PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 14 -- Chap. 1 of F&W

- Summary of scattering theory of a single particle from a stationary target
- Analysis of two particle system; center of mass and laboratory frames
- Differential cross section for in the center of mass reference frame

Physics Colloquium

- Thursday -

September 26, 2024

Two-dimensional Chalcogenide Topological Insulators

NANOTEQ
Center for Nanotechnology and Quantum Materials

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(with Kenneth Kohn on Zoom)

Over the last few years, the Nanotech group led by Dr. Carroll has been extensively working on the synthesis, characterization and application of chalcogenides-based 2D topological insulators. Why are topological insulators important and what makes the research exciting?

A topological insulator (TI) is an insulating material that always has a metallic boundary when placed next to a vacuum or an 'ordinary' insulator. These metallic boundaries originate from topological invariants, which cannot change as long as a material remains insulating. This unusual metallic edge gives rise to spin-momentum-locked electrons, thereby leading to dissipationless transport that is robust to disorder and even thermal fluctuations. In other words, the metallic states become superconducting – a phenomenon restricted to low temperatures, reduced dimensionality, and high magnetic fields. Manipulation of these topologically protected edge states could lead to a new architecture for quantum bits at room temperature, an application that is most sought-after in current research.

The metallic states on TIs arise from two primary classes: 2D TIs that are metallic only on the edges of a nanostructure, and the 3D class that is metallic on the crystal surface while its bulk is insulating. By screening materials with a spin-orbit coupling that is larger than their bulk-band gap, a pre-requisite to observe TI properties close to ambient temperatures, layered, bismuth and antimony-based chalcogenides proved most accessible and promising. Moreover, it is predicted that 2D TIs have more addressable states, so our group is spending significant efforts studying chalcogenides-based 2D TIs.

The talk is divided into the following sections to give an overall picture of our ongoing activities in this direction: synthesis of

Reception 3:30
Olin Lobby

Colloquium 4:00
Olin 101

	Date	F&W	Topic	HW
1	Mon, 8/26/2024		Introduction and overview	#1
2	Wed, 8/28/2024	Chap. 3(17)	Calculus of variation	#2
3	Fri, 8/30/2024	Chap. 3(17)	Calculus of variation	#3
4	Mon, 9/02/2024	Chap. 3	Lagrangian equations of motion	#4
5	Wed, 9/04/2024	Chap. 3 & 6	Lagrangian equations of motion	#5
6	Fri, 9/06/2024	Chap. 3 & 6	Lagrangian equations of motion	#6
7	Mon, 9/09/2024	Chap. 3 & 6	Lagrangian to Hamiltonian formalism	#7
8	Wed, 9/11/2024	Chap. 3 & 6	Phase space	#8
9	Fri, 9/13/2024	Chap. 3 & 6	Canonical Transformations	
10	Mon, 9/16/2024	Chap. 5	Dynamics of rigid bodies	#9
11	Wed, 9/18/2024	Chap. 5	Dynamics of rigid bodies	#10
12	Fri, 9/20/2024	Chap. 5	Dynamics of rigid bodies	#11
13	Mon, 9/23/2024	Chap. 1	Scattering analysis	#12
14	Wed, 9/25/2024	Chap. 1	Scattering analysis	#13
15	Fri, 9/27/2024	Chap. 1	Scattering analysis	#14

PHY 711 – Assignment #13

Assigned: 09/25/2024 Due: 9/30/2024

1. Suppose that a particle is scattered by a very massive target particle such that energy and angular momentum are conserved. The trajectory of the scattering particle is found to have an impact parameter b which depends on the scattering angle θ according to the formula

$$b(\theta) = K \left| \frac{1}{\sin(\theta/2)} \right|,$$

where K denotes a constant which depends on energy and other parameters. What is the differential cross section for this process?

PHY 711 -- Assignment #14

Assigned: 9/27/2024 Due: 9/30/2024

Continue reading Chapter 1 in **Fetter & Walecka**.

- Work Problem #1.16 at the end of Chapter 1 in **Fetter and Walecka**. Note that you might want to use the equation in FW #1.15 or the equivalent equation derived in class.

Scattering theory:

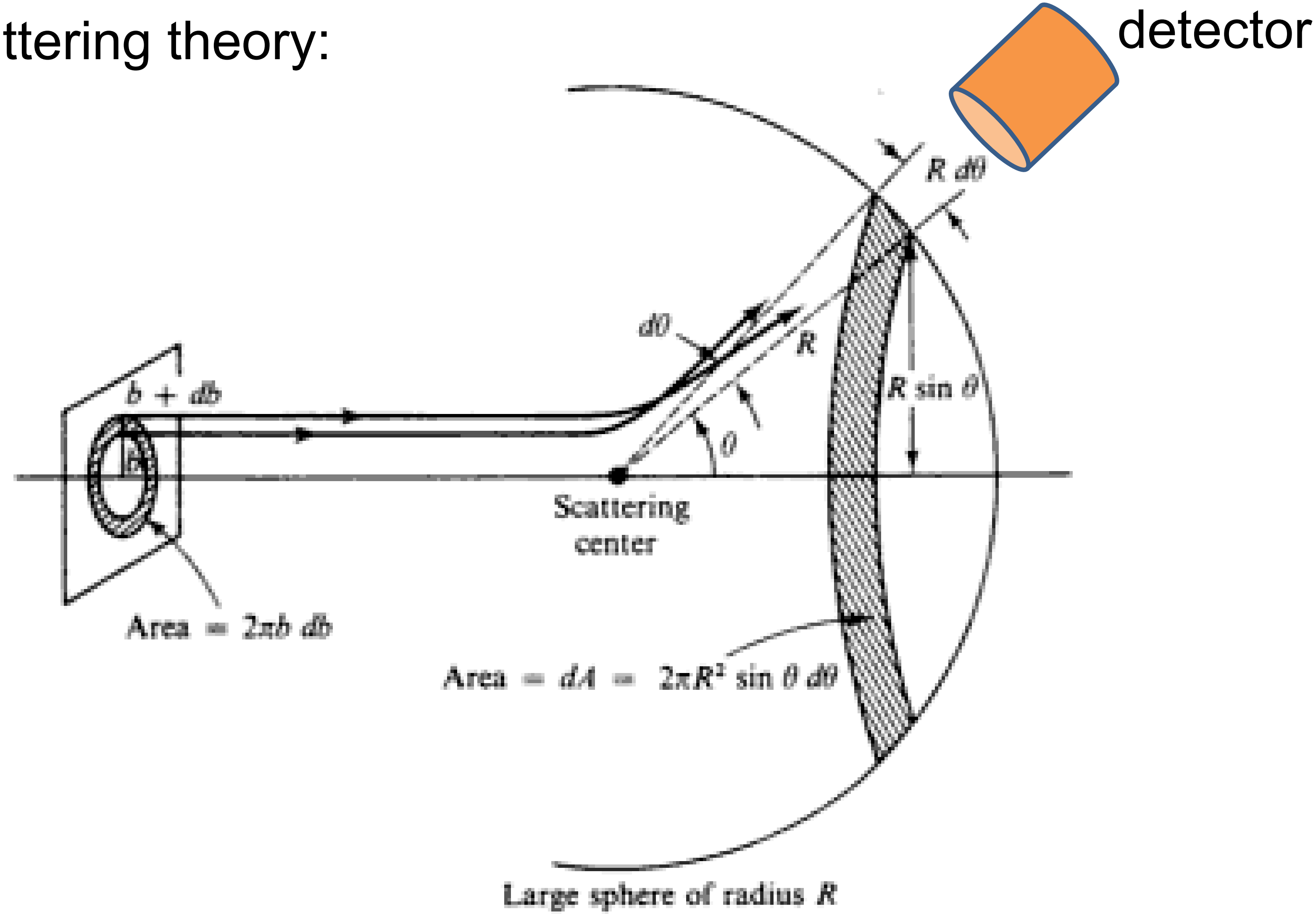


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Can you think of examples of such an experimental setup?

Other experimental designs –

At CERN https://home.cern/science/experiments/totem

the study of highly energetic proton-proton scattering is designed in the center of mass frame of reference by accelerating two proton beams focused to collide head on in the Large Hadron Collider LHC facility.

Figure from CERN website

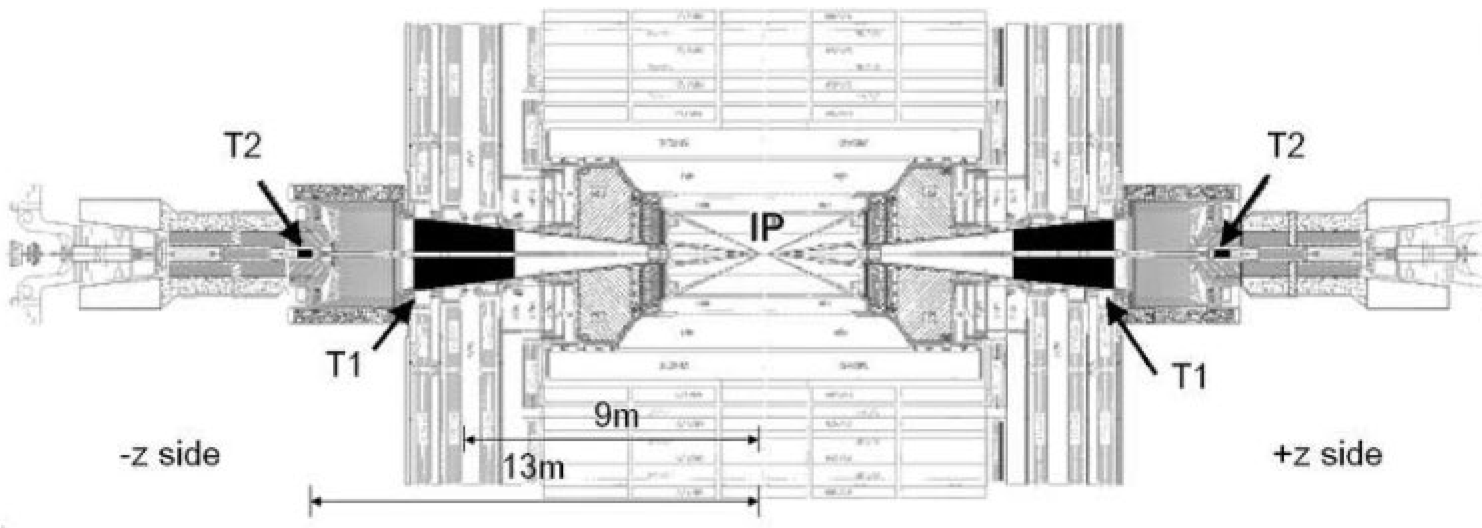


Figure 1.17: View of the inelastic forward trackers T1 and T2 inside the CMS detector.

What might be the advantage/disadvantage of this design?

What are the benefits/disadvantages of expressing the scattering cross section in the laboratory frame of reference vs center of mass frame of reference? (When or why to use a particular frame of reference)

<u>Advantages of Lab frame</u>

1. Natural experimental design.
2. Some targets are more naturally at rest.
3. ??

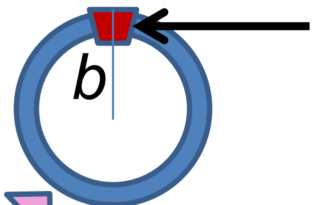
<u>Advantages of CM frame</u>

1. Analysis is done in CM frame.
2. Experiment is more energy efficient.
3. ??

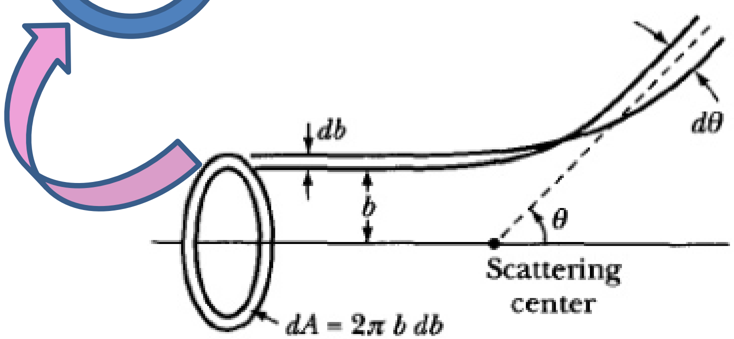
Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$$= \text{Area of incident beam that is scattered into detector at angle } \theta$$



$$d\varphi b db$$



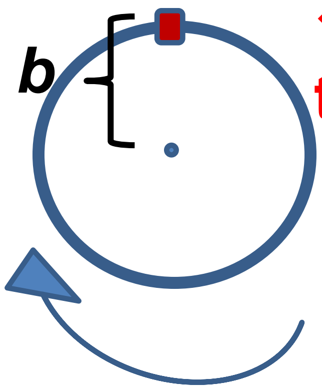
$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

More details --

We imagine that the beam of particles has a cylindrical geometry and that the physics is totally uniform in the azimuthal direction. The cross section of the beam is a circle.

View of beam cross section:

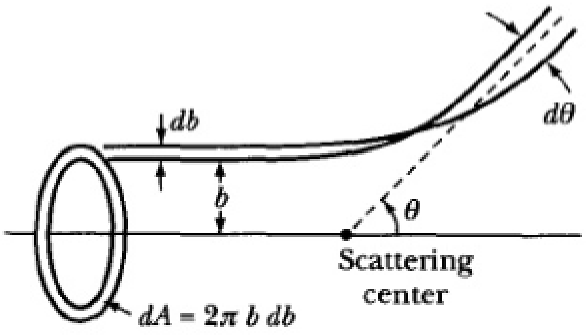


←This piece of the beam scatters into the detector at angle θ

This logic leads to the notion that b is a function of theta and we will try to find b(θ) for various cases.

$\varphi \equiv$ azimuthal angle

Note: The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.



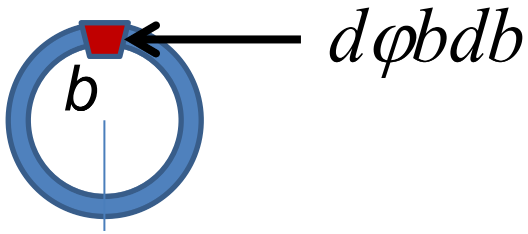
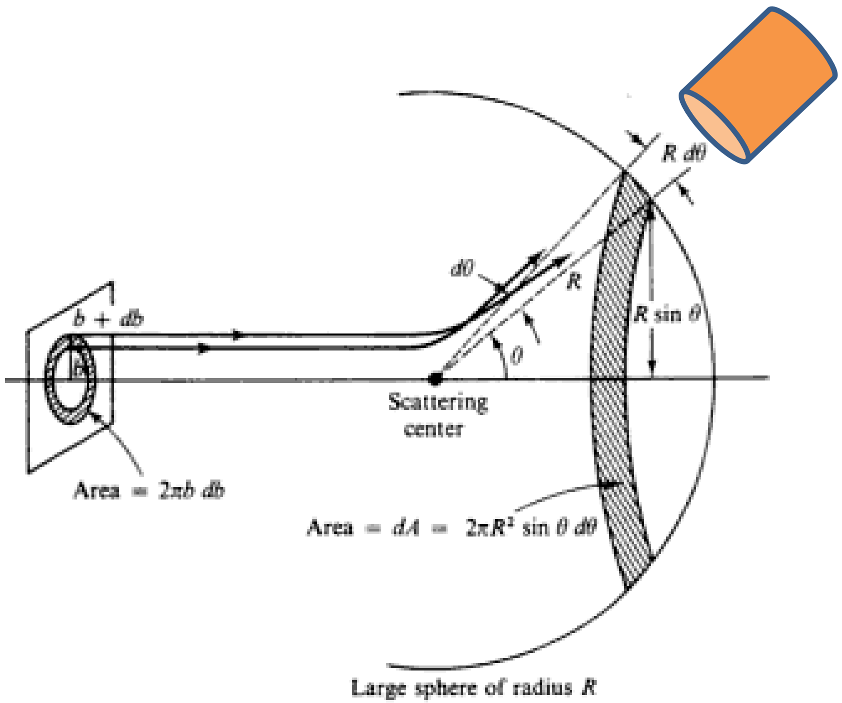


Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

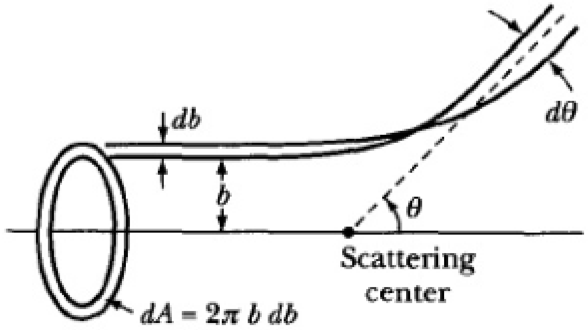
Note: We are assuming that the process is isotropic in ϕ

Elaboration on how we know that $b\ db\ d\phi$ is the relevant piece of beam ending up in our detector?



Comment: The interaction potential will determine the detailed shape of the particle trajectory which we can express as $r(\theta)$, which in principle can be related to the impact parameter as a function of scattering angle $b(\theta)$.

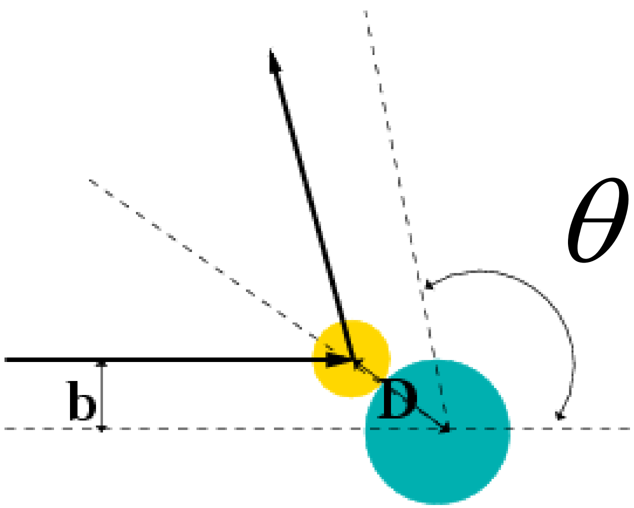
Simple example – collision of hard spheres



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Microscopic view:

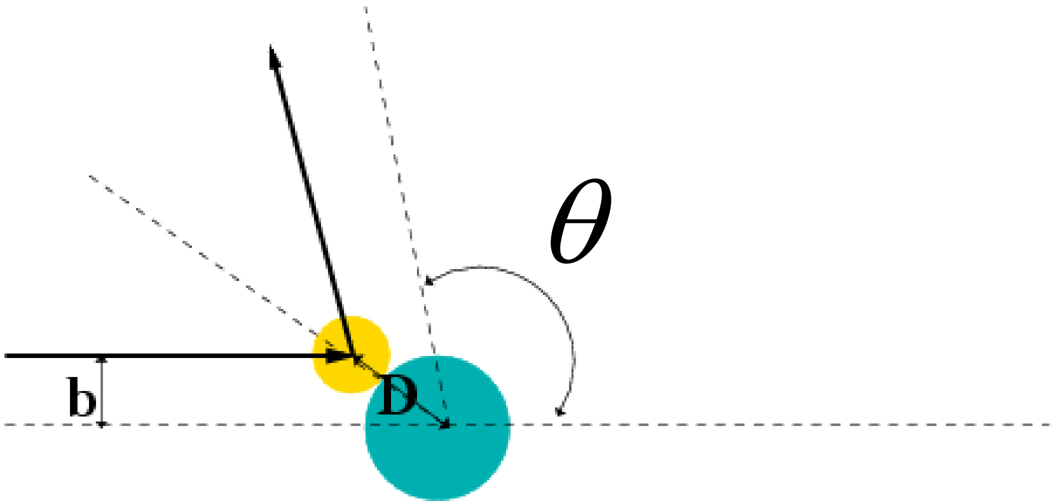
$$b(\theta) = ?$$

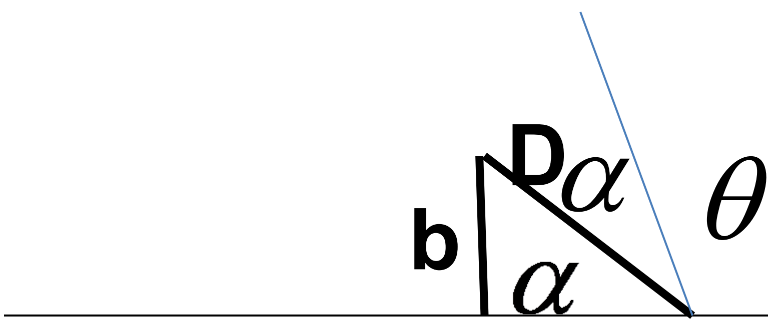


$$b(\theta) = D\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{D^2}{4}$$

Some more details of form of *b(θ)*

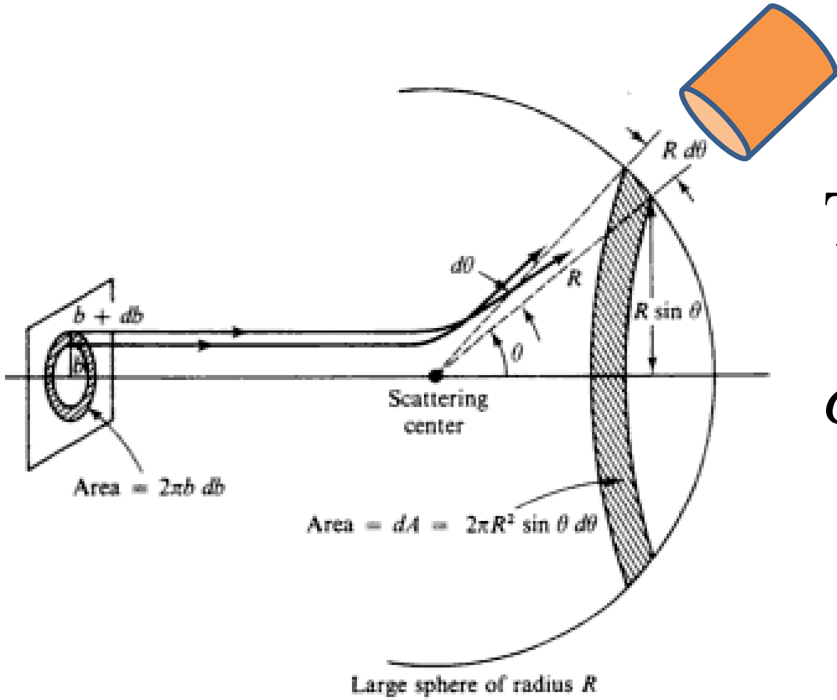




$$b = D\sin\alpha = D\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$2\alpha + \theta = \pi$$

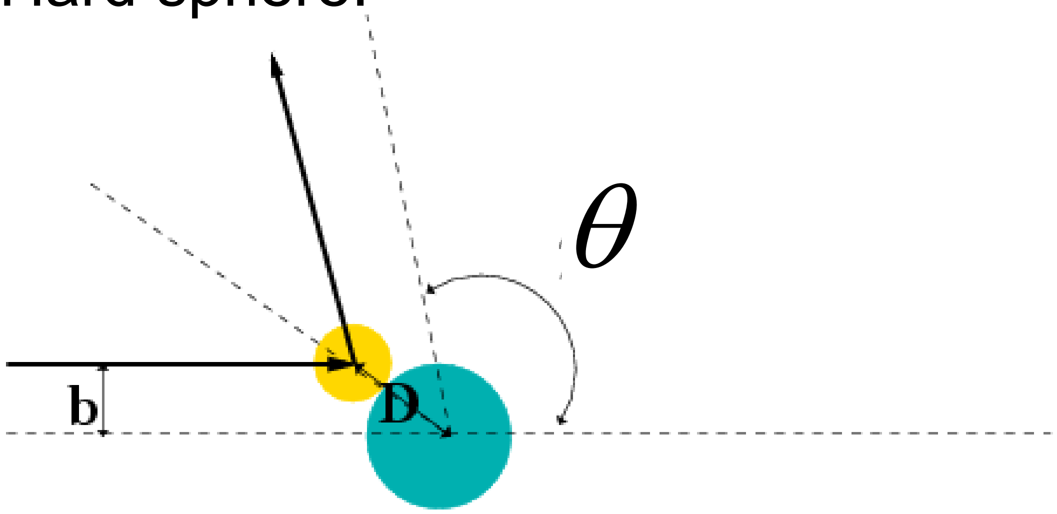
Simple example – collision of hard spheres -- continued



Total scattering cross section:

$$\sigma = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega$$

Hard sphere:



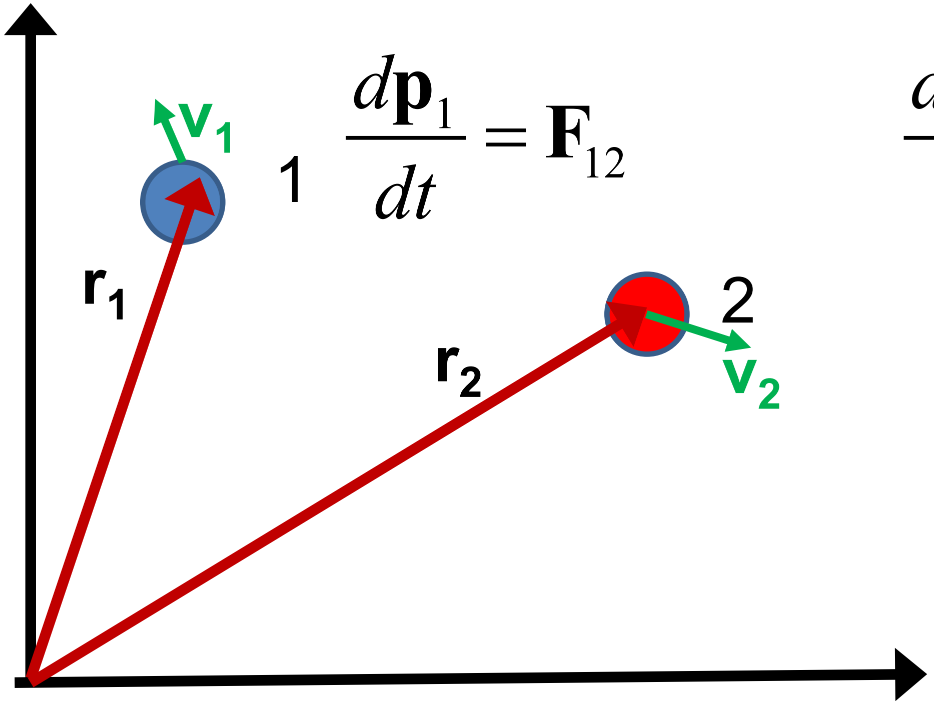
$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{D^2}{4}$$

$$\sigma = \pi D^2$$

Now consider the more general case of particle interactions and the corresponding scattering analysis.

Scattering theory can help us analyze the interaction potential $V(r)$. First, we need to simplify the number of variables.

Relationship of scattering cross-section to particle interactions -- Classical mechanics of a conservative 2-particle system.



$$\frac{d\mathbf{p}_1}{dt} = \mathbf{F}_{12} \qquad \frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{21}$$

$$\mathbf{F}_{12} = -\nabla_1 V\left(\mathbf{r}_1 - \mathbf{r}_2\right) \quad \Rightarrow E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

Relationship between center of mass and laboratory frames of reference. At a time *t*, the following relationships apply --

Definition of center of mass $\mathbf{R}_{CM}$

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = (m_1 + m_2)\mathbf{R}_{CM}$$

$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = (m_1 + m_2)\dot{\mathbf{R}}_{CM} = (m_1 + m_2)\mathbf{V}_{CM}$$

Note that $\dot{\mathbf{r}} \equiv \frac{d\mathbf{r}}{dt}$

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

$$= \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu|\mathbf{v}_1 - \mathbf{v}_2|^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

where: $\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$

Why do this? We need to make the mathematics tractible…

Classical mechanics of a conservative 2-particle system -- continued

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu|\mathbf{v}_1 - \mathbf{v}_2|^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

For central potentials: $V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r_{12})$

Relative angular momentum is also conserved:

$$\mathbf{L}_{12} \equiv \mathbf{r}_{12} \times \mu \mathbf{v}_{12}$$

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu v_{12}^2 + \frac{L_{12}^2}{2\mu r_{12}^2} + V(r_{12})$$

Simpler notation:

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Simpler notation:

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

constants

vary in time

For scattering analysis only need to know trajectory **before** and **after** the collision. We also generally assume that the interaction between particle and target *V(r)* conserves energy and angular momentum.

Comment: The impact parameter *b* is a useful concept in the general case.

$$E_{total} = \underbrace{\frac{1}{2}(m_1 + m_2)V_{CM}^2}_{E_{CM}} + \underbrace{\frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)}_{E_{rel}}$$

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{E_{rel}b^2}{r^2} + V(r)$$

In what situations do particles undergo inelastic scattering, rather than elastic scattering?

Comment – elastic scattering means $E_{initial}=E_{final}$

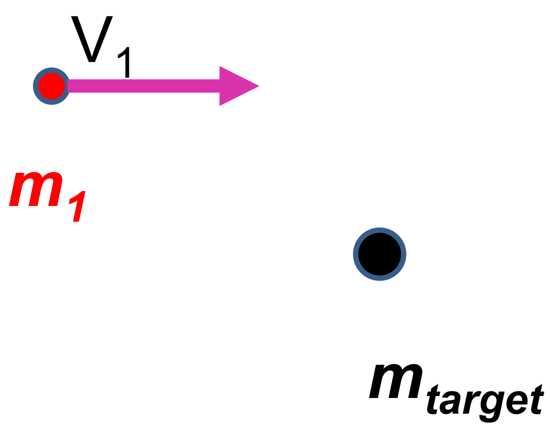
Typically, elastic scattering occurs when two fundamental particles interact (as long as the final kinetic energy of both particles is taken into account).

Elastically bouncing ball

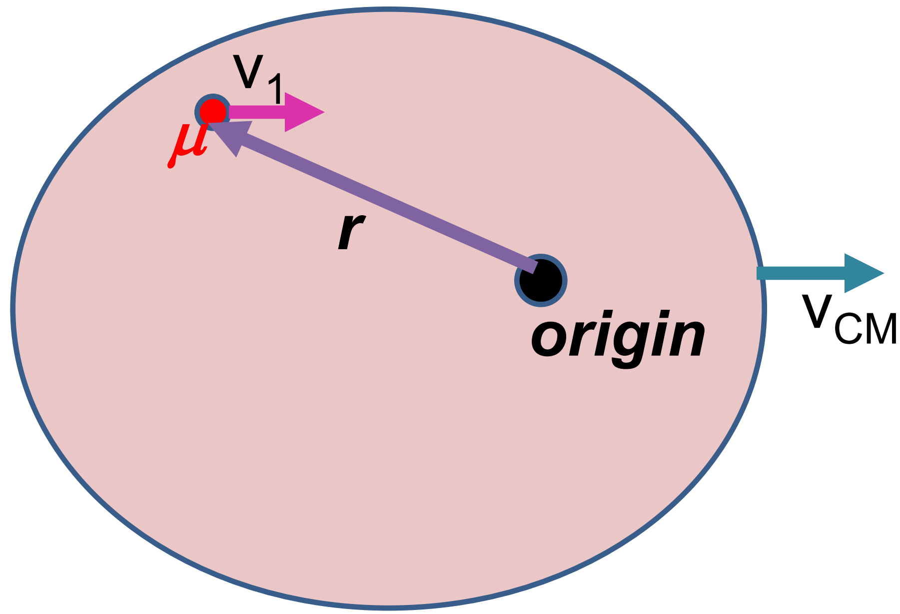
Inelastically collision

Note: The following analysis will be carried out in the center of mass frame of reference.

In laboratory frame:



In center-of-mass frame:



$$\mu = \frac{m_1 m_{\text{target}}}{m_1 + m_{\text{target}}}$$

$$\ell = \left| \mathbf{r} \times \mu \mathbf{v}_1 \right|$$

Typically, the laboratory frame is where the data is taken, but the center of mass frame is where the analysis is most straightforward.

Previous equations --

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

↑ constant

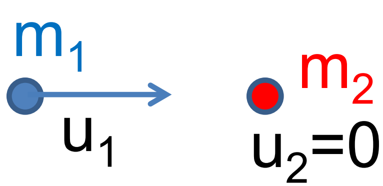
↑ relative coordinate system; visualize as “in” CM frame

It is often convenient to analyze the scattering cross section in the center of mass reference frame.

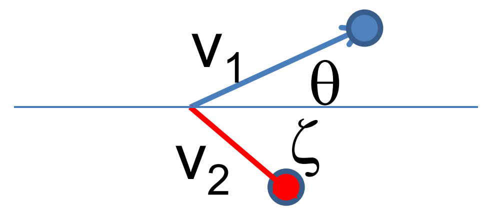
Relationship between normal laboratory reference and center of mass:

Laboratory reference frame:

Before

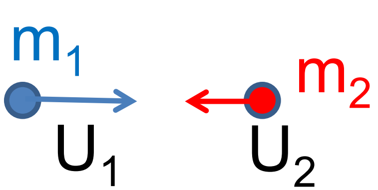


After

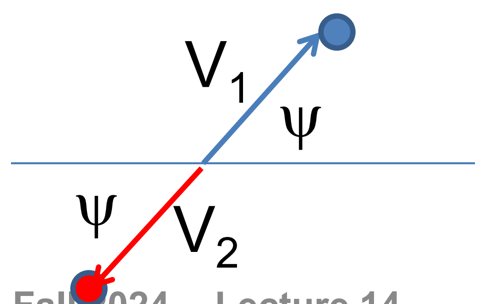


Center of mass reference frame:

Before



After



Relationship between center of mass and laboratory frames of reference -- continued

Since m_2 is initially at rest in lab frame:

Before collision:

$$\mathbf{V}_{CM} = \frac{m_1}{m_1 + m_2}\mathbf{u}_1 \qquad \mathbf{u}_1 = \mathbf{U}_1 + \mathbf{V}_{CM} \qquad \Rightarrow \mathbf{U}_1 = \frac{m_2}{m_1 + m_2}\mathbf{u}_1 = \frac{m_2}{m_1}\mathbf{V}_{CM}$$

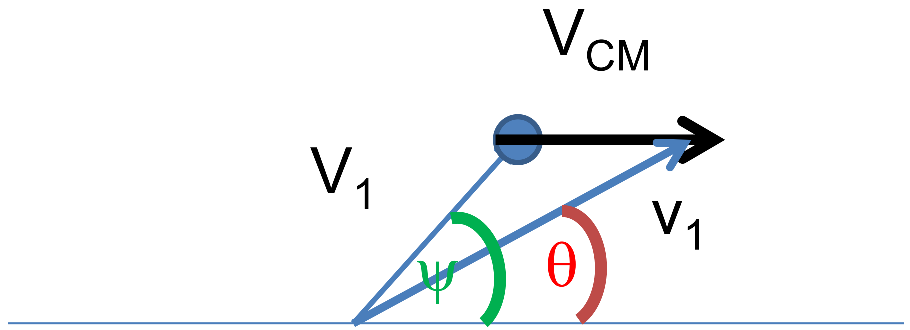
$$\mathbf{u}_2 = \mathbf{U}_2 + \mathbf{V}_{CM} \qquad \Rightarrow \mathbf{U}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{u}_1 = -\mathbf{V}_{CM}$$

After collision:

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$\mathbf{v}_2 = \mathbf{V}_2 + \mathbf{V}_{CM}$$

Relationship between center of mass and laboratory frames of reference for the scattering particle 1



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\theta = V_1 \sin\psi$$

$$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$$

$$\tan\theta = \frac{\sin\psi}{\cos\psi + V_{CM}/V_1} = \frac{\sin\psi}{\cos\psi + m_1/m_2}$$

For elastic scattering

Digression – elastic scattering

$$\tfrac{1}{2}m_1U_1^2+\tfrac{1}{2}m_2U_2^2+\tfrac{1}{2}(m_1+m_2)V_{CM}^2$$
$$=\tfrac{1}{2}m_1V_1^2+\tfrac{1}{2}m_2V_2^2+\tfrac{1}{2}(m_1+m_2)V_{CM}^2$$

Also note:

$$m_1\mathbf{U}_1+m_2\mathbf{U}_2=0 \qquad\qquad m_1\mathbf{V}_1+m_2\mathbf{V}_2=0$$

$$\mathbf{U}_1=\frac{m_2}{m_1}\mathbf{V}_{CM} \qquad\qquad \mathbf{U}_2=-\mathbf{V}_{CM}$$

$$\Rightarrow |\mathbf{U_1}|=|\mathbf{V_1}| \quad \text{and} \quad |\mathbf{U_2}|=|\mathbf{V_2}|=|\mathbf{V_{CM}}|$$

$$\text{Also note that :}\quad m_1|\mathbf{U_1}|=m_2|\mathbf{U_2}|$$

$$\text{So that :}\qquad V_{CM}/V_1=V_{CM}/U_1=m_1/m_2$$

Summary of results --

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\theta = V_1 \sin\psi$$

$$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$$

$$\tan\theta = \frac{\sin\psi}{\cos\psi + V_{CM} / V_1} = \frac{\sin\psi}{\cos\psi + m_1 / m_2}$$

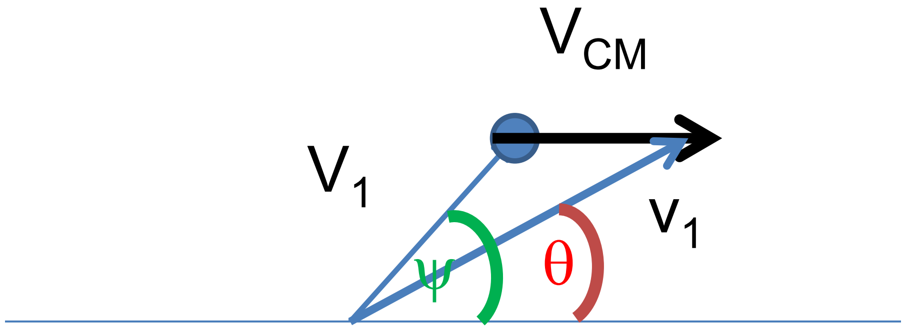
General case

Special case of elastic scattering

For elastic scattering

$$V_{CM} / V_1 = m_1 / m_2$$

Relationship between center of mass and laboratory frames of reference – continued (elastic scattering)



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\theta = V_1 \sin\psi$$

$$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$$

$$\tan\theta = \frac{\sin\psi}{\cos\psi + V_{CM}/V_1} = \frac{\sin\psi}{\cos\psi + m_1/m_2}$$

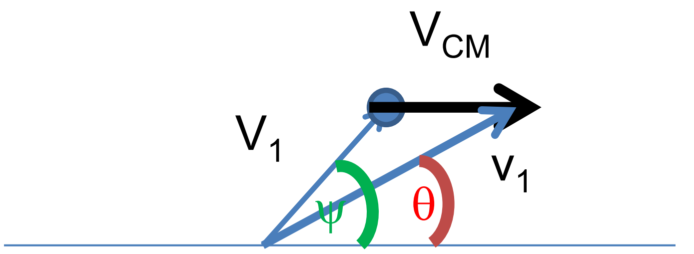
Also: $$\cos\theta = \frac{\cos\psi + m_1/m_2}{\sqrt{1 + 2m_1/m_2 \cos\psi + (m_1/m_2)^2}}$$

More details -- from the diagram and equations --

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\theta = V_1 \sin\psi$$

$$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$$



Take the dot product of the first equation with itself

$$v_1^2 = V_1^2 + 2V_1V_{CM}\cos\psi + V_{CM}^2$$

$$\text{or } \frac{v_1}{V_1} = \sqrt{1 + 2\frac{V_{CM}}{V_1}\cos\psi + \frac{V_{CM}^2}{V_1^2}} = \sqrt{1 + 2\frac{m_1}{m_2}\cos\psi + \left(\frac{m_1}{m_2}\right)^2}$$

$$\Rightarrow \quad \cos\theta = \frac{\cos\psi + m_1/m_2}{\sqrt{1 + 2m_1/m_2\cos\psi + (m_1/m_2)^2}}$$

Differential cross sections in different reference frames

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{d\Omega_{CM}}{d\Omega_{LAB}}$$

$$\frac{d\Omega_{CM}}{d\Omega_{LAB}}=\left|\frac{\sin\psi}{\sin\theta}\frac{d\psi}{d\theta}\right|=\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

Using:

$$\cos\theta=\frac{\cos\psi+m_1/m_2}{\sqrt{1+2(m_1/m_2)\cos\psi+(m_1/m_2)^2}}$$

$$\left|\frac{d\cos\theta}{d\cos\psi}\right|=\frac{(m_1/m_2)\cos\psi+1}{\left(1+2(m_1/m_2)\cos\psi+(m_1/m_2)^2\right)^{3/2}}$$

Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\psi+1}$$

where: $\tan\theta=\dfrac{\sin\psi}{\cos\psi+m_1/m_2}$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\psi+1}$$

where: $\tan\theta=\dfrac{\sin\psi}{\cos\psi+m_1/m_2}$

Example: suppose $m_1 = m_2$

In this case: $\tan\theta=\dfrac{\sin\psi}{\cos\psi+1} \quad \Rightarrow \theta=\dfrac{\psi}{2}$

note that $0\le\theta\le\dfrac{\pi}{2}$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(2\theta)}{d\Omega_{CM}}\right)\cdot 4\cos\theta$$

Summary --
Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\psi+1}$$

where: $$\tan\theta=\frac{\sin\psi}{\cos\psi+m_1/m_2}$$

For elastic scattering

Hard sphere example – continued

$m_1=m_2$

Center of mass frame

Lab frame

$$\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)=\frac{D^2}{4}$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=D^2\cos\theta \quad \theta=\frac{\psi}{2}$$

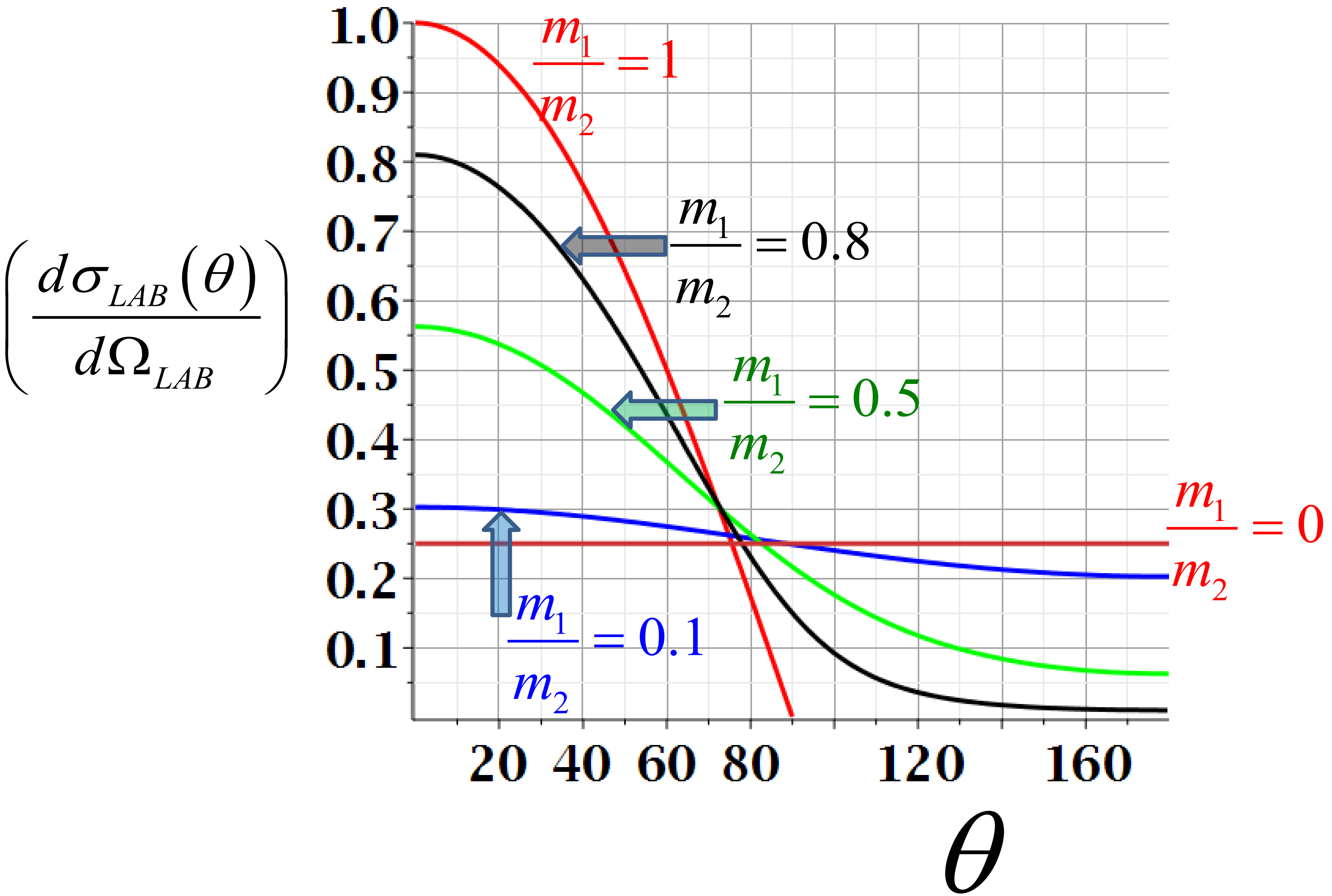
$$\int\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}d\Omega_{CM}=$$

$$\frac{D^2}{4}4\pi=\pi D^2$$

$$\int\frac{d\sigma_{lab}(\theta)}{d\Omega_{lab}}d\Omega_{lab}=$$

$$2\pi D^2\int_0^{\pi/2}\cos\theta\ \sin\theta d\theta=\pi D^2$$

Scattering cross section for hard sphere in lab frame for various mass ratios:



For visualization, is convenient to make a "parametric" plot of

$$\left(\frac{d\sigma_{LAB}}{d\Omega}(\theta)\right) \text{ vs } \theta(\psi)$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\psi+1}$$

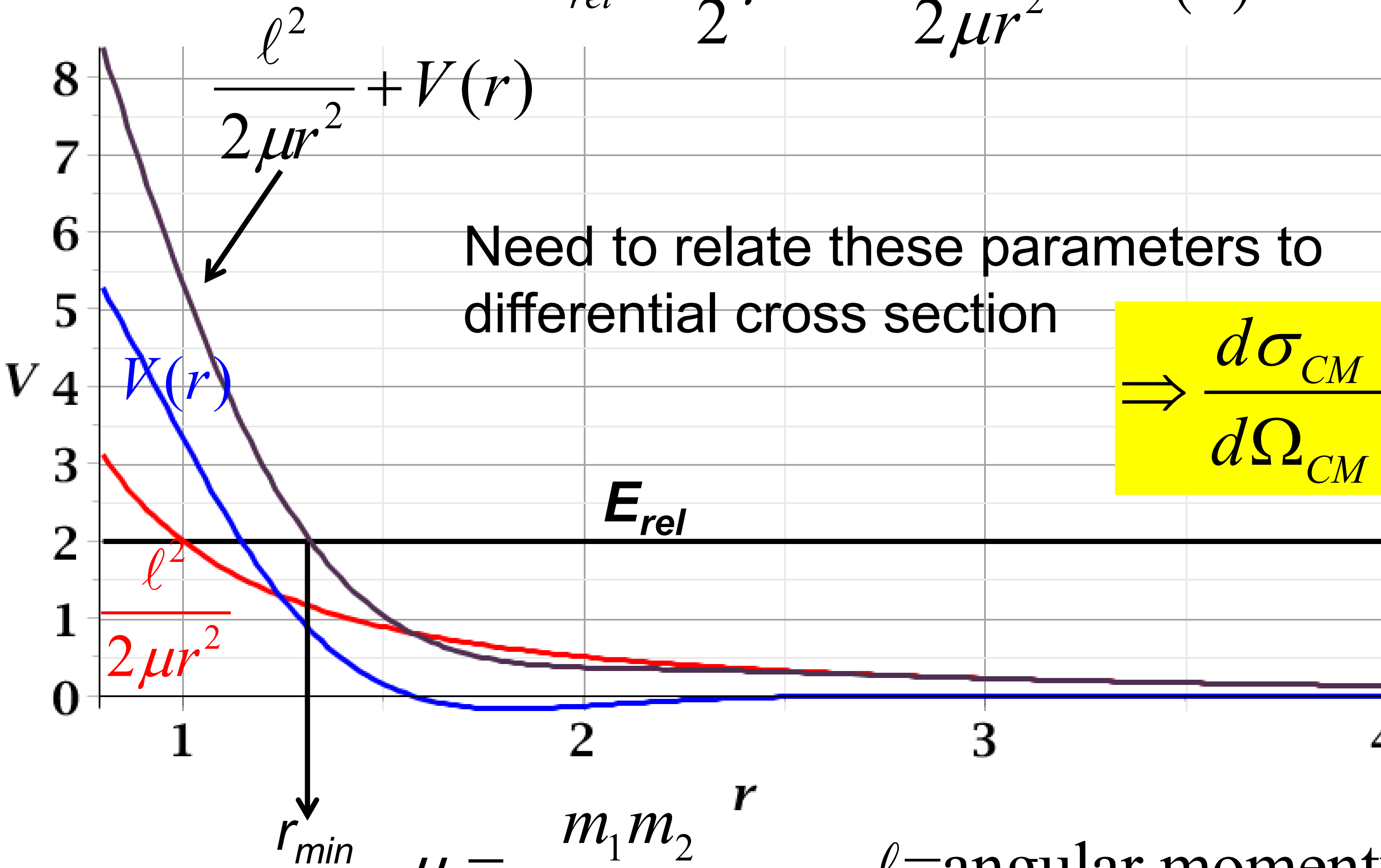
where: $$\tan\theta=\frac{\sin\psi}{\cos\psi+m_1/m_2}$$

Maple syntax:

```
> plot({[psi(theta, 0), sigma(theta, 0), theta = 0.001 ..3.14], [psi(theta, .1), sigma(theta, .1), theta
   = 0.001 ..3.14], [psi(theta, .5), sigma(theta, .5), theta = 0.001 ..3.14], [psi(theta, .8),
   sigma(theta, .8), theta = 0.001 ..3.14], [psi(theta, 1), sigma(theta, 1), theta = 0.001 ..3.14]},
   thickness = 3, font = ['Times','bold', 24], gridlines = true, color = [red, blue, green, black,
   orange])
```

For a continuous potential interaction in center of mass reference frame:

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



Need to relate these parameters to differential cross section

$$\Rightarrow \frac{d\sigma_{CM}}{d\Omega_{CM}}$$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

ℓ=angular momentum