# PHY 711 Classical Mechanics and Mathematical Methods
# 10-10:50 AM MWF in Olin 103

## Lecture 21 – Chap. 7 (F&W)

## Ordinary and partial differential equations

1. The wave equation – traveling wave solutions
2. The wave equation – standing wave solutions
3. The Sturm-Liouville equation

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|---|---|---|---|---|
| **11** | Wed, 9/18/2024 | Chap. 5 | Dynamics of rigid bodies | #10 |
| **12** | Fri, 9/20/2024 | Chap. 5 | Dynamics of rigid bodies | #11 |
| **13** | Mon, 9/23/2024 | Chap. 1 | Scattering analysis | #12 |
| **14** | Wed, 9/25/2024 | Chap. 1 | Scattering analysis | #13 |
| **15** | Fri, 9/27/2024 | Chap. 1 | Scattering analysis | #14 |
| **16** | Mon, 9/30/2024 | Chap. 4 | Small oscillations near equilibrium | |
| **17** | Wed, 10/2/2024 | Chap. 1-6 | Review | THE-10/3-9/24 |
| **18** | Fri, 10/4/2024 | Chap. 4 | Normal mode analysis | THE-10/3-9/24 |
| **19** | Mon, 10/7/2024 | Chap. 4 | Normal mode analysis in multiple dimensions | THE-10/3-9/24 |
| **20** | Wed, 10/9/2024 | Chap. 4&7 | Normal modes of continuous strings | THE-10/3-9/24 |
| **21** | Fri, 10/11/2024 | Chap. 7 | The wave and other partial differential equations | |
| **22** | Mon, 10/14/2024 | Chap. 7 | Sturm-Liouville equations | |
| **23** | Wed, 10/16/2024 | Chap. 7 | Sturm-Liouville equations | |
| | Fri, 10/18/2024 | Fall Break | | |
| **24** | Mon, 10/21/2024 | Chap. 7 | Laplace transforms and complex functions | |

One-dimensional wave equation
representing longitudinal or transverse displacements as a function of *x* and *t* , an example of a partial differential equation --

Traveling wave solutions thanks to D'Alembert --

For the displacement function, $\mu(x,t)$, the wave equation has the form:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$ :

$$\mu(x,t) = f(x - ct) + g(x + ct)$$

satisfies the wave equation.

Initial value traveling wave solutions $\mu(x,t)$ to the wave equation; attributed to D'Alembert:

These functions would be given

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = \varphi(x) \text{ and } \frac{\partial \mu}{\partial t}(x,0) = \psi(x)$$

Assume:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

then: $\mu(x,0) = \varphi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

Solution -- continued : $\mu(x,t) = f(x-ct) + g(x+ct)$

then : $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

For each $x$, find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$g(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\big(\phi(x-ct) + \phi(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

Checking that D'Alembert's solution solves the wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

$$\mu(x,t) = \frac{1}{2}\Big(\varphi(x-ct) + \varphi(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

$$\frac{\partial \mu(x,t)}{\partial x} = \frac{1}{2}\Big(\varphi'(x-ct) + \varphi'(x+ct)\Big) + \frac{1}{2c}\Big(\psi(x-ct) + \psi(x+ct)\Big)$$

$$\frac{\partial^2 \mu(x,t)}{\partial x^2} = \frac{1}{2}\Big(\varphi''(x-ct) + \varphi''(x+ct)\Big) + \frac{1}{2c}\Big(\psi'(x-ct) + \psi'(x+ct)\Big)$$

$$\frac{\partial \mu(x,t)}{\partial t} = \frac{c}{2}\Big(-\varphi'(x-ct) + \varphi'(x+ct)\Big) + \frac{c}{2c}\Big(-\psi(x-ct) + \psi(x+ct)\Big)$$

$$\frac{\partial^2 \mu(x,t)}{\partial t^2} = \frac{c^2}{2}\Big(\varphi''(x-ct) + \varphi''(x+ct)\Big) + \frac{c^2}{2c}\Big(\psi'(x-ct) + \psi'(x+ct)\Big)$$

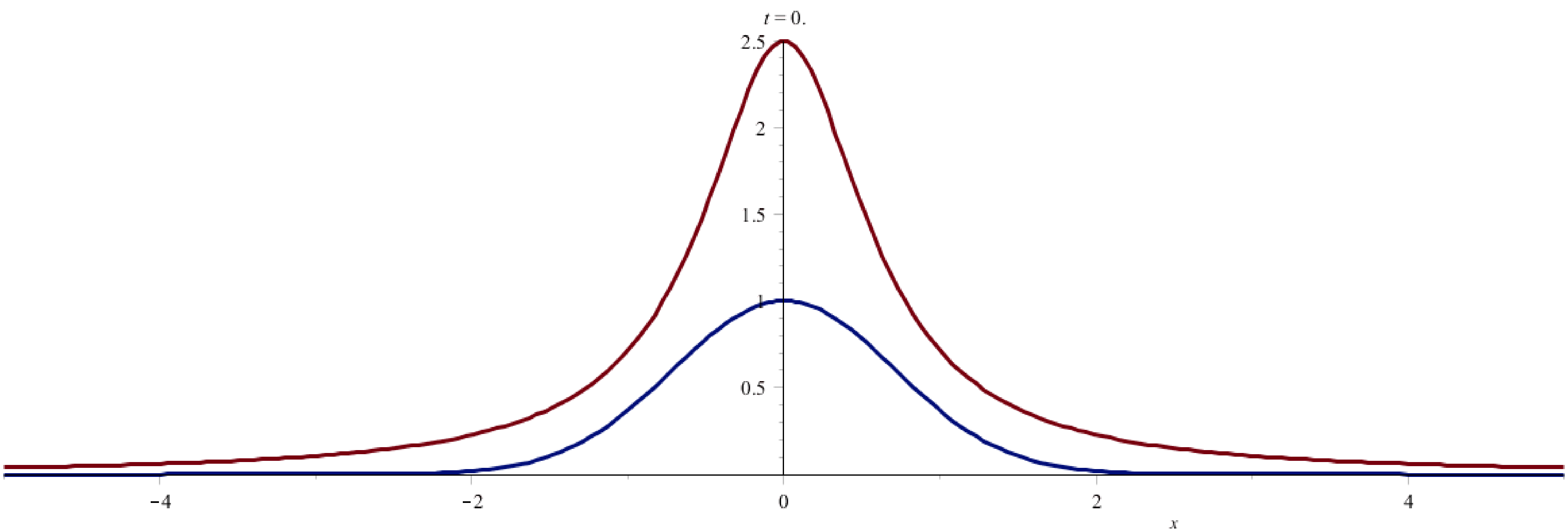
Here we have assumed that $\varphi(u)$ and $\psi(u)$ are continuous functions and

$$\varphi'(u) \equiv \frac{d\varphi(u)}{du}, \qquad \varphi''(u) \equiv \frac{d^2\varphi(u)}{du^2}, \qquad \text{etc.}$$

Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = e^{-x^2/\sigma^2} \text{ and } \frac{\partial \mu}{\partial t}(x,0) = 0$$

$$\Rightarrow \mu(\mathrm{x},\mathrm{t}) = \frac{1}{2}\left(e^{-(x+ct)^2/\sigma^2} + e^{-(x-ct)^2/\sigma^2}\right)$$

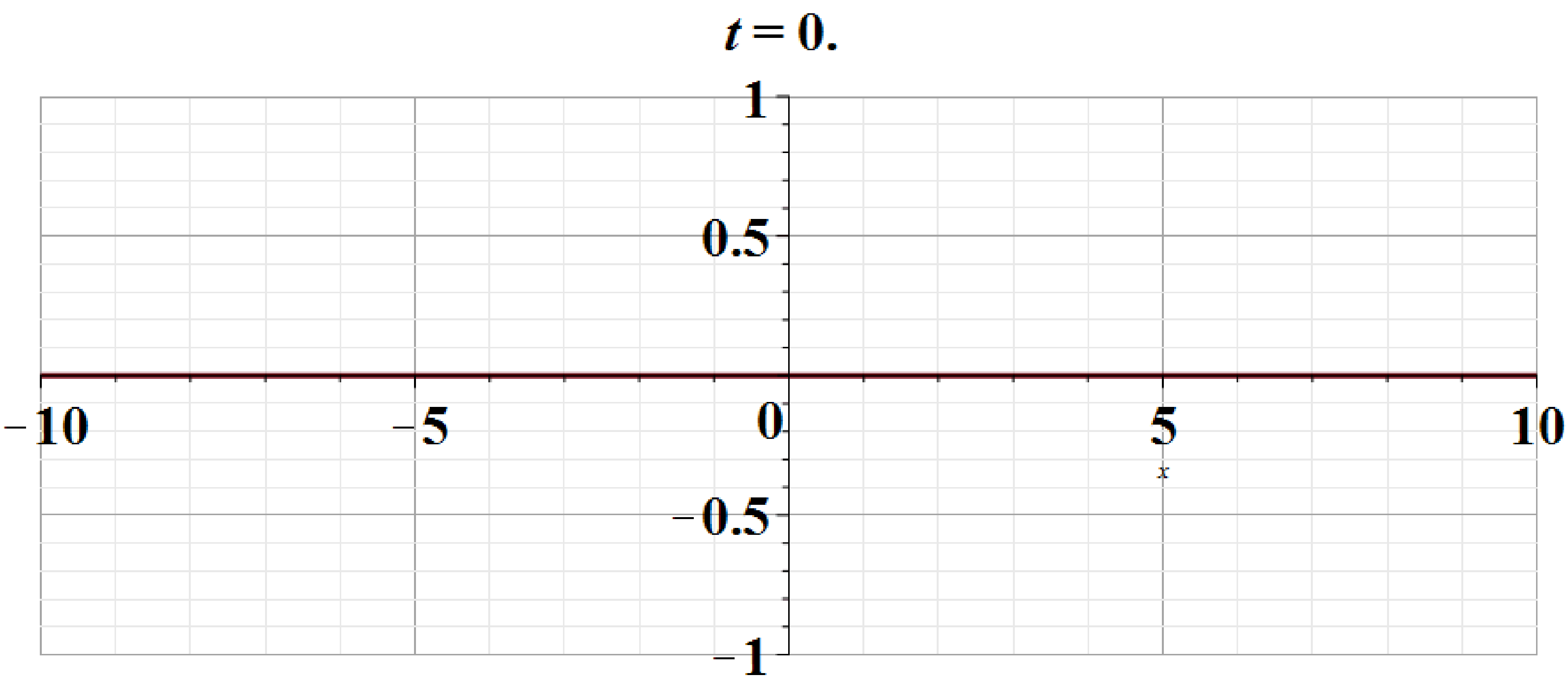


Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = 0 \text{ and } \frac{\partial \mu}{\partial t}(x,0) = -\frac{2x}{\sigma^2} e^{-x^2/\sigma^2}$$

$$\Rightarrow \mu(x,t) = \frac{1}{2c}\left(e^{-(x+ct)^2/\sigma^2} - e^{-(x-ct)^2/\sigma^2}\right)$$

Note that $\dfrac{\partial \mu(x,t)}{\partial t} = -\dfrac{1}{\sigma^2}\left((x+ct)e^{-(x+ct)^2/\sigma^2} + (x-ct)e^{-(x-ct)^2/\sigma^2}\right)$



Other types of solutions to the wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that because of the way that the equation is written, it is possible to find "separable" solutions of the form

$$\mu(x,t) = X(x)T(t)$$

or more generally, a linear combination of separable solutions:

$$\mu(x,t) = \sum_n X_n(x)T_n(t)$$

Separable solutions to the wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{for} \qquad \mu(x,t) = X(x)T(t)$$

$$\Rightarrow \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = \frac{1}{c^2 T(t)} \frac{d^2 T(t)}{dt^2}$$

For example, suppose the time function is harmonic in time with frequency $\omega$: $\quad T(t) = \cos(\omega t + \eta)$

Then the spacial function must statisfy the ordinary differential equation:

$$\frac{d^2 X(x)}{dx^2} = -\frac{\omega^2}{c^2} X(x)$$

$$\Rightarrow X(x) = A\sin(kx + \nu) \quad \text{where} \quad k = \frac{\omega}{c}$$

It is often the case, there are boundary values specified for *X(x)*.

$$X(x) = A\sin(kx+\nu) \quad \text{where} \qquad k = \frac{\omega}{c}$$

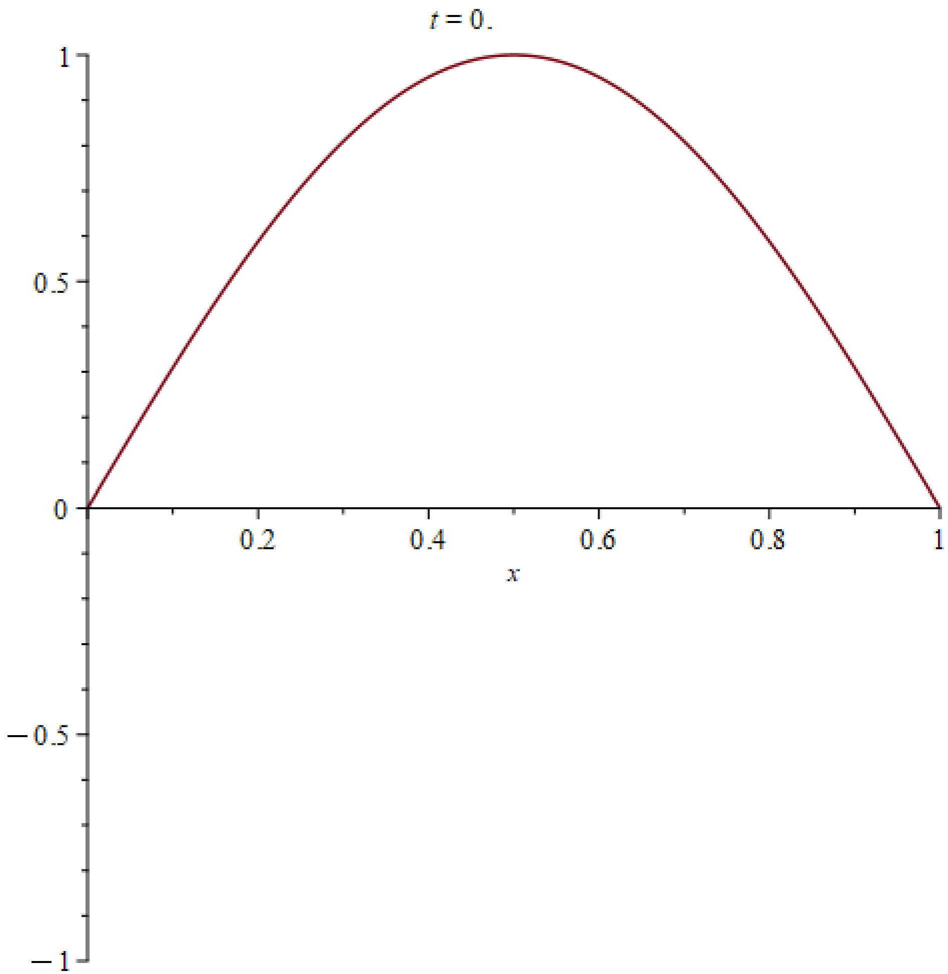
For example, suppose $X(0) = 0$ and $X(a) = 0$ $\quad --$ assume $\nu$=0

$$A\sin(kx)\Big|_{x=0} = 0 \qquad A\sin(kx)\Big|_{x=a} = 0$$

$$\Rightarrow k = \frac{n\pi x}{a} \quad \text{for } n = 0,1,2,.....$$

$$\Rightarrow X(x) = A\sin\left(\frac{n\pi x}{a}\right) \quad \text{and } \omega = kc = \frac{n\pi c}{a}$$

Standing wave -- $\mu(x,t) = A\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi ct}{a}\right)$



How are the traveling wave and standing wave solutions to the wave equations related?

A. They are exactly the same
B. They are not related
C. ???

More general solution with boundaries at *x=0,a* :

$$\mu(x,t) = \sum_n X_n(x) T_n(t) = \sum_n A_n \sin\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi c t}{a} + \alpha_n\right)$$

# The wave equation and related linear PDE's

One dimensional wave equation for $\mu(x,t)$:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Generalization for spacially dependent tension and mass density plus an extra potential energy density:

$$\sigma(x)\frac{\partial^2 \mu(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left(\tau(x)\frac{\partial \mu(x,t)}{\partial x}\right) + v(x)\mu(x,t) = 0$$

Factoring time and spatial variables:

$$\mu(x,t) = \phi(x)\,\cos(\omega t + \alpha)$$

Sturm-Liouville equation for spatial function $\phi(x)$:

$$-\frac{d}{dx}\left(\tau(x)\frac{d\phi(x)}{dx}\right) + v(x)\phi(x) = \omega^2 \sigma(x)\phi(x)$$

Linear second-order ordinary differential equations
Sturm-Liouville equations

Inhomogenous problem: $$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

given functions

applied force

solution to be determined

When applicable, it is assumed that the form of the applied force is known.

Homogenous problem: *F(x)=0*

# Examples of Sturm-Liouville eigenvalue equations --

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=0$$

Bessel functions: $0 \le x \le \infty$

$$\tau(x)=-x \quad v(x)=x \quad \sigma(x)=\frac{1}{x} \quad \lambda=\nu^2 \quad \varphi(x)=J_\nu(x)$$

Legendre functions: $-1 \le x \le 1$

$$\tau(x)=-\left(1-x^2\right) \quad v(x)=0 \quad \sigma(x)=1 \quad \lambda=l(l+1) \quad \varphi(x)=P_l(x)$$

Fourier functions: $0 \le x \le 1$

$$\tau(x)=1 \quad v(x)=0 \quad \sigma(x)=1 \quad \lambda=n^2\pi^2 \quad \varphi(x)=\sin(n\pi x)$$

## Solution methods of Sturm-Liouville equations (assume all functions and constants are real):

$$\text{Homogenous problem:}\quad \left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi_0(x)=0$$

$$\text{Inhomogenous problem:}\quad \left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi(x)=F(x)$$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

$$\text{Orthogonality of eigenfunctions:}\quad \int_a^b \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n,$$

$$\text{where}\quad N_n\equiv\int_a^b\sigma(x)(f_n(x))^2dx.$$

Completeness of eigenfunctions:

$$\sigma(x)\sum_n\frac{f_n(x)f_n(x')}{N_n}=\delta(x-x')$$

Why all of the fuss about eigenvalues and eigenvectors?

a. They are sometimes useful in finding solutions to differential equations
b. Not all eigenfunctions have analytic forms.
c. It is possible to solve a differential equation without the use of eigenfunctions.
d. Eigenfunctions have some useful properties.

# Comment on orthogonality of eigenfunctions

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_m(x)=\lambda_m\sigma(x)f_m(x)$$

$$f_m(x)\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)-f_n(x)\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_m(x)$$

$$=\left(\lambda_n-\lambda_m\right)\sigma(x)f_n(x)f_m(x)$$

$$-\frac{d}{dx}\left(f_m(x)\tau(x)\frac{df_n(x)}{dx}-f_n(x)\tau(x)\frac{df_m(x)}{dx}\right)=\left(\lambda_n-\lambda_m\right)\sigma(x)f_n(x)f_m(x)$$

# Comment on orthogonality of eigenfunctions -- continued

$$-\frac{d}{dx}\left( f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right) = \left(\lambda_n - \lambda_m\right)\sigma(x) f_n(x) f_m(x)$$

Now consider integrating both sides of the equation in the interval

$$a \leq x \leq b:$$

$$-\left( f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right)\Bigg|_a^b = \left(\lambda_n - \lambda_m\right)\int_a^b dx\sigma(x) f_n(x) f_m(x)$$

Vanishes for various boundary conditions at *x=a and x=b*

# Comment on orthogonality of eigenfunctions -- continued

$$-\left( f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right)\Bigg|_a^b = \left(\lambda_n - \lambda_m\right)\int_a^b dx\sigma(x) f_n(x) f_m(x)$$

Possible boundary values for Sturm-Liouville equations:

1. $f_m(a) = f_m(b) = 0$

2. $\tau(x)\frac{df_m(x)}{dx}\Big|_a = \tau(x)\frac{df_m(x)}{dx}\Big|_b = 0$

3. $f_m(a) = f_m(b)$ and $\frac{df_m(a)}{dx} = \frac{df_m(b)}{dx}$

In any of these cases, we can conclude that:

$$\int_a^b dx\sigma(x) f_n(x) f_m(x) = 0 \text{ for } \lambda_n \neq \lambda_m$$

## Comment on "completeness"

It can be shown that for any reasonable function *h(x)*, defined within the interval $a < x < b$, we can expand that function as a linear combination of the eigenfunctions $f_n(x)$

$$h(x) \approx \sum_n C_n f_n(x),$$

$$\text{where} \quad C_n = \frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx'.$$

These ideas lead to the notion that the set of eigenfunctions $f_n(x)$ form a ``complete'' set in the sense of ``spanning'' the space of all functions in the interval $a < x < b$, as summarized by the statement:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta(x - x').$$

# Comment on "completeness" -- continued

Suppose that: $h(x) \approx \sum_n C_n f_n(x),$

where $C_n = \frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx'.$

Consider the squared error of the expansion:

$$\epsilon^2 = \int_a^b dx\sigma(x)\left(h(x) - \sum_n C_n f_n(x)\right)^2$$

$\epsilon^2$ can be minimized:

$$\frac{\partial\epsilon^2}{\partial C_m} = 0 = -2\int_a^b dx\sigma(x)\left(h(x) - \sum_n C_n f_n(x)\right) f_m(x)$$

$$\Rightarrow C_m = \frac{1}{N_m}\int_a^b dx\sigma(x)h(x)f_m(x)$$

Comment on "completeness"

Note that we have "proven" that

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta(x-x').$$

which follows from the identities

$$h(x) = \sum_n C_n f_n(x), \text{where} \quad C_n = \frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx'.$$

$$\Rightarrow h(x) = \sum_n \left( \frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx' \right) f_n(x)$$

From the definition of the Dirac delta function:

For $a \le x \le b$: $\quad h(x) = \int_a^b \delta(x-x')h(x')dx'$

$$\Rightarrow \delta(x-x') = \sigma(x')\sum_n \frac{f_n(x)f_n(x')}{N_n}$$

# Eigenvalues and eigenfunctions of Sturm-Liouville equations

In the domain $a \le x \le b$:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Alternative boundary conditions; 1. $f_m(a)=f_m(b)=0$

or 2. $\tau(x)\left.\frac{df_m(x)}{dx}\right|_a=\tau(x)\left.\frac{df_m(x)}{dx}\right|_b=0$

or 3. $f_m(a)=f_m(b)$ and $\frac{df_m(a)}{dx}=\frac{df_m(b)}{dx}$

Properties:

Eigenvalues $\lambda_n$ are real

Eigenfunctions are orthogonal: $\int_a^b \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n$,

where $N_n \equiv \int_a^b \sigma(x)(f_n(x))^2dx.$

# Variation approximation to lowest eigenvalue

In general, there are several techniques to determine the eigenvalues $\lambda_n$ and eigenfunctions $f_n(x)$. When it is not possible to find the ``exact'' functions, there are several powerful approximation techniques. For example, the lowest eigenvalue can be approximated by minimizing the function

$$\lambda_0 \le \frac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle}, \qquad S(x) \equiv -\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x)$$

where $\tilde{h}(x)$ is a variable function which satisfies the correct boundary values. The ``proof'' of this inequality is based on the notion that $\tilde{h}(x)$ can in principle be expanded in terms of the (unknown) exact eigenfunctions $f_n(x)$:

$\tilde{h}(x) = \sum_n C_n f_n(x)$, where the coefficients $C_n$ can be assumed to be real.

Estimation of the lowest eigenvalue – continued:

From the eigenfunction equation, we know that

$$S(x)\tilde{h}(x) = S(x)\sum_n C_n f_n(x) = \sum_n C_n \lambda_n \sigma(x) f_n(x).$$

It follows that:

$$\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle = \int_a^b \tilde{h}(x) S(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n \lambda_n.$$

It also follows that:

$$\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle = \int_a^b \tilde{h}(x) \sigma(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n,$$

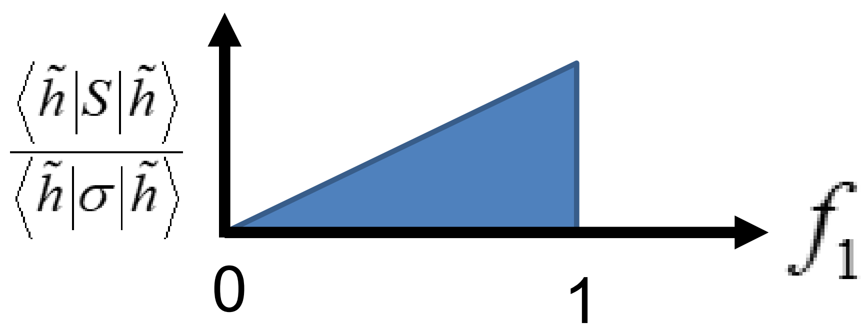
Therefore $$\frac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle} = \frac{\sum_n |C_n|^2 N_n \lambda_n}{\sum_n |C_n|^2 N_n} \geq \lambda_0.$$

Some additional comments -- $\dfrac{\langle \tilde{h}|S|\tilde{h}\rangle}{\langle \tilde{h}|\sigma|\tilde{h}\rangle} = \dfrac{\sum_n |C_n|^2 N_n \lambda_n}{\sum_n |C_n|^2 N_n} \geq \lambda_0.$

$$\frac{\langle \tilde{h}|S|\tilde{h}\rangle}{\langle \tilde{h}|\sigma|\tilde{h}\rangle} = \sum_{n=0}^{\infty} f_n \lambda_n \quad \text{where } f_n \equiv \frac{|C_n|^2 N_n}{\sum_m |C_m|^2 N_m} \text{ and } \sum_{n=0}^{\infty} f_n = 1$$

For the case of only two non-trivial eigenvalues:

$$\frac{\langle \tilde{h}|S|\tilde{h}\rangle}{\langle \tilde{h}|\sigma|\tilde{h}\rangle} = f_0\lambda_0 + f_1\lambda_1 = \lambda_0 + (\lambda_1 - \lambda_0) f_1$$



# Rayleigh-Ritz method of estimating the lowest eigenvalue

$$\lambda_0 \leq \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle},$$

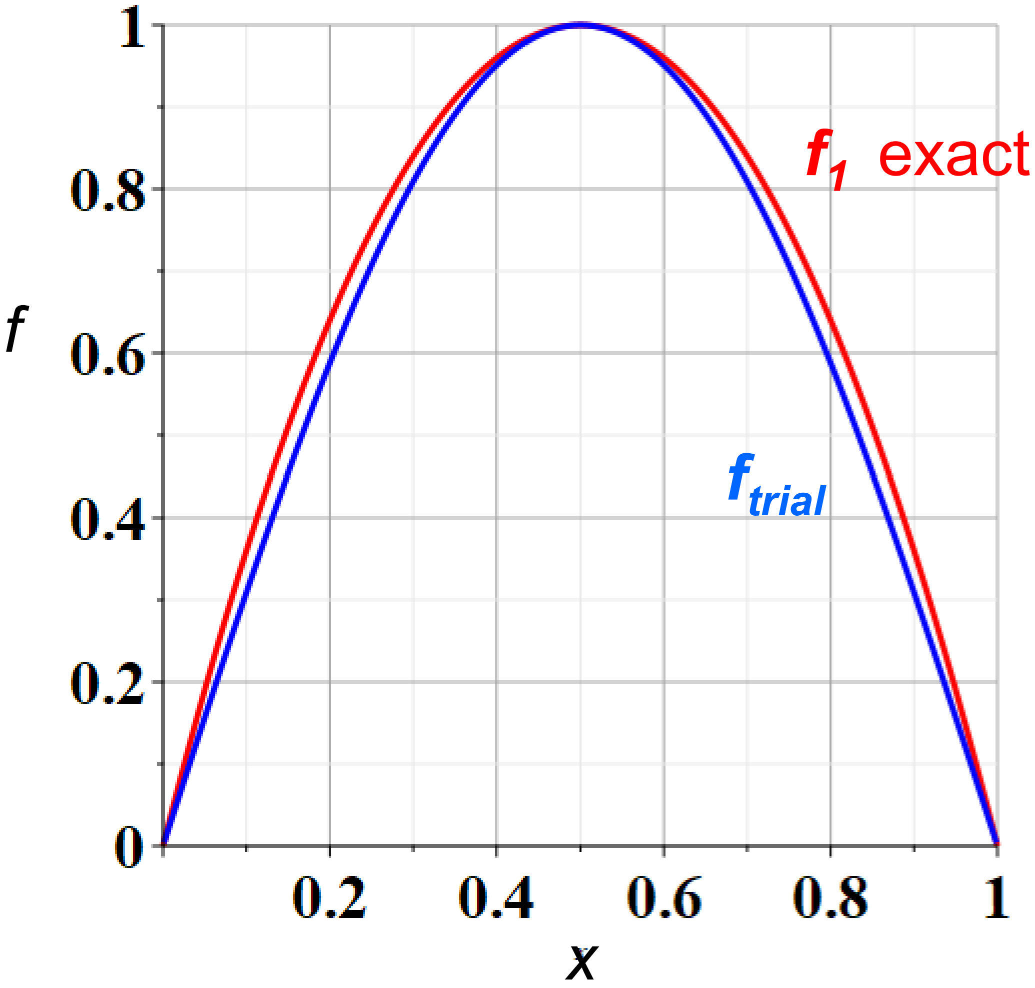
Example: $-\frac{d^2}{dx^2} f_n(x) = \lambda_n f_n(x) \quad \text{with } f_n(0) = f_n(a) = 0$

Exact eigenfunctions: $f_n(x) = \sin\left(\frac{n\pi x}{a}\right) \quad n = 1, 2, 3....$

Exact eigenvalues: $\lambda_n = \left(\frac{n\pi}{a}\right)^2 \quad n = 1, 2, 3.... \quad \frac{\pi^2}{a^2} = \frac{9.869604404}{a^2}$

Trial function $f_{\text{trial}}(x) = x(x-a)$

Raleigh-Ritz estimate: $\frac{\langle x(a-x) | -\frac{d^2}{dx^2} | x(a-x) \rangle}{\langle x(a-x) | x(a-x) \rangle} = \frac{10}{a^2}$



f
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
x
f1 exact
ftrial

# Rayleigh-Ritz method of estimating the lowest eigenvalue

$$\lambda_0 \le \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle},$$

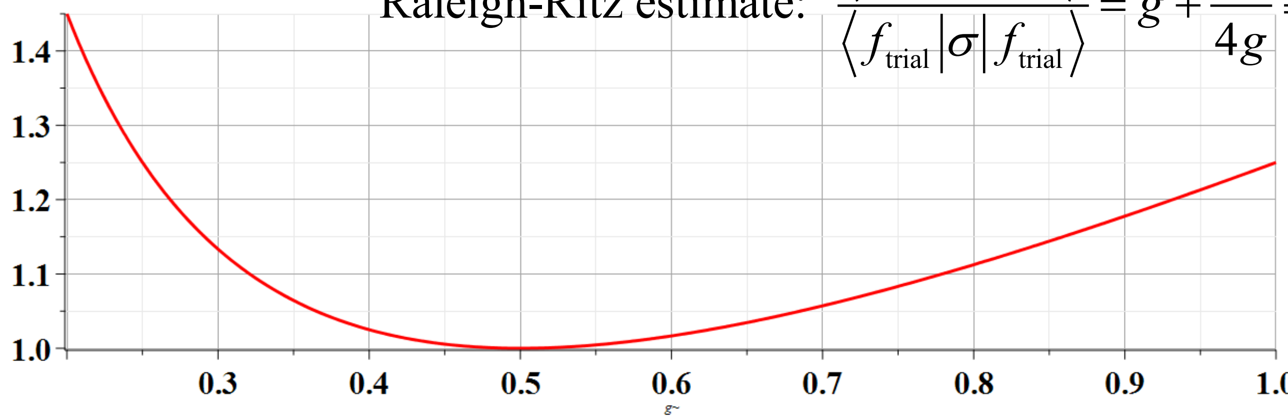
Another example – this time with a variable parameter

Example: $$-\frac{d^2 f_n(x)}{dx^2} + Gx^2 f_n(x) = \lambda_n f_n(x) \qquad \text{with } f_n(-\infty) = f_n(\infty) = 0$$

trial function $\quad f_{\text{trial}}(x) = e^{-gx^2}$

Raleigh-Ritz estimate: $$\frac{\langle f_{\text{trial}} | S | f_{\text{trial}} \rangle}{\langle f_{\text{trial}} | \sigma | f_{\text{trial}} \rangle} = g + \frac{G}{4g} \equiv \lambda_{\text{trial}}(g)$$

$$\frac{\lambda_{\text{trial}}(g)}{\sqrt{G}}$$




$$g / \sqrt{G}$$

$$g_0 = \frac{1}{2}\sqrt{G} \qquad \lambda_{\text{trial}}(g_0) = \sqrt{G}$$

Note that for differential equation of the Schoedinger equation of the harmonic oscillator:

$$\sqrt{G} = \frac{m\omega}{\hbar} \qquad \lambda_{\text{trial}} = \frac{2m}{\hbar^2} E_0 \qquad \Rightarrow E_0 = \frac{\hbar\omega}{2}$$

Recap -- Rayleigh-Ritz method of estimating the lowest eigenvalue

Example from Schroedinger equation for one-dimensional harmonic oscillator:

$$-\frac{\hbar^2}{2m}\frac{d^2 f_n(x)}{dx^2}+\frac{1}{2}m\omega^2 x^2 f_n(x)=E_n f_n(x) \qquad \text{with } f_n(-\infty)=f_n(\infty)=0$$

Trial function $\quad f_{\text{trial}}(x)=e^{-gx^2}$

Raleigh-Ritz estimate: $\dfrac{\langle f_{\text{trial}}|S|f_{\text{trial}}\rangle}{\langle f_{\text{trial}}|\sigma|f_{\text{trial}}\rangle}=\dfrac{\hbar^2}{2m}\left(g+\dfrac{m^2\omega^2/\hbar^2}{4g}\right)\equiv E_{\text{trial}}(g)$

$$g_0=\frac{m\omega}{\hbar} \qquad E_{\text{trial}}(g_0)=\frac{1}{2}\hbar\omega$$

Exact answer

Do you think that there is a reason for getting the correct answer from this method?

a. Chance only
b. Skill