$= I_1\dot{\alpha}\sin^2\beta + p_\gamma\cos\beta$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2}I_1\dot{\beta}^2 + \frac{\left(p_\alpha - p_\gamma\cos\beta\right)^2}{2I_1\sin^2\beta} + Mgl\cos\beta$$