PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion for Lecture 18 – Chap. 4 (F & W)

Analysis of motion near equilibrium –

Normal Mode Analysis

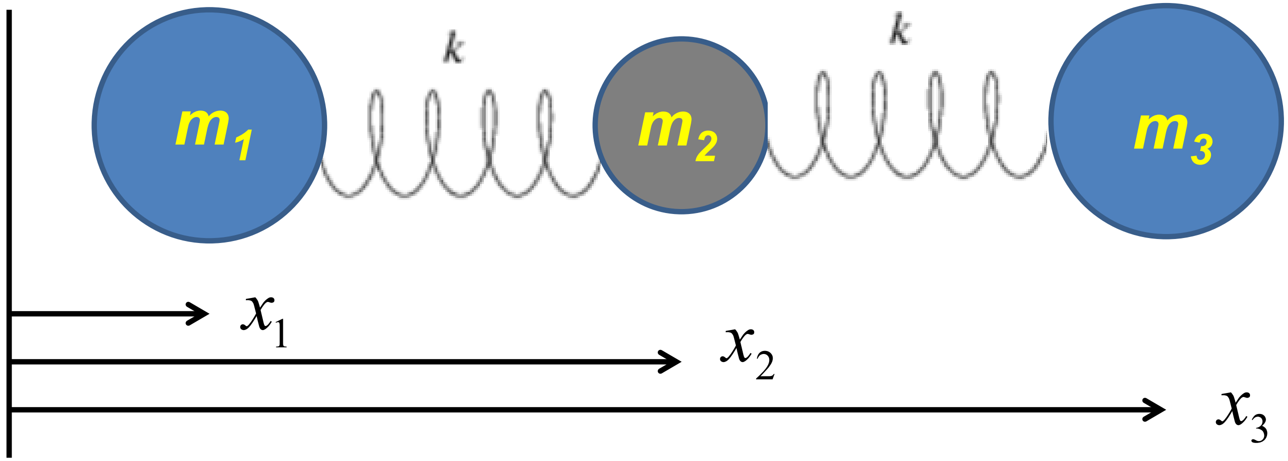
1. Normal modes of vibration for simple systems
2. Some concepts of linear algebra
3. Normal modes of vibration for more complicated systems

11	Wed, 9/18/2024	Chap. 5	Dynamics of rigid bodies	#10
12	Fri, 9/20/2024	Chap. 5	Dynamics of rigid bodies	#11
13	Mon, 9/23/2024	Chap. 1	Scattering analysis	#12
14	Wed, 9/25/2024	Chap. 1	Scattering analysis	#13
15	Fri, 9/27/2024	Chap. 1	Scattering analysis	#14
16	Mon, 9/30/2024	Chap. 4	Small oscillations near equilibrium	
17	Wed, 10/2/2024	Chap. 1-6	Review	THE-10/3-9/24
18	Fri, 10/4/2024	Chap. 4	Normal mode analysis	THE-10/3-9/24
19	Mon, 10/7/2024	Chap. 4	Normal mode analysis in multiple dimensions	THE-10/3-9/24
20	Wed, 10/9/2024	Chap. 4&7	Normal modes of continuous strings	THE-10/3-9/24
21	Fri, 10/11/2024	Chap. 7	The wave and other partial differential equations	
22	Mon, 10/14/2024	Chap. 7	Sturm-Liouville equations	
23	Wed, 10/16/2024	Chap. 7	Sturm-Liouville equations	
	Fri, 10/18/2024	Fall Break		
24	Mon, 10/21/2024	Chap. 7	Laplace transforms and complex functions	

From last time – example of system near equilibrium

Coupled oscillators --

Example – linear molecule



Simplifying the analysis:

1. Measure the displacements of each mass x_i relative to equilibrium position.
2. Introduced mass-scaled position $y_i = \sqrt{m_i} x_i$

Analysis using linear algebra methods

General matrix form:

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

for $m_1 = m_3 \equiv m_O$ and $m_2 \equiv m_C$ (CO_2)

$$\begin{pmatrix} \kappa_{OO} & -\kappa_{OC} & 0 \\ -\kappa_{OC} & 2\kappa_{CC} & -\kappa_{OC} \\ 0 & -\kappa_{OC} & \kappa_{OO} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

Finding eigenvalues/eigenvectors by hand --

$$\mathbf{M}\mathbf{y}^\alpha = \lambda^\alpha \mathbf{y}^\alpha$$

$$\left(\mathbf{M} - \lambda^\alpha \mathbf{I}\right)\mathbf{y}^\alpha = 0$$

$$\left|\mathbf{M} - \lambda^\alpha \mathbf{I}\right| \equiv \det\left(\mathbf{M} - \lambda^\alpha \mathbf{I}\right) = 0 \quad \Rightarrow \text{polynomial for solutions } \lambda^\alpha$$

For each α and λ^α solve for the eigenvector coefficients $\mathbf{y}^\alpha$

Example

$$\mathbf{M} = \begin{pmatrix} A & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A \end{pmatrix} \qquad A \equiv \frac{k}{m_O} \quad B \equiv \frac{k}{m_C}$$

$$\left|\mathbf{M} - \lambda^\alpha \mathbf{I}\right| = \begin{vmatrix} A - \lambda^\alpha & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B - \lambda^\alpha & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A - \lambda^\alpha \end{vmatrix} = \lambda^\alpha \left(\lambda^\alpha - A\right)\left(\lambda^\alpha - (A + 2B)\right) = 0$$

Example -- continued

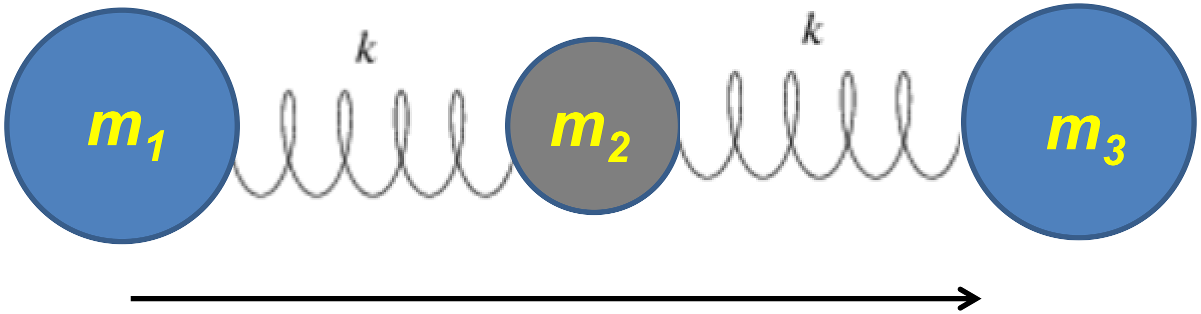
$$\mathbf{M} = \begin{pmatrix} A & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A \end{pmatrix} \qquad A \equiv \frac{k}{m_O} \quad B \equiv \frac{k}{m_C}$$

$$\left|\mathbf{M} - \lambda^\alpha \mathbf{I}\right| = \begin{vmatrix} A-\lambda^\alpha & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B-\lambda^\alpha & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A-\lambda^\alpha \end{vmatrix} = \lambda^\alpha \left(\lambda^\alpha - A\right)\left(\lambda^\alpha - (A+2B)\right)$$

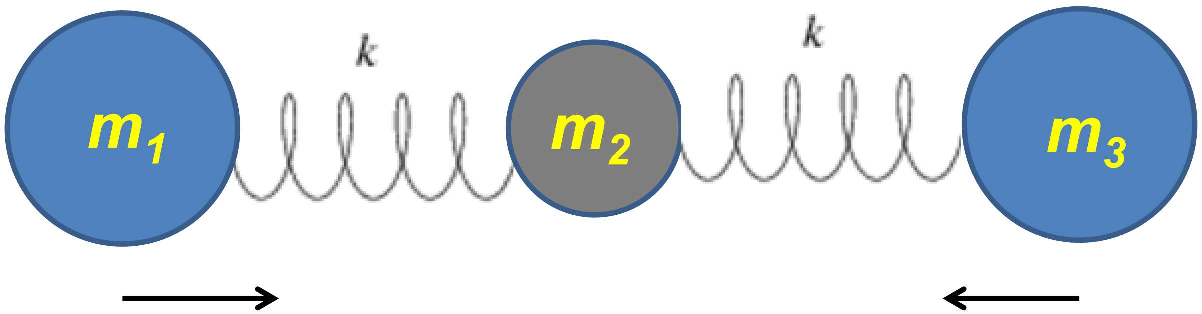
Solving for eigenvector corresponding to $\lambda^\alpha \equiv \lambda^1 = 0$

$$\begin{pmatrix} A & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A \end{pmatrix} \begin{pmatrix} y_{O1}^1 \\ y_C^1 \\ y_{O2}^1 \end{pmatrix} = 0 \qquad \Rightarrow \frac{y_{O1}^1}{y_C^1} = \frac{y_{O2}^1}{y_C^1} = \sqrt{\frac{B}{A}}$$

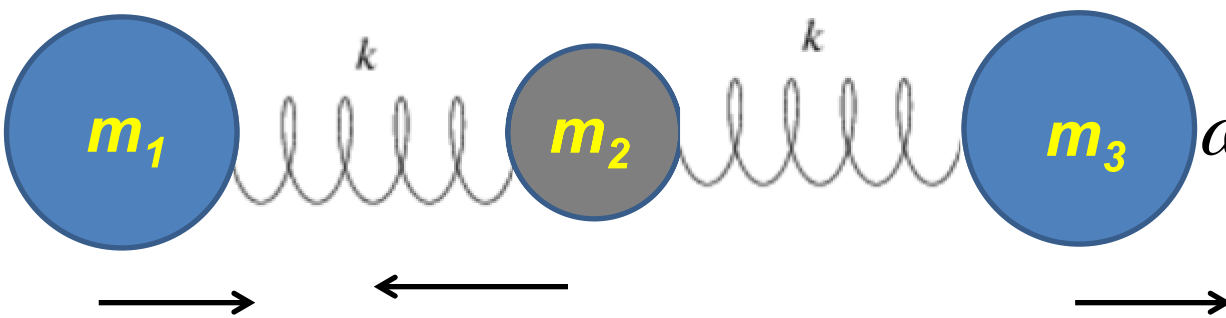
Note that the normalization of the eigenvector is arbitrary.



$$\omega_1 = 0$$



$$\omega_2 = \sqrt{\frac{k}{m_O}}$$



$$\omega_3 = \sqrt{\frac{k}{m_O} + \frac{2k}{m_C}}$$

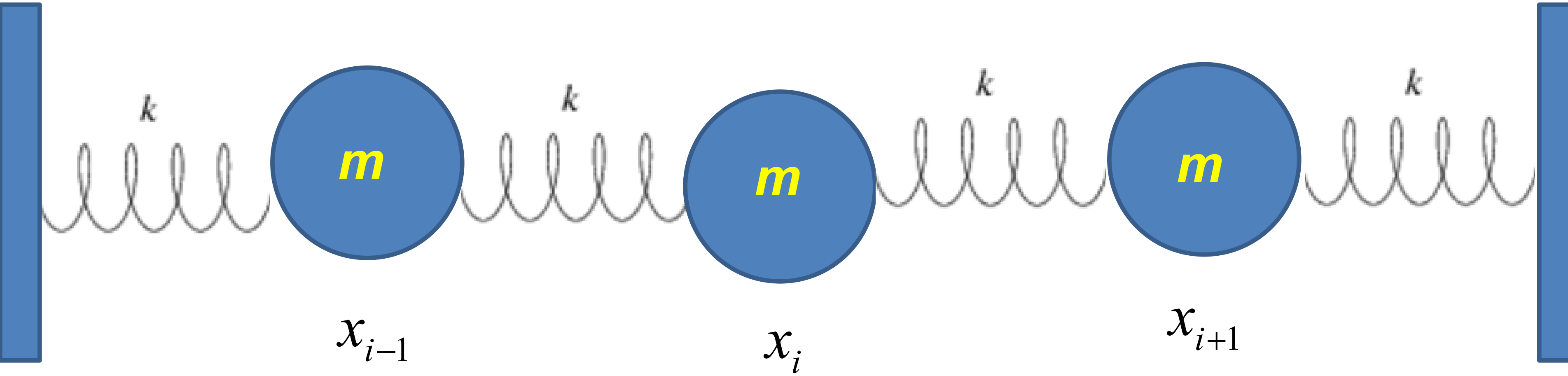
Eigenvalues and eigenvectors :

$$\omega_1^2 = 0 \qquad \begin{pmatrix} Y_1^1 \\ Y_2^1 \\ Y_3^1 \end{pmatrix} = N_1 \begin{pmatrix} \sqrt{\frac{m_O}{m_C}} \\ 1 \\ \sqrt{\frac{m_O}{m_C}} \end{pmatrix}, \quad \begin{pmatrix} X_1^1 \\ X_2^1 \\ X_3^1 \end{pmatrix} = N'_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\omega_2^2 = \frac{k}{m_O} \qquad \begin{pmatrix} Y_1^2 \\ Y_2^2 \\ Y_3^2 \end{pmatrix} = N_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} X_1^2 \\ X_2^2 \\ X_3^2 \end{pmatrix} = N'_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\omega_3^2 = \frac{k}{m_O} + \frac{2k}{m_C} \qquad \begin{pmatrix} Y_1^3 \\ Y_2^3 \\ Y_3^3 \end{pmatrix} = N_3 \begin{pmatrix} 1 \\ -2\sqrt{\frac{m_O}{m_C}} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} X_1^3 \\ X_2^3 \\ X_3^3 \end{pmatrix} = N'_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Consider an extended system of masses and springs:



Note : each mass coordinate is measured relative to its equilibrium position x_i^0

$$L = T - V = \frac{1}{2} m \sum_{i=1}^{N} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{N} \left(x_{i+1} - x_i \right)^2$$

Note: In fact, we have N masses; x_0 and x_{N+1} will be treated using boundary conditions.

$$L = T - V = \frac{1}{2} m \sum_{i=1}^{N} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{N} \left(x_{i+1} - x_i \right)^2$$

$$x_0 \equiv 0 \;\text{ and }\; x_{N+1} \equiv 0$$

From Euler - Lagrange equations :

$$m\ddot{x}_1 = k\left(x_2 - 2x_1\right)$$

$$m\ddot{x}_2 = k\left(x_3 - 2x_2 + x_1\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1}\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$m\ddot{x}_N = k\left(x_{N-1} - 2x_N\right)$$

Matrix formulation --

Assume $x_i(t) = X_i e^{-i\omega t}$

$$\frac{m}{k}\omega^2 \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-1} \\ X_N \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & -1 & 2 & -1 \\ \cdots & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-1} \\ X_N \end{pmatrix}$$

Can solve as an eigenvalue problem –

(Why did we not have to transform the equations as we did in the previous example?)

Because of its very regular form, this example also has an algebraic solution --

From Euler - Lagrange equations :

$$m\ddot{x}_j = k\left(x_{j+1} - 2x_j + x_{j-1}\right) \qquad \text{with } x_0 = 0 = x_{N+1}$$

Try : $\quad x_j(t) = Ae^{-i\omega t + iqaj}$

Here "*a*" is the equilibrium length of a spring and *q* has the units of 1/length.

$$-\omega^2 Ae^{-i\omega t+iqaj} = \frac{k}{m}\left(e^{iqa} - 2 + e^{-iqa}\right)Ae^{-i\omega t+iqaj}$$

$$-\omega^2 = \frac{k}{m}\left(2\cos(qa) - 2\right)$$

$$\Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$$

Is this treatment cheating?

a. Yes.
b. No cheating, but we are not done.

From Euler - Lagrange equations - - continued :

$$m\ddot{x}_j = k\left(x_{j+1} - 2x_j + x_{j-1}\right) \qquad \text{with } x_0 = 0 = x_{N+1}$$

Try : $x_j(t) = Ae^{-i\omega t + iqaj}$ $\Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$

Note that : $x_j(t) = Be^{-i\omega t - iqaj}$ $\Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$

General solution :

$$x_j(t) = \Re\left(Ae^{-i\omega t + iqaj} + Be^{-i\omega t - iqaj}\right)$$

Impose boundary conditions :

$$x_0(t) = \Re\left(Ae^{-i\omega t} + Be^{-i\omega t}\right) = 0$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t + iqa(N+1)} + Be^{-i\omega t - iqa(N+1)}\right) = 0$$

Impose boundary conditions -- continued:

$$x_0(t) = \Re\left(Ae^{-i\omega t} + Be^{-i\omega t} \right) = 0$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t + iqa(N+1)} + Be^{-i\omega t - iqa(N+1)} \right) = 0$$

$$\Rightarrow B = \text{-}A$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t}\left(e^{iqa(N+1)} - e^{-iqa(N+1)} \right)\right) = 0$$

$$\Rightarrow \sin\left(qa\left(N+1 \right)\right) = 0$$

$$\Rightarrow \; qa\left(N+1 \right) = \nu\pi \quad \text{where } \nu = 1,2\cdots N$$

$$qa = \frac{\nu\pi}{N+1}$$

Recap -- solution for integer parameter ν

$$x_j(t) = \Re\left(2iAe^{-i\omega_\nu t} \sin\left(\frac{\nu\pi j}{N+1} \right) \right)$$

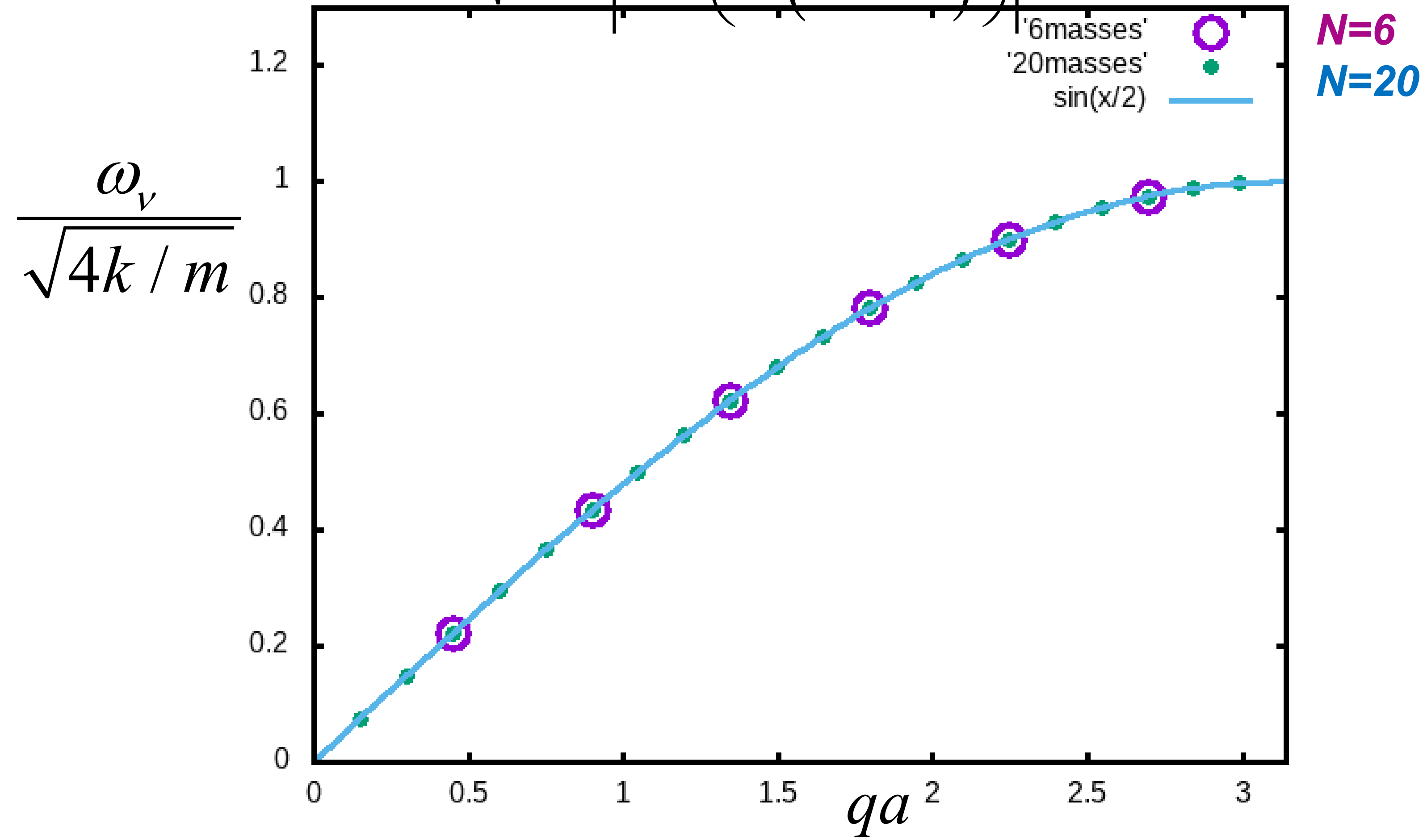
$$\omega_\nu^{\ 2} = \frac{4k}{m} \sin^2\left(\frac{\nu\pi}{2(N+1)} \right)$$

Note that non-trivial, unique values are

$$\nu = 1,2,\cdots N$$

Examples

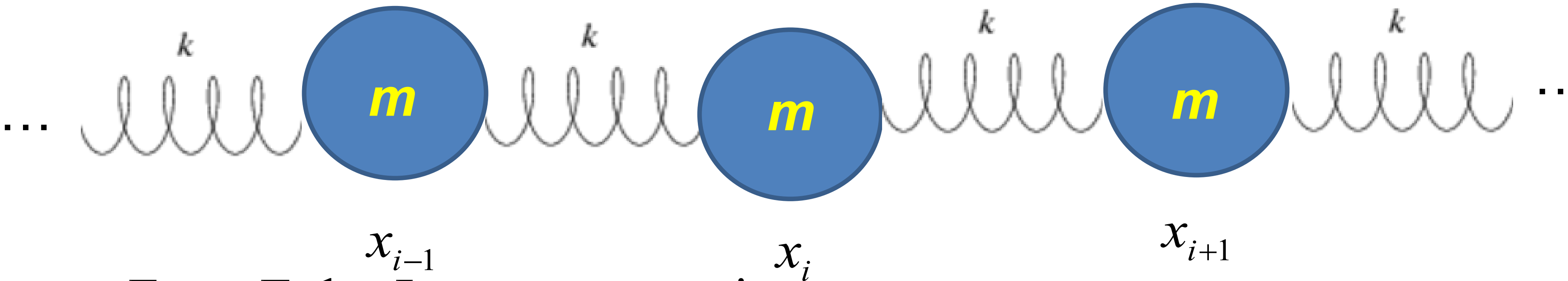
$$\omega_\nu = \sqrt{\frac{4k}{m}} \left| \sin\left(\frac{\nu\pi}{2(N+1)} \right) \right|$$



Note that solution form remains correct for $N \to \infty$

$$\omega(qa) = \sqrt{4k/m} \left| \sin\left(\frac{qa}{2} \right) \right|$$

For extended (infinite) chain without boundaries:



From Euler-Lagrange equations:

$$m\ddot{x}_j = k\left(x_{j+1} - 2x_j + x_{j-1}\right) \qquad \text{for all } x_j$$

Try: $x_j(t) = Ae^{-i\omega t + iqaj}$

$$-\omega^2 Ae^{-i\omega t+iqaj} = \frac{k}{m}\left(e^{iqa} - 2 + e^{-iqa}\right)Ae^{-i\omega t+iqaj}$$

$$-\omega^2 = \frac{k}{m}\left(2\cos(qa) - 2\right)$$

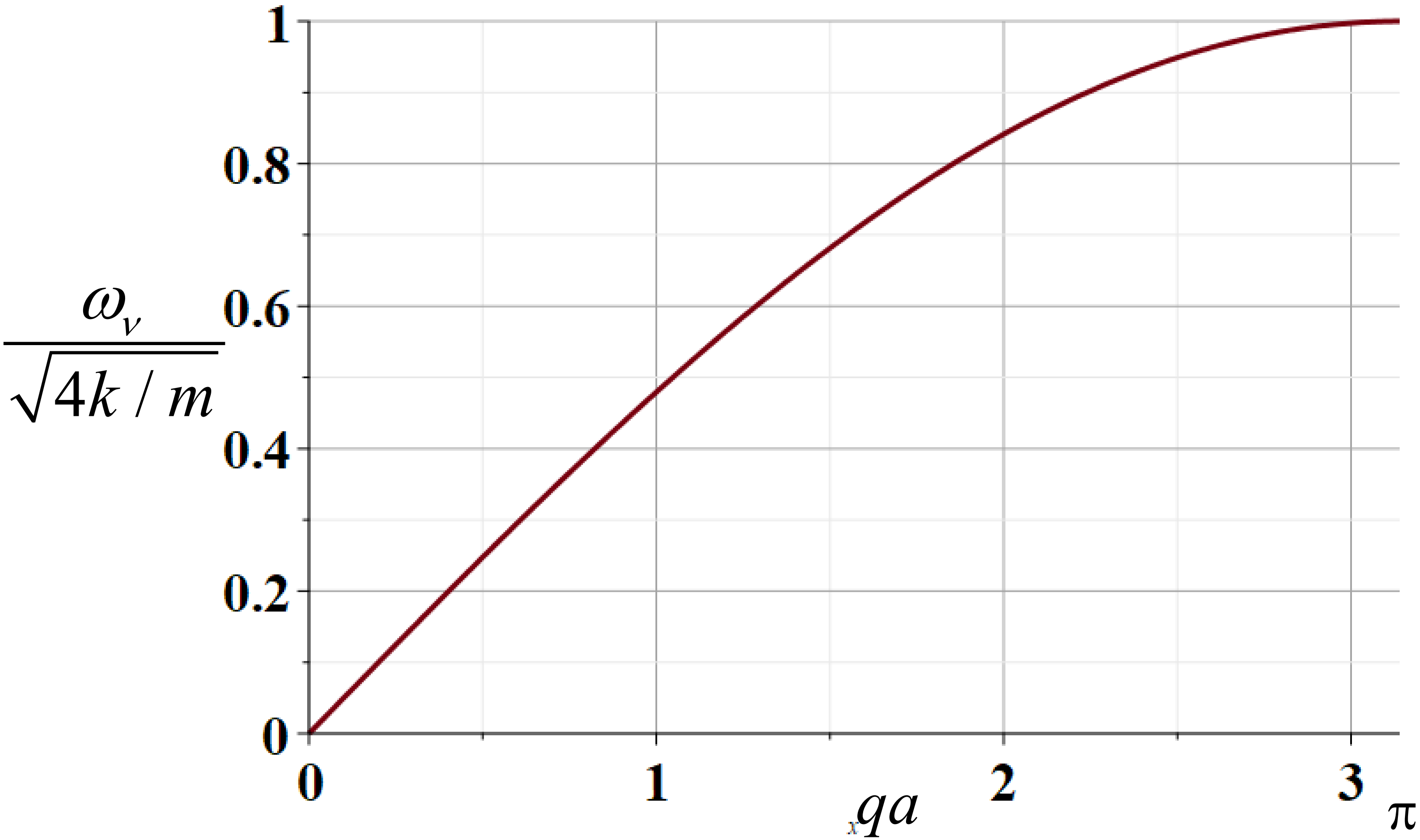
$$\Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$$

Note that we are assuming that all masses and springs are identical here.

Here "*a*" is the equilibrium length of a spring and *q* has the units of 1/length.

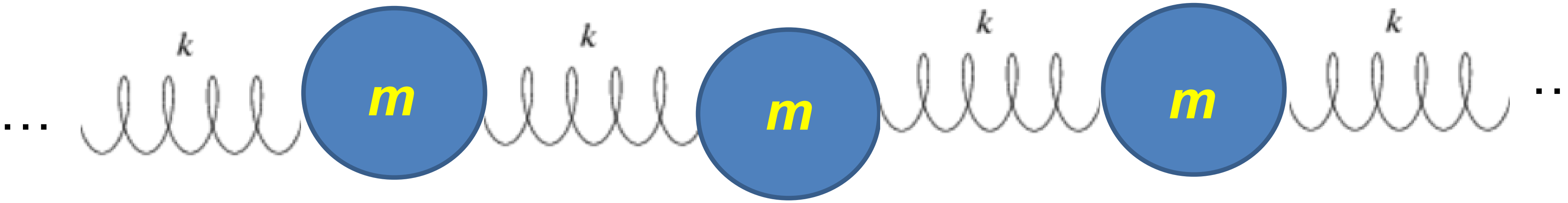
distinct values for $0 \le qa \le \pi$

Plot of distinct values of $\omega_\nu(q)$



Note that for $N \rightarrow \infty$, q becomes a continuous variable within the range $0 < qa < \pi$.

For extended (infinite) chain without boundaries:



x_{i-1} x_i x_{i+1}

From Euler-Lagrange equations:

$$m\ddot{x}_j = k\left(x_{j+1} - 2x_j + x_{j-1}\right) \qquad \text{for all } x_j \qquad \text{Try:} \quad x_j(t) = Ae^{-i\omega t + iqaj}$$

$$-\omega^2 Ae^{-i\omega t+iqaj} = \frac{k}{m}\left(e^{iqa} - 2 + e^{-iqa}\right)Ae^{-i\omega t+iqaj} \qquad -\omega^2 = \frac{k}{m}\left(2\cos(qa) - 2\right)$$

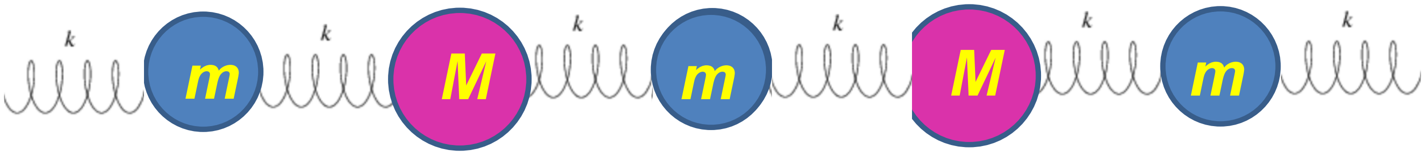
$$\Rightarrow \omega = \sqrt{\frac{4k}{m}}\left|\sin\left(\frac{qa}{2}\right)\right| \qquad \text{distinct values for } 0 \le qa \le \pi$$

Note that there are an infinite number of normal mode frequencies!

Does this make sense?

(A) Yes (B) No

Consider an infinite system of masses and springs now with two kinds of masses:



x_i y_i x_{i+1} y_{i+1} x_{i+2}

Note: each mass coordinate is measured relative to its equilibrium position $x_i^0 \equiv 0, y_i^0 \equiv 0, \cdots$

$$L = T - V$$

$$= \frac{1}{2} m \sum_{i=0}^{\infty} \dot{x}_i^2 + \frac{1}{2} M \sum_{i=0}^{\infty} \dot{y}_i^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(x_{i+1} - y_i\right)^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(y_i - x_i\right)^2$$

$$L = T - V$$

$$= \frac{1}{2} m \sum_{i=0}^{\infty} \dot{x}_i^2 + \frac{1}{2} M \sum_{i=0}^{\infty} \dot{y}_i^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(x_{i+1} - y_i \right)^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(y_i - x_i \right)^2$$

Euler - Lagrange equations :

$$m\ddot{x}_j = k\left(y_{j-1} - 2x_j + y_j \right)$$

$$M\ddot{y}_j = k\left(x_j - 2y_j + x_{j+1} \right)$$

Trial solution :

$$x_j(t) = Ae^{-i\omega t + i2qaj}$$

$$y_j(t) = Be^{-i\omega t + i2qaj}$$

Note that 2qa is an unknown parameter.

Does this form seem reasonable?

$$\begin{pmatrix} m\omega^2 - 2k & k\left(e^{-i2qa} + 1\right) \\ k\left(e^{i2qa} + 1\right) & M\omega^2 - 2k \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

Comment on notation --

x_i y_i x_{i+1} y_{i+1} x_{i+2}

Trial solution:

$$x_j(t) = Ae^{-i\omega t + i2qaj}$$

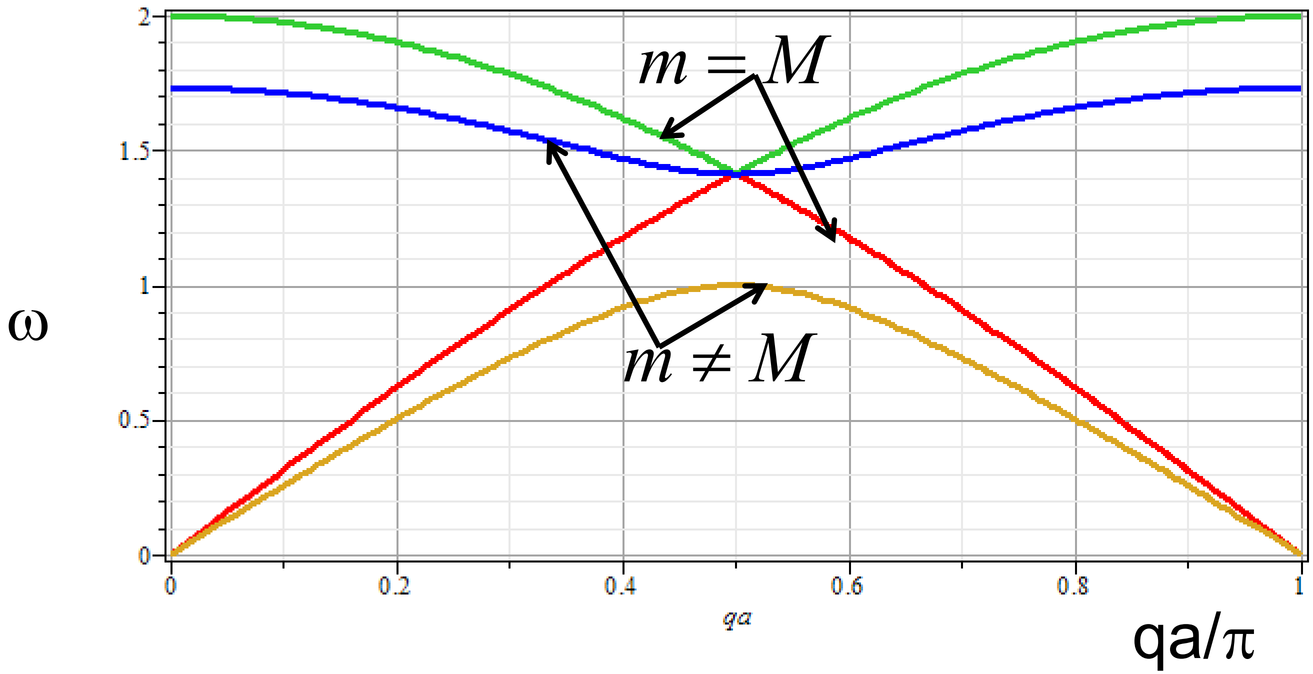
$$y_j(t) = Be^{-i\omega t + i2qaj}$$

Using 2qa as our unknown parameter is a convenient choice so that we can easily relate our solution to the m=M case.

$$\begin{pmatrix} m\omega^2 - 2k & k\left(e^{-i2qa}+1\right) \\ k\left(e^{i2qa}+1\right) & M\omega^2 - 2k \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0$$

Solutions:

$$\omega_{\pm}{}^2 = \frac{k}{m} + \frac{k}{M} \pm k\sqrt{\frac{1}{m^2} + \frac{1}{M^2} + \frac{2\cos(2qa)}{mM}}$$



Next time –

1. Extension of these ideas to 2 and 3 dimensions
2. Extension of these ideas to continuous elastic media.