PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 20: Chap. 4&7 (F&W)

One dimensional motion of many coupled masses ➔continuous elastic string and related systems

1. Comments on linear vs. non-linear differential equations – considering beyond harmonic oscillations
2. Back to linear analyses -- masses coupled by springs ←→mass continuum coupled by string
3. Mechanics and mathematics of one-dimensional continuous system

11	Wed, 9/18/2024	Chap. 5	Dynamics of rigid bodies	#10
12	Fri, 9/20/2024	Chap. 5	Dynamics of rigid bodies	#11
13	Mon, 9/23/2024	Chap. 1	Scattering analysis	#12
14	Wed, 9/25/2024	Chap. 1	Scattering analysis	#13
15	Fri, 9/27/2024	Chap. 1	Scattering analysis	#14
16	Mon, 9/30/2024	Chap. 4	Small oscillations near equilibrium	
17	Wed, 10/2/2024	Chap. 1-6	Review	THE-10/3-9/24
18	Fri, 10/4/2024	Chap. 4	Normal mode analysis	THE-10/3-9/24
19	Mon, 10/7/2024	Chap. 4	Normal mode analysis in multiple dimensions	THE-10/3-9/24
20	Wed, 10/9/2024	Chap. 4&7	Normal modes of continuous strings	THE-10/3-9/24
21	Fri, 10/11/2024	Chap. 7	The wave and other partial differential equations	
22	Mon, 10/14/2024	Chap. 7	Sturm-Liouville equations	
23	Wed, 10/16/2024	Chap. 7	Sturm-Liouville equations	
	Fri, 10/18/2024	Fall Break		
24	Mon, 10/21/2024	Chap. 7	Laplace transforms and complex functions	

Physics Colloquium

- Thursday -

October 10, 2024

4 PM

Olin 101

A journey to SPArc: Spot-scanning Proton Arc therapy

Radiation therapy is one of the main treatment modalities in the management of cancer. Proton beam therapy uses a charged particle, Hydrogen, to irradiate the tumor. One of the biggest advantages of using proton compared to X-ray is to reduce the integral dose to the patient's body which is critical to many patients, especially the pediatric cancer population. As of today, more than 40 proton therapy centers are operating in the United States, and over 200 particle therapy centers are either operating clinically or under construction globally. Today, Atrium Health and Wake Forest School of Medicine bring the first proton therapy center to North Carolina, downtownl Charlotte, which directly benefits the cancer patient

The concept of Spot scanning Proton Arc therapy (SPArc) was first introduced by Dr Ding and his team in at William Beaumont University Hospital, Corewell Health in 2015. It has become one of the most important merging treatment modalities in radiation oncology. The technique enables the dynamic rotational proton gantry while irradiating the proton spot and switching energy layers in a submillimeter accuracy. Through an

Dr. Xuanfeng (Leo) Ding

Beaumont Proton Therapy Center

Digression – comment on linear vs non-linear equations

Linear oscillator equations (ODE example from one dimension)

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left.\frac{d^2V}{dx^2}\right|_{x_{eq}} + \cdots$$

$$\Rightarrow \frac{1}{2}kx^2 \equiv \frac{1}{2}m\omega^2 x^2$$

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

Euler-Lagrange equations: $\ddot{x} = -\omega^2 x$

Superposition property of linear equations: --

Suppose that the functions $x_1(t)$ and $x_2(t)$ are solutions

$$\Rightarrow Ax_1(t) + Bx_2(t) \text{ are also solutions (all } A, B)$$

Non - linear oscillator equations (example from one dimension)

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left.\frac{d^2V}{dx^2}\right|_{x_{eq}} + \frac{1}{4!}(x - x_{eq})^4 \left.\frac{d^4V}{dx^4}\right|_{x_{eq}} + \cdots$$

$$\Rightarrow \frac{1}{2}m\omega^2\left(x^2 + \frac{1}{2}\varepsilon x^4\right)$$

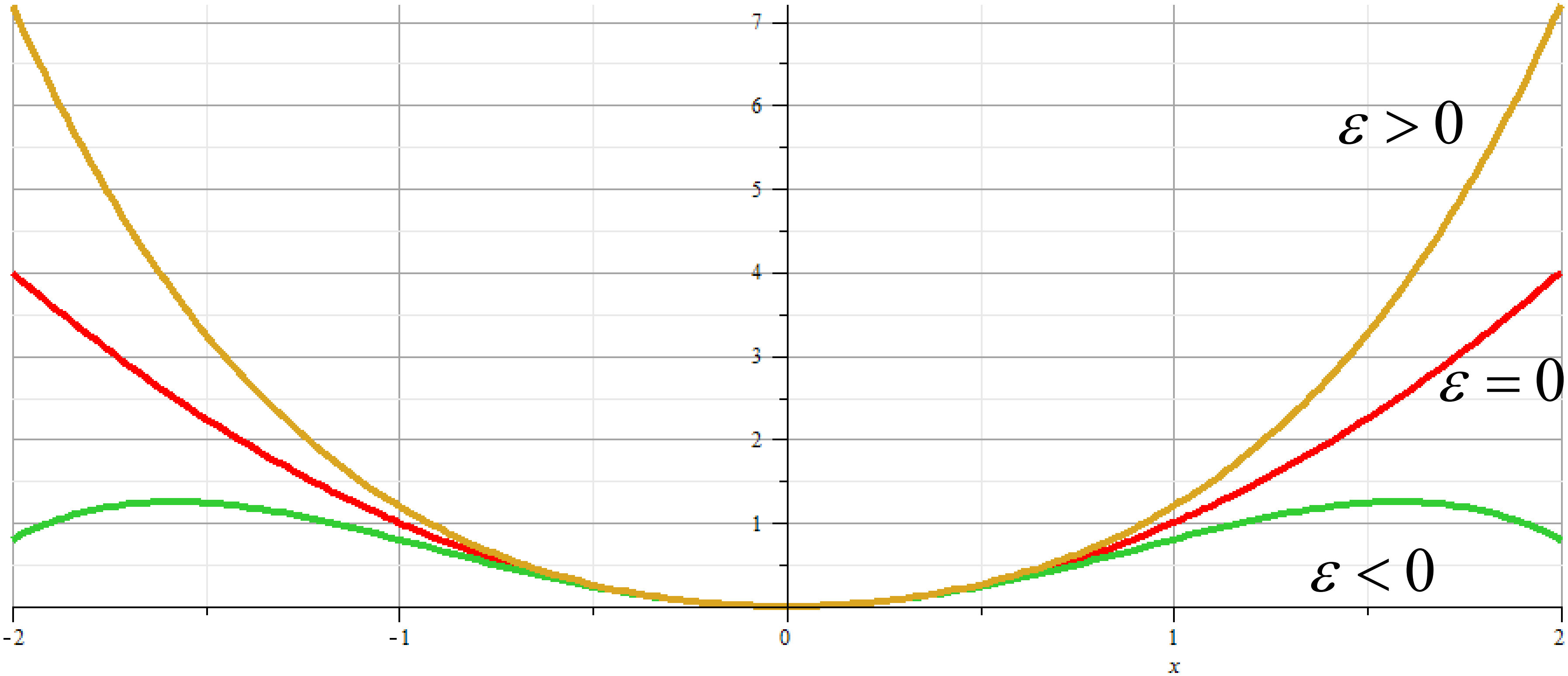
$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2\left(x^2 + \frac{1}{2}\varepsilon x^4\right)$$

Euler - Lagrange equations :

$$\ddot{x} = -\omega^2\left(x + \varepsilon x^3\right)$$

Superposition-- no longer applies

$$V(x) \approx \frac{1}{2} m\omega^2 \left(x^2 + \frac{1}{2}\varepsilon x^4 \right)$$



x

Non-linear example -- continued

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2\left(x^2 + \frac{1}{2}\varepsilon x^4\right)$$

Euler-Lagrange equations:

$$\ddot{x} + \omega^2\left(x + \varepsilon x^3\right) = 0$$

Perturbation expansion:

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots$$

Euler-Lagrange equations:

zero order (factor of ε^0): $\ddot{x}_0 + \omega^2 x_0 = 0$

first order (factor of ε^1): $\ddot{x}_1 + \omega^2 x_1 + \omega^2 x_0^3 = 0$

Non-linear example -- continued

$$\ddot{x}+\omega^2\left(x+\varepsilon x^3\right)=0$$

Initial conditions:

$$x(0)=X_0 \qquad \dot{x}(0)=0$$

Perturbation expansion:

$$x(t)=x_0(t)+\varepsilon x_1(t)+\ldots$$

Euler-Lagrange equations:

zero order: $\ddot{x}_0+\omega^2 x_0=0 \qquad \Rightarrow x_0(t)=X_0\cos(\omega t)$

first order: $\ddot{x}_1+\omega^2 x_1+\omega^2 x_0^3=0$

$$\Rightarrow \ddot{x}_1(t)+\omega^2 x_1(t)=-X_0^3\cos^3(\omega t)=-\frac{X_0^3}{4}\left(3\cos(\omega t)+\cos(3\omega t)\right)$$

$$\Rightarrow x_1(t)=-\frac{X_0^3}{8\omega^2}\left\{3\omega t\sin(\omega t)+\frac{1}{4}\left[\cos(\omega t)-\cos(3\omega t)\right]\right\}$$

$$x(t)=X_0\cos(\omega t)-\varepsilon\frac{X_0^3}{8\omega^2}\left\{3\omega t\sin(\omega t)+\frac{1}{4}\left[\cos(\omega t)-\cos(3\omega t)\right]\right\}+O\left(\varepsilon^2\right)$$

Non - linear example - - continued

$$\ddot{x} + \omega^2\left(x + \varepsilon x^3\right) = 0$$

Initial conditions:

$$x(0) = X_0 \qquad \dot{x}(0) = 0$$

Perturbation expansion:

$$x\left(t\right) = x_0\left(t\right) + \varepsilon x_1\left(t\right) + \ldots$$

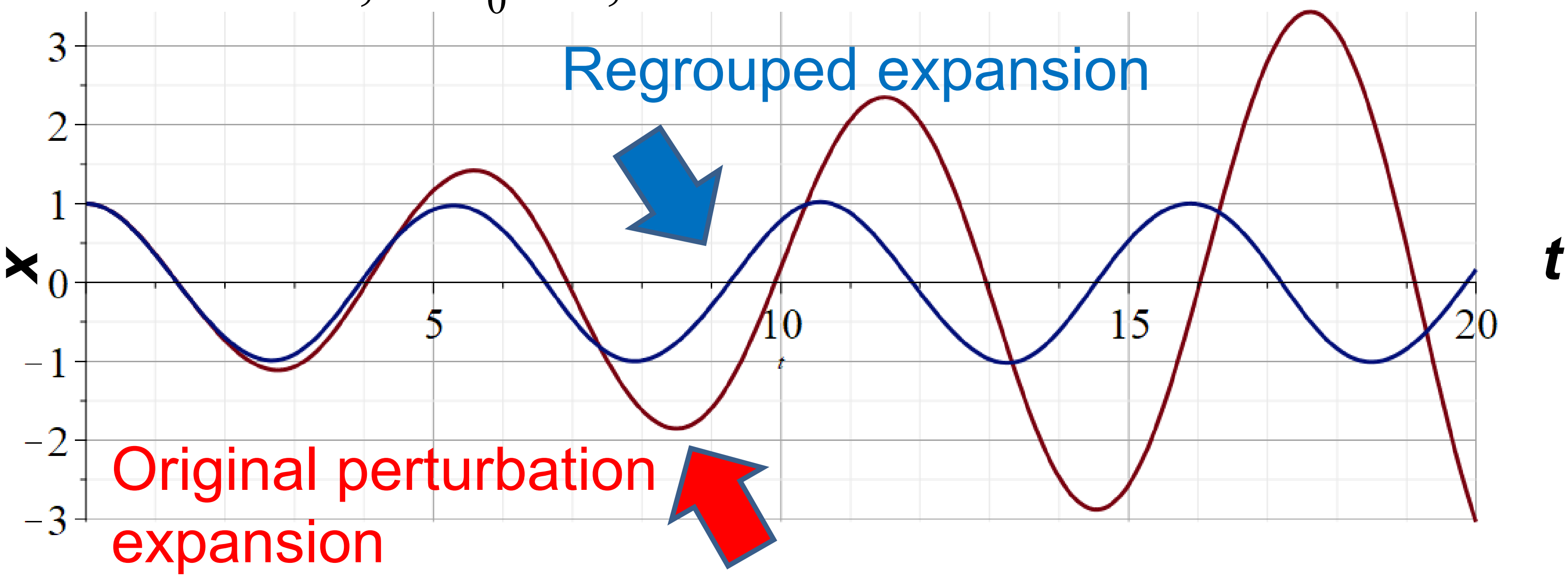
Previous result (blows up at large t):

$$x(t) = X_0 \cos\left(\omega t\right) - \varepsilon \frac{X_0^3}{8\omega^2}\left\{3\omega t\ \sin\left(\omega t\right) + \frac{1}{4}\left[\cos\left(\omega t\right) - \cos\left(3\omega t\right)\right]\right\} + O\left(\varepsilon^2\right)$$

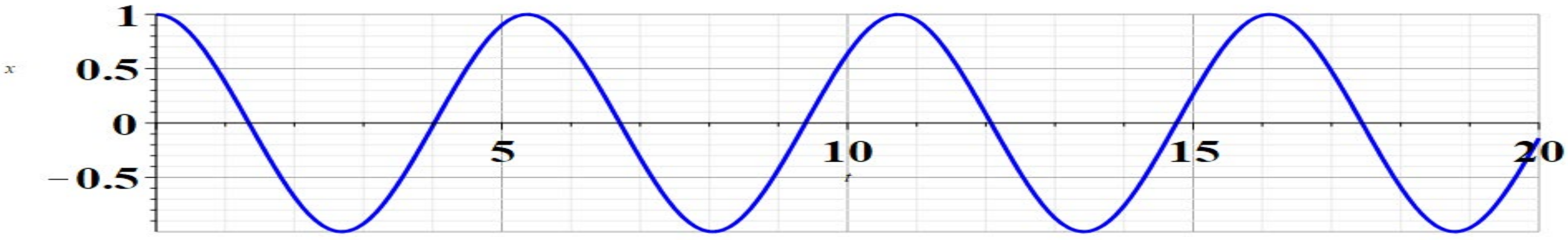
By rearranging terms (allowing effective frequency to vary):

$$x(t) = X_0 \cos\left(\omega\left(1 + \varepsilon\frac{3X_0^2}{8\omega}\right)t\right) - \varepsilon\frac{X_0^3}{32\omega^2}\left\{\cos\left(\omega t\right) - \cos\left(3\omega t\right)\right\} + O\left(\varepsilon^2\right)$$

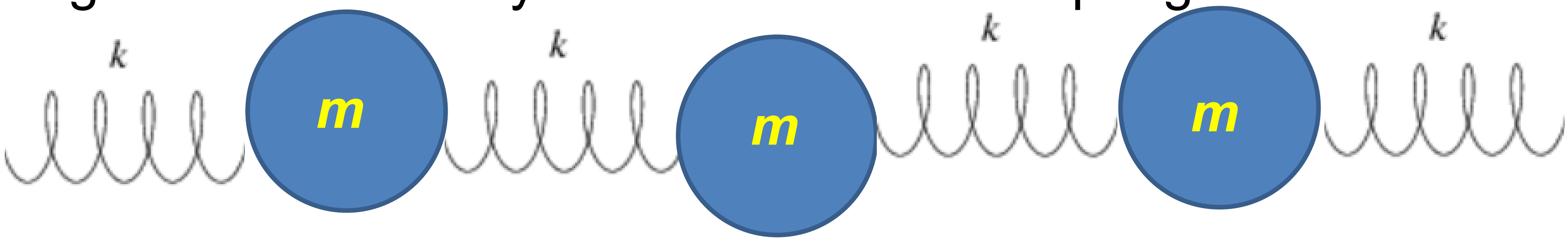
For $\omega = 1, \quad X_0 = 1, \quad \epsilon = 0.5$



Numerical solution according to Maple



Back to linear equations –
Longitudinal case: a system of masses and springs:



$$x_{i-1} \qquad x_i \qquad x_{i+1}$$

$$L = T - V = \frac{1}{2} m \sum_{i=0}^{\infty} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(x_{i+1} - x_i \right)^2$$

$$\Rightarrow m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1} \right)$$

Now imagine the continuum version of this system:

$$x_i(t) \Rightarrow \mu(x,t) \qquad \ddot{x}_i \Rightarrow \frac{\partial^2 \mu}{\partial t^2}$$

$$x_{i+1} - 2x_i + x_{i-1} \Rightarrow \approx \frac{\partial^2 \mu}{\partial x^2} (\Delta x)^2 \qquad \text{where } \Delta x \equiv x_{i+1} - x_i = x_i - x_{i-1}$$

More details

Longitudinal case

Consider Taylor's series (focussing on x-dependence)

$$\mu(x+\Delta x) = \mu(x) + \Delta x \left.\frac{d\mu}{dx}\right|_x + \frac{1}{2}(\Delta x)^2 \left.\frac{d^2\mu}{dx^2}\right|_x + \frac{1}{6}(\Delta x)^3 \left.\frac{d^3\mu}{dx^3}\right|_x + \frac{1}{24}(\Delta x)^4 \left.\frac{d^4\mu}{dx^4}\right|_x + ...$$

$$\mu(x-\Delta x) = \mu(x) - \Delta x \left.\frac{d\mu}{dx}\right|_x + \frac{1}{2}(\Delta x)^2 \left.\frac{d^2\mu}{dx^2}\right|_x - \frac{1}{6}(\Delta x)^3 \left.\frac{d^3\mu}{dx^3}\right|_x + \frac{1}{24}(\Delta x)^4 \left.\frac{d^4\mu}{dx^4}\right|_x + ...$$

Therefore $(\Delta x)^2 \left.\frac{d^2\mu}{dx^2}\right|_x = \mu(x+\Delta x) + \mu(x-\Delta x) - 2\mu(x) - \frac{1}{12}(\Delta x)^4 \left.\frac{d^4\mu}{dx^4}\right|_x + ...$

$$\Rightarrow \left.\frac{d^2\mu}{dx^2}\right|_x \approx \frac{\mu(x+\Delta x) + \mu(x-\Delta x) - 2\mu(x)}{(\Delta x)^2}$$

Discrete equation: $m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1}\right)$

Continuum equation: $m\dfrac{\partial^2 \mu}{\partial t^2} = k(\Delta x)^2 \dfrac{\partial^2 \mu}{\partial x^2}$

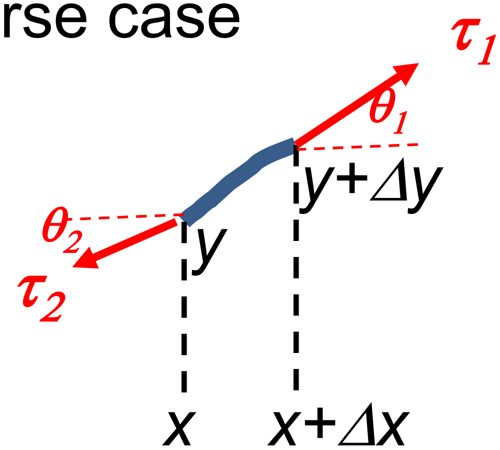
$$\frac{\partial^2 \mu}{\partial t^2} = \left(\frac{k\Delta x}{m/\Delta x}\right)\frac{\partial^2 \mu}{\partial x^2}$$

system parameter with units of (velocity)^2

For transverse oscillations on a string with tension τ and mass/length σ:

$$\left(\frac{k\Delta x}{m/\Delta x}\right) \Rightarrow \frac{\tau}{\sigma}$$

More details
Transverse case



Net vertical force on increment of string:

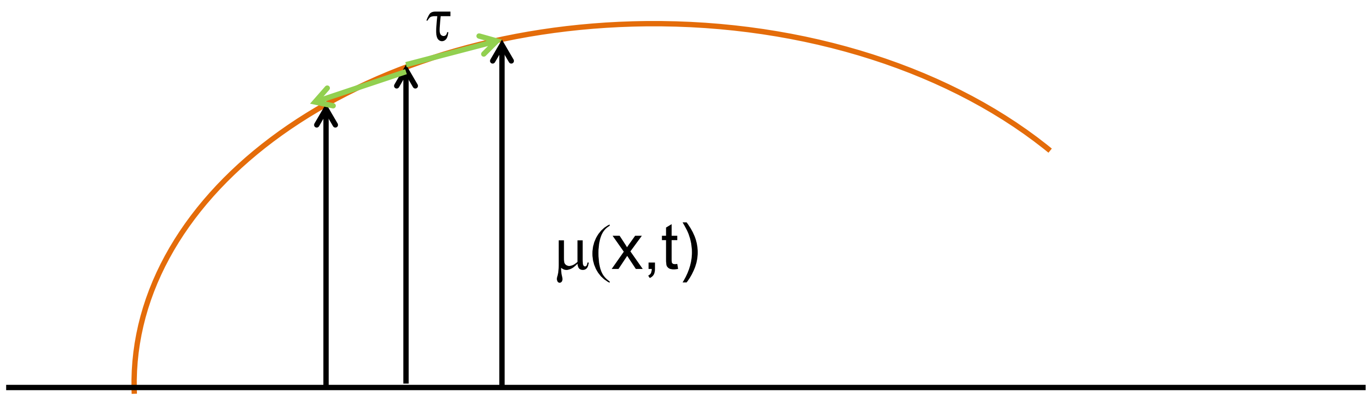
$$\tau_1 \sin\theta_1 - \tau_2 \sin\theta_2 \approx \tau_1 \tan\theta_1 - \tau_2 \tan\theta_2$$

$$\approx \tau\left(\left.\frac{dy}{dx}\right|_{x+\Delta x} - \left.\frac{dy}{dx}\right|_{x} \right)$$

$$= \tau \Delta x \frac{d}{dx}\left(\frac{dy}{dx} \right)$$

$$= \tau\left(\Delta x \frac{d^2 y}{dx^2} \right)$$

Transverse displacement:



$$\frac{\partial^2 \mu}{\partial t^2} = \frac{\tau}{\sigma}\frac{\partial^2 \mu}{\partial x^2}$$

Wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} = c^2 \frac{\partial^2 \mu}{\partial x^2}$$

Lagrangian for continuous system:

Denote the generalized displacement by $\mu(x,t)$:

$$L = L\left(\mu, \frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial t}; x, t\right)$$

Hamilton's principle:

$$\delta \int_{t_i}^{t_f} dt \int_{x_i}^{x_f} dx\, L\left(\mu, \frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial t}; x, t\right) = 0$$

$$\Rightarrow \frac{\partial L}{\partial \mu} - \frac{\partial}{\partial x}\frac{\partial L}{\partial(\partial \mu / \partial x)} - \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial \mu / \partial t)} = 0$$

Euler - Lagrange equations for continuous system:

$$\frac{\partial L}{\partial \mu} - \frac{\partial}{\partial x}\frac{\partial L}{\partial(\partial \mu / \partial x)} - \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial \mu / \partial t)} = 0$$

Example:

$$L = \frac{\sigma}{2}\left(\frac{\partial \mu}{\partial t}\right)^2 - \frac{\tau}{2}\left(\frac{\partial \mu}{\partial x}\right)^2$$

$$\Rightarrow \sigma \frac{\partial^2 \mu}{\partial t^2} - \tau \frac{\partial^2 \mu}{\partial x^2} = 0$$

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{for} \quad c^2 = \frac{\tau}{\sigma}$$

Note that this is an example of a partial differential equation

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } c = \sqrt{\frac{\tau}{\sigma}} \text{ or } \sqrt{\frac{k\Delta x}{m/\Delta x}}$$

Transverse motion

Longitudinal motion

Often, it is useful to seek a solution in terms of ordinary differential equations:

$$\mu(x,t) = f(x)g(t)$$

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \Rightarrow \frac{1}{g(t)} \frac{d^2 g(t)}{dt^2} - \frac{c^2}{f(x)} \frac{d^2 f(x)}{dx^2} = 0$$

Solving partial differential equation using ordinary differential equations – continued.

Suppose: $\mu(x,t) = f(x)g(t)$

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \Rightarrow \frac{1}{g(t)} \frac{d^2 g(t)}{dt^2} - \frac{c^2}{f(x)} \frac{d^2 f(x)}{dx^2} = 0$$

The equality is possible if

$$\frac{1}{g(t)} \frac{d^2 g(t)}{dt^2} = \lambda \quad \text{and} \quad \frac{c^2}{f(x)} \frac{d^2 f(x)}{dx^2} = \lambda$$

Eigenvalue equations to be solved:

$$\frac{d^2 g(t)}{dt^2} = \lambda g(t) \quad \text{and} \quad \frac{d^2 f(x)}{dx^2} = \frac{\lambda}{c^2} f(x)$$

We will work out additional details of this separation of variables method a little later.

Digression on tools for solving ordinary differential equations – Method of Frobenius

https://mathshistory.st-andrews.ac.uk/Biographies/Frobenius/

Ferdinand Georg Frobenius

Born: 26 October 1849
Berlin-Charlottenburg, Prussia (now Germany)

Died: 3 August 1917
Berlin, Germany

Summary: Georg Frobenius combined results from the theory of algebraic equations, geometry, and number theory, which led him to the study of abstract groups, the representation theory of groups and the character theory of groups. He also developed methods for solving linear differential equations.

Simple example of ordinary differential equation:

Solutions of the differential equation: $\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right)f(r)=0$

Frobenius method for finding solutions near $r=0$:

Guess series solution form: $f(r)=\sum_{n=0} A_n r^{s+n}$

Evaluate: $\hat{O}f(r)=\sum_{n=0} A_n \hat{O} r^{s+n}=0$ for each power of r^{s+m} to find

relationships between coefficients A_m

and the condition for non-trivial A_0.

Example (thanks to F. B. Hildebrand):

$$\hat{O}=2r\frac{d^2}{dr^2}+(1-2r)\frac{d}{dr}-1$$

$$\sum_{n=0} A_n \hat{O} r^{s+n}=0=\sum_{n=0} A_n\left((s+n)(2s+2n-1)r^{s+n-1}-(2s+2n+1)r^{s+n}\right)$$

Condition for non-trivial A_0: $s(2s-1)=0$

Continued --

Example (thanks to F. B. Hildebrand):

$$\hat{O} = 2r\frac{d^2}{dr^2} + (1-2r)\frac{d}{dr} - 1$$

$$\sum_{n=0} A_n \hat{O} r^{s+n} = 0 = \sum_{n=0} A_n \left((s+n)(2s+2n-1) r^{s+n-1} - (2s+2n+1) r^{s+n} \right)$$

Condition for non-trivial A_0 : $s(2s-1) = 0$

First solution: $s = 0$

Coefficient of r^m : $A_{m+1}(2m+1)(m+1) - A_m(2m+1) = 0$

$$f_1(r) = A_0\left(1 + r + \frac{r^2}{2} + \frac{r^3}{3!} + \ldots\right) = A_0 e^r$$

Second solution: $s = \frac{1}{2}$

Coefficient of r^m : $A_{m+1}(2m+3)(m+1) - A_m 2(m+1) = 0$

$$f_2(r) = A_0 r^{1/2}\left(1 + \frac{2}{3}r + \frac{2^2}{3\cdot 5}r^2 + \frac{2^3}{3\cdot 5\cdot 7}r^3 \ldots\right)$$

(infinite series, converges slowly)

Simple example of ordinary differential equation:

Solutions of the differential equation: $\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right)f(r)=0$

We can use the Frobenius method for this example; in this case the series truncates.

Special properties of particular partial differential equations

General solutions $\mu(x,t)$ to the wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$ satisfies the wave equation.

Because $$\frac{\partial^2 \mu}{\partial t^2} = c^2 \left(\left.\frac{d^2 f(w)}{dw^2}\right|_{w=x-ct} + \left.\frac{d^2 g(w)}{dw^2}\right|_{w=x+ct} \right)$$

$$\frac{\partial^2 \mu}{\partial x^2} = \left(\left.\frac{d^2 f(w)}{dw^2}\right|_{w=x-ct} + \left.\frac{d^2 g(w)}{dw^2}\right|_{w=x+ct} \right)$$

Initial value solutions $\mu(x,t)$ to the wave equation;

attributed to D'Alembert:

These functions would be given

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = \phi(x) \text{ and } \frac{\partial \mu}{\partial t}(x,0) = \psi(x)$$

Assume:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

Solution -- continued: $\mu(x,t) = f(x-ct)+g(x+ct)$

then: $\mu(x,0) = \varphi(x) = f(x)+g(x)$

$$\frac{\partial\mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx}-\frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x)-g(x) = -\frac{1}{c}\int^{x}\psi(x')dx'$$

For each x, find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left(\varphi(x)-\frac{1}{c}\int^{x}\psi(x')dx'\right)$$

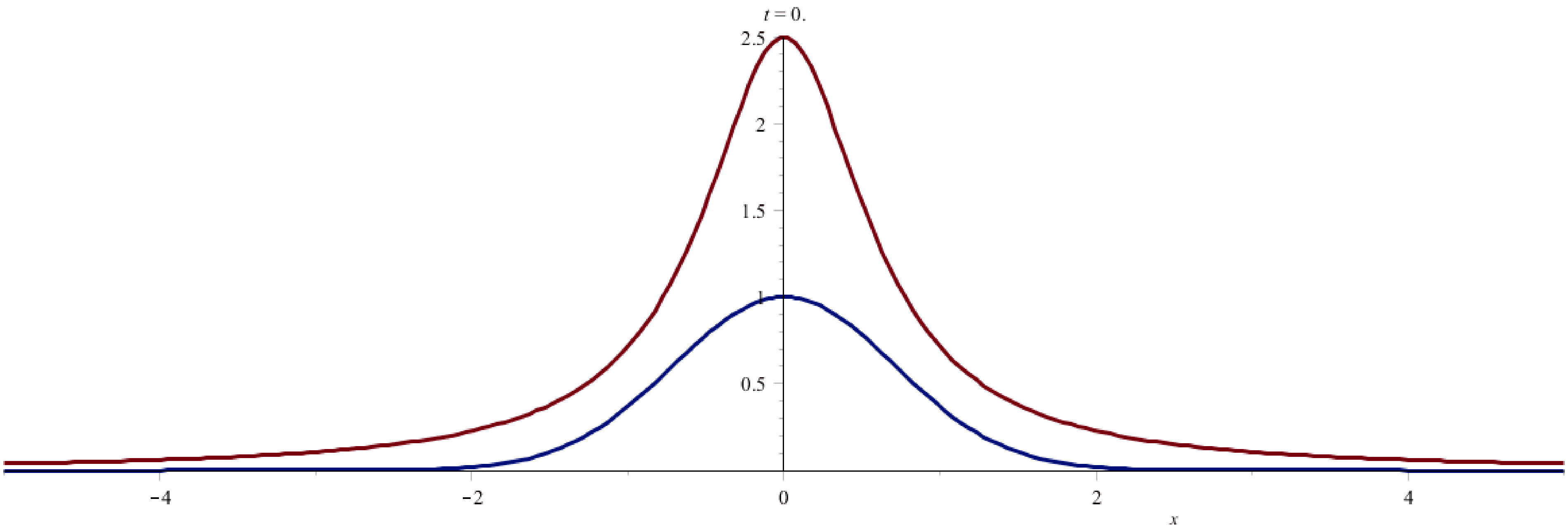
$$g(x) = \frac{1}{2}\left(\varphi(x)+\frac{1}{c}\int^{x}\psi(x')dx'\right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\left(\varphi(x-ct)+\varphi(x+ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(x')dx'$$

Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = e^{-x^2/\sigma^2} \text{ and } \frac{\partial \mu}{\partial t}(x,0) = 0$$

$$\Rightarrow \mu(\mathrm{x},\mathrm{t}) = \frac{1}{2}\left(e^{-(x+ct)^2/\sigma^2} + e^{-(x-ct)^2/\sigma^2}\right)$$



Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = 0 \text{ and } \frac{\partial \mu}{\partial t}(x,0) = -\frac{2x}{\sigma^2} e^{-x^2/\sigma^2}$$

$$\Rightarrow \mu(x,t) = \frac{1}{2c}\left(e^{-(x+ct)^2/\sigma^2} - e^{-(x-ct)^2/\sigma^2}\right)$$

Note that $\dfrac{\partial \mu(x,t)}{\partial t} = -\dfrac{1}{\sigma^2}\left((x+ct)e^{-(x+ct)^2/\sigma^2} + (x-ct)e^{-(x-ct)^2/\sigma^2}\right)$