PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 22 – Chap. 7 (F&W)

Sturm-Liouville equations

1. Eigenvalues and eigenfunctions
2. Rayleigh-Ritz approximation method
3. Green's function solution methods based on eigenfunction expansions
4. Green's function solution methods based on solutions of the homogeneous equations

$\left.\begin{matrix} 1. \\ 2. \end{matrix}\right\}$ review

13	Mon, 9/23/2024	Chap. 1	Scattering analysis	#12
14	Wed, 9/25/2024	Chap. 1	Scattering analysis	#13
15	Fri, 9/27/2024	Chap. 1	Scattering analysis	#14
16	Mon, 9/30/2024	Chap. 4	Small oscillations near equilibrium	
17	Wed, 10/2/2024	Chap. 1-6	Review	THE-10/3-9/24
18	Fri, 10/4/2024	Chap. 4	Normal mode analysis	THE-10/3-9/24
19	Mon, 10/7/2024	Chap. 4	Normal mode analysis in multiple dimensions	THE-10/3-9/24
20	Wed, 10/9/2024	Chap. 4&7	Normal modes of continuous strings	THE-10/3-9/24
21	Fri, 10/11/2024	Chap. 7	The wave and other partial differential equations	
22	Mon, 10/14/2024	Chap. 7	Sturm-Liouville equations	#15
23	Wed, 10/16/2024	Chap. 7	Sturm-Liouville equations	#16
	Fri, 10/18/2024	Fall Break		
24	Mon, 10/21/2024	Chap. 7	Laplace transforms and complex functions	

PHY 711 – Assignment #16

Assigned: 10/14/2024   Due: 10/23/2024

Continue reading Chapter 7 in **Fetter and Walecka**.

1. Consider the differential eigenvalue problem

$$-\frac{d^2}{dx^2}f_n(x) = \lambda_n f_n(x),$$

with boundary values $f_n(x=0) = 0 = f_n(x=a)$.

(a) Find the first few eigenvalues λ_n and eigenfunctions $f_n(x)$.

(b) Consider a trial function

$$f_{trial}(x) \equiv x(a^2 - x^2)$$

to estimate the lowest eigenvalue of this system using the Rayleigh-Ritz method. How well does it do?

PHY 711 -- Assignment #16

Assigned: 10/14/2024 Due: 10/23/2024

Continue reading Chapter 7 in **Fetter & Walecka**.

1. Consider the function $f(x) = x\ (1-x^2)$ in the interval $0 \le x \le 1$. Find the coefficients A_n of the Fourier series based on the terms sin($n\ \pi\ x$). Extra credit: Plot $f(x)$ and the Fourier series including 3 terms for example.

Review – Sturm-Liouville equations defined over a range of x.

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi_0(x)=0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Note that, because Sturm-Liouville operator is Hermitian, the eigenvalues are real and the eigenfunctions are orthogonal. In the last lecture, we argued that the eigenfunctions form a "complete" set over the range of x defined for the particular system.

Eigenvalues and eigenfunctions of Sturm-Liouville equations

In the domain $a \leq x \leq b$:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Alternative boundary conditions; 1. $f_m(a)=f_m(b)=0$

or 2. $\left.\tau(x)\frac{df_m(x)}{dx}\right|_a=\left.\tau(x)\frac{df_m(x)}{dx}\right|_b=0$

or 3. $f_m(a)=f_m(b)$ and $\frac{df_m(a)}{dx}=\frac{df_m(b)}{dx}$

Properties:

Eigenvalues λ_n are real

Eigenfunctions are orthogonal: $\int_a^b \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n,$

where $N_n \equiv \int_a^b \sigma(x)(f_n(x))^2dx.$

Formal statement of completeness of eigenfunctions:

Completeness of eigenfunctions:

$$\sigma\left(x\right)\sum_{n}\frac{f_n(x)f_n(x')}{N_n}=\delta\left(x-x'\right)$$

Variation approximation to lowest eigenvalue

In general, there are several techniques to determine the eigenvalues λ_n and eigenfunctions $f_n(x)$. When it is not possible to find the ``exact'' functions, there are several powerful approximation techniques. For example, the lowest eigenvalue can be approximated by minimizing the function

$$\lambda_0 \leq \frac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle},$$

$$S(x) \equiv -\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x)$$

where $\tilde{h}(x)$ is a variable function which satisfies the correct boundary values. The ``proof'' of this inequality is based on the notion that $\tilde{h}(x)$ can in principle be expanded in terms of the (unknown) exact eigenfunctions $f_n(x)$:

$\tilde{h}(x) = \sum_n C_n f_n(x),$ where the coefficients C_n can be assumed to be real.

Estimation of the lowest eigenvalue – continued:

From the eigenfunction equation, we know that

$$S(x)\tilde{h}(x) = S(x)\sum_n C_n f_n(x) = \sum_n C_n \lambda_n \sigma(x) f_n(x).$$

It follows that:

$$\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle = \int_a^b \tilde{h}(x) S(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n \lambda_n.$$

It also follows that:

$$\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle = \int_a^b \tilde{h}(x) \sigma(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n,$$

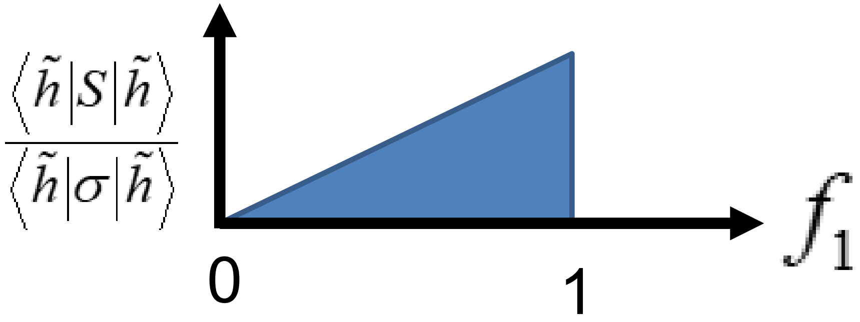
Therefore $$\frac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle} = \frac{\sum_n |C_n|^2 N_n \lambda_n}{\sum_n |C_n|^2 N_n} \geq \lambda_0.$$

Some additional comments -- $\frac{\langle\tilde{h}|S|\tilde{h}\rangle}{\langle\tilde{h}|\sigma|\tilde{h}\rangle} = \frac{\sum_n |C_n|^2 N_n \lambda_n}{\sum_n |C_n|^2 N_n} \geq \lambda_0.$

$$\frac{\langle\tilde{h}|S|\tilde{h}\rangle}{\langle\tilde{h}|\sigma|\tilde{h}\rangle} = \sum_{n=0}^{\infty} f_n \lambda_n \quad \text{where } f_n \equiv \frac{|C_n|^2 N_n}{\sum_m |C_m|^2 N_m} \text{ and } \sum_{n=0}^{\infty} f_n = 1$$

For the case of only two non-trivial eigenvalues:

$$\frac{\langle\tilde{h}|S|\tilde{h}\rangle}{\langle\tilde{h}|\sigma|\tilde{h}\rangle} = f_0\lambda_0 + f_1\lambda_1 = \lambda_0 + (\lambda_1 - \lambda_0) f_1$$



Rayleigh-Ritz method of estimating the lowest eigenvalue

$$\lambda_0 \leq \frac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle},$$

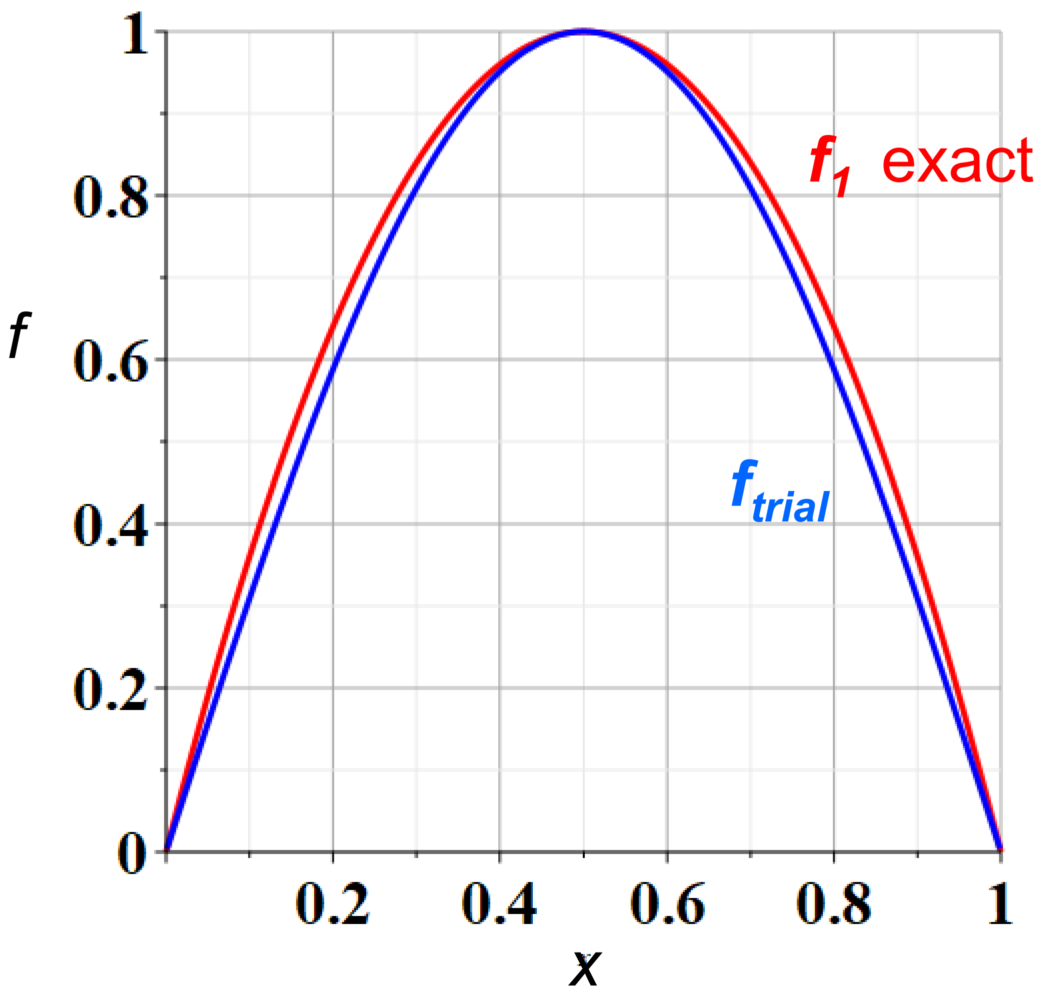
Example: $-\frac{d^2}{dx^2} f_n(x) = \lambda_n f_n(x) \quad$ with $f_n(0) = f_n(a) = 0$

Exact eigenfunctions: $f_n(x) = \sin\left(\frac{n\pi x}{a}\right) \quad n = 1,2,3....$

Exact eigenvalues: $\lambda_n = \left(\frac{n\pi}{a}\right)^2 \quad n = 1,2,3.... \quad \frac{\pi^2}{a^2} = \frac{9.869604404}{a^2}$

Trial function $f_{\text{trial}}(x) = x(x-a)$

Raleigh-Ritz estimate: $\frac{\left\langle x(a-x) \middle| -\frac{d^2}{dx^2} \middle| x(a-x) \right\rangle}{\left\langle x(a-x) \middle| x(a-x) \right\rangle} = \frac{10}{a^2}$


1
0.8
0.6
0.4
0.2
0
f
0.2
0.4
0.6
0.8
1
x
f_1 exact
f_{trial}

Rayleigh-Ritz method of estimating the lowest eigenvalue

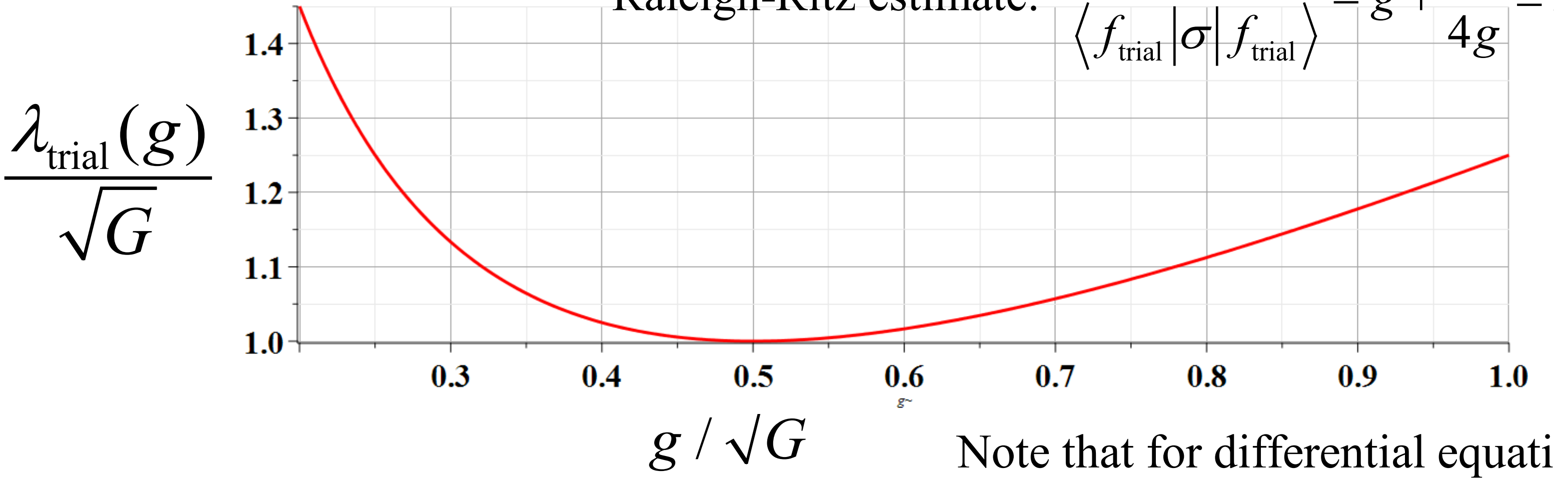
$$\lambda_0 \leq \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle},$$

Another example – this time with a variable parameter

Example: $$-\frac{d^2 f_n(x)}{dx^2} + Gx^2 f_n(x) = \lambda_n f_n(x) \qquad \text{with } f_n(-\infty) = f_n(\infty) = 0$$

trial function $f_{\text{trial}}(x) = e^{-gx^2}$

Raleigh-Ritz estimate: $$\frac{\langle f_{\text{trial}} | S | f_{\text{trial}} \rangle}{\langle f_{\text{trial}} | \sigma | f_{\text{trial}} \rangle} = g + \frac{G}{4g} \equiv \lambda_{\text{trial}}(g)$$



$$g_0 = \frac{1}{2}\sqrt{G} \qquad \lambda_{\text{trial}}(g_0) = \sqrt{G}$$

Note that for differential equation of the Schoedinger equation of the harmonic oscillator:

$$\sqrt{\text{G}} = \frac{m\omega}{\hbar} \qquad \lambda_{\text{trial}} = \frac{2m}{\hbar^2} E_0 \qquad \Rightarrow E_0 = \frac{\hbar\omega}{2}$$

Recap -- Rayleigh-Ritz method of estimating the lowest eigenvalue

Example from Schroedinger equation for one-dimensional harmonic oscillator:

$$-\frac{\hbar^2}{2m}\frac{d^2 f_n(x)}{dx^2}+\frac{1}{2}m\omega^2 x^2 f_n(x)=E_n f_n(x) \qquad \text{with}\, f_n(-\infty)=f_n(\infty)=0$$

Trial function $\quad f_{\text{trial}}(x)=e^{-gx^2}$

Raleigh-Ritz estimate: $\displaystyle \frac{\langle f_{\text{trial}}|S|f_{\text{trial}}\rangle}{\langle f_{\text{trial}}|\sigma|f_{\text{trial}}\rangle}=\frac{\hbar^2}{2m}\left(g+\frac{m^2\omega^2/\hbar^2}{4g}\right)\equiv E_{\text{trial}}(g)$

$$g_0=\frac{m\omega}{\hbar} \qquad E_{\text{trial}}(g_0)=\frac{1}{2}\hbar\omega$$

Exact answer

Do you think that there is a reason for getting the correct answer from this method?

a. Chance only
b. Skill

The Green of Green Functions

In 1828, an English miller from Nottingham published a mathematical essay that generated little response. George Green's analysis, however, has since found applications in areas ranging from classical electrostatics to modern quantum field theory.

Lawrie Challis

Check for updates

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The Green of Green Functions

In 1828, an English miller from Nottingham published a mathematical essay that generated little response. George Green's analysis, however, has since found applications in areas ranging from classical electrostatics to modern quantum field theory.

Lawrie Challis and Fred Sheard

Nottingham, an attractive and thriving town in the English Midlands, is famous for its association with Robin Hood, whose statue stands in the shadow of the castle wall. The Sheriff of Nottingham still has a special role in the city government although happily no longer strikes terror into the hearts of the good citizens.

Recently a new attraction, a windmill, has appeared on the Nottingham skyline (see figure 1). The sails turn on windy days and the adjoining mill shop sells packets of stone ground flour but also, more surprisingly, tracts on mathematical physics. The connection between the flour and the physics is part of the mill's unique character and is explained by a plaque once attached to the side of the mill tower that said,

HERE LIVED AND LABOURED
GEORGE GREEN
MATHEMATICIAN
B.1793–D.1841.

That is the Green of Green's theorem, which is familiar to physics undergraduate students worldwide, and of the Green functions that are used in many branches of both classical and quantum physics.

his family built a house next to the mill, Green spent most of his days and many of his nights working and indeed living in the mill. When he was 31, Jane Smith bore him a daughter. They had seven children in all but never married. It was said that Green's father felt that Jane was not a suitable wife for the son of a prosperous tradesman and landowner and threatened to disinherit him.

Little is known about Green's life from 1802 until 1823. In particular, it is not known whether he received any help in his mathematical development or if he was entirely self-taught. He may have received help from John Toplis, a fellow of Queens' College in the University of Cambridge and headmaster of the Nottingham Grammar School. Toplis's translation of Pierre-Simon Laplace's book *Mécanique Céleste*, published in Nottingham in 1814, seems a likely source of Green's interest in potential theory. The work was unusual in Britain at that time inasmuch as Toplis used Gottfried Leibniz's more convenient notation for differentials rather than Isaac Newton's. Because Green adapted the Leibniz notation, it seems plausible that Green was influenced by Toplis, but there is no evidence that Toplis acted in any way as his tutor.

In 1823, Green joined the Nottingham Subscription Library, the center of intellectual activity in the town. The library was situated in Bromley House (see figure 2). Library membership provided Green with encouragement, support, and access to the *Philosophical Transactions of the Royal Society* and other scientific journals. These did not include overseas journals, but the *Transactions* listed the contents of those journals, and that would have al-

Solution to inhomogeneous problem by using Green's functions

Inhomogenous problem:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_{\lambda}(x,x')=\delta\left(x-x'\right)$$

Formal solution:

$$\varphi_{\lambda}(x)=\varphi_{\lambda 0}(x)+\int_a^b G_{\lambda}(x,x')F(x')dx'$$

Solution to homogeneous problem

Formal solution:

$$\varphi_\lambda(x) = \varphi_{\lambda 0}(x) + \int_a^b G_\lambda(x,x')F(x')dx'$$

Solution to homogeneous problem

What is the homogeneous equation psi_0(x)?

Homogenous problem:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x) \right)\varphi_{\lambda 0}(x) = 0$$

In this lecture, we will discuss several methods of finding this Green's function. This topic will also appear in PHY 712

How do we arrive at the formal solution?

Formal solution:

$$\varphi_\lambda(x) = \varphi_{\lambda 0}(x) + \int_a^b G_\lambda(x,x')F(x')dx'$$

Note that this form satisfies the inhomogenous equation

Define $S(x) \equiv -\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x)$

$$S(x)\varphi_\lambda(x) = S(x)\varphi_{\lambda 0}(x) + S(x)\int_a^b G(x,x')F(x')dx'$$

$$S(x)\varphi_\lambda(x) = \qquad 0 \qquad + \int_a^b \delta(x-x')F(x')dx' = F(x)$$

Using complete set of eigenfunctions to form Green's function --

Suppose that we can find a Green's function defined as follows:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta\left(x-x'\right)$$

Recall:

Completeness of eigenfunctions:

$$\sigma\left(x\right)\sum_n \frac{f_n(x)f_n(x')}{N_n}=\delta\left(x-x'\right)$$

In terms of eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\sigma\left(x\right)\sum_n \frac{f_n(x)f_n(x')}{N_n}$$

$$\Rightarrow G_\lambda(x,x')=\sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}$$

By construction

Example Sturm-Liouville problem:

Example: $\tau(x)=1; \quad \sigma(x)=1; \quad v(x)=0; \quad a=0$ and $b=L$

$$\lambda=1; \qquad F(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

Inhomogenous equation:

$$\left(-\frac{d^2}{dx^2}-1\right)\varphi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

Eigenvalue equation :

$$\left(-\frac{d^2}{dx^2}\right)f_n(x)=\lambda_n f_n(x)$$

Eigenfunctions

$$f_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

Eigenvalues :

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2$$

Completeness of eigenfunctions:

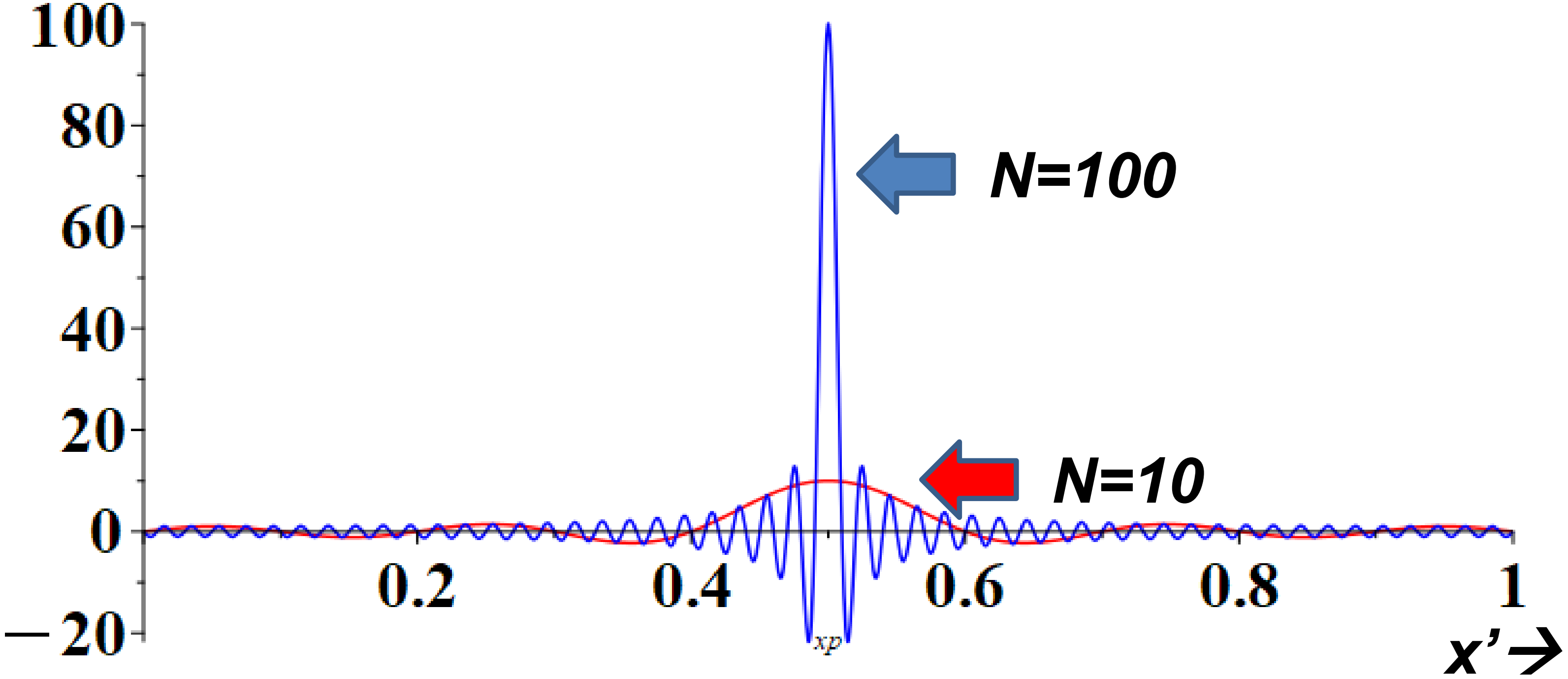
$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n}=\delta(x-x')$$

In this example:

$$\frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)=\delta(x-x')$$

In reality, for finite summation $$\frac{2}{L}\sum_{n=1}^{N}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)=\delta\left(x-x'\right)$$

x=1/2, L=1



N=100

N=10

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_{\lambda}(x,x')=\delta(x-x')$$

Green's function for the example :

$$G(x,x')=\sum_{n}\frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}=\frac{2}{L}\sum_{n}\frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}$$

Using Green's function to solve inhomogenous equation:

$$\left(-\frac{d^2}{dx^2}-1\right)\varphi(x)=F_0\sin\left(\frac{\pi x}{L}\right) \qquad \text{with boundary values } \varphi(0)=\varphi(L)=0$$

$$\varphi(x)=\varphi_0(x)+\int_0^L G(x,x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$\varphi(x)=\varphi_0(x)+\frac{2}{L}\sum_n\left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}\int_0^L\sin\left(\frac{n\pi x'}{L}\right)F_0\sin\left(\frac{\pi x'}{L}\right)dx'\right]$$

$$\varphi(x)=\varphi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

Another method of constructing Green's functions -- using two solutions to the homogeneous problem

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta\left(x-x'\right)$$

Two homogeneous solutions

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)g_i(x)=0 \qquad \text{for} \quad i=a,b$$

Let $G_\lambda(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$

where $W \equiv \tau(x')\left(g_a(x')\frac{d}{dx'}g_b(x')-g_b(x')\frac{d}{dx'}g_a(x')\right)$

Some details:

For $\epsilon \to 0$:

$$\int_{x'-\epsilon}^{x'+\epsilon} dx\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\int_{x'-\epsilon}^{x'+\epsilon} dx\delta\left(x-x'\right)$$

$$\int_{x'-\epsilon}^{x'+\epsilon} dx\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}\right)\frac{1}{W}g_a(x_<)g_b(x_>)=1$$

$$-\frac{\tau(x)}{W}\left(\frac{d}{dx}g_a(x_<)g_b(x_>)\right)\Bigg]_{x'-\epsilon}^{x'+\epsilon}=\frac{\tau(x')}{W}\left(g_a(x')\frac{d}{dx'}g_b(x')-g_b(x')\frac{d}{dx'}g_a(x')\right)$$

$$\Rightarrow W=\tau(x')\left(g_a(x')\frac{d}{dx'}g_b(x')-g_b(x')\frac{d}{dx'}g_a(x')\right)$$

Note -- W (Wronskian) is constant, since $\dfrac{dW}{dx'}=0.$

$\Rightarrow$ Useful Green's function construction in one dimension:

$$G_\lambda(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$$

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function solution:

$$\varphi_\lambda(x)=\varphi_{\lambda 0}(x)+\int_a^b G_\lambda(x,x')F(x')dx'$$

$$=\varphi_{\lambda 0}(x)+\frac{g_b(x)}{W}\int_a^x g_a(x')F(x')dx'+\frac{g_a(x)}{W}\int_x^b g_b(x')F(x')dx'$$

Note that the integral has to be performed in two parts. While the eigenfunction expansion method can be generalized to 2 and 3 dimensions, this method only works for one dimension.

Example from previous discussion:

$$\left(-\frac{d^2}{dx^2}-1\right)\varphi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$ with boundary values $\varphi(0)=\varphi(L)=0$

Using: $G(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$ for $0\le x\le L$

$$\left(-\frac{d^2}{dx^2}-1\right)g_i(x)=0 \qquad \Rightarrow g_a(x)=\sin(x); \qquad g_b(x)=\sin(L-x);$$

$$W=g_b(x)\frac{dg_a(x)}{dx}-g_a(x)\frac{dg_b(x)}{dx}=\sin(L-x)\cos(x)+\sin(x)\cos(L-x)$$

$$=\sin(L)$$

$$\varphi(x)=\varphi_0(x)+\frac{\sin(L-x)}{\sin(L)}\int_0^x \sin(x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$+\frac{\sin(x)}{\sin(L)}\int_x^L \sin(L-x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$\varphi(x)=\varphi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

(Actually the algebra is painful).
But, hurray! Same result as before.

Another example --

$$\frac{d^2}{dx^2}\Phi(x) = -\rho(x)/\epsilon_0$$ electrostatic potential for charge density $\rho(x)$

Homogeneous equation:

$$\frac{d^2}{dx^2}g_{a,b}(x) = 0$$

Let $g_a(x) = x \qquad g_b(x) = 1$

Wronskian:

$$W = g_a(x)\frac{dg_b(x)}{dx} - g_b(x)\frac{dg_a(x)}{dx} = -1$$

Green's function:

$$G(x,x') = -x_<$$

$$\Phi(x) = \Phi_0(x) + \frac{1}{\epsilon_0}\int_{-\infty}^{x} dx'\,x'\rho(x') + \frac{x}{\epsilon_0}\int_{x}^{\infty} dx'\,\rho(x')$$

Example -- continued

$$\frac{d^2}{dx^2}\Phi(x) = -\rho(x)/\epsilon_0$$ electrostatic potential for charge density $\rho(x)$

$$\Phi(x) = \Phi_0(x) + \frac{1}{\epsilon_0}\int_{-\infty}^{x} dx' x' \rho(x') + \frac{x}{\epsilon_0}\int_{x}^{\infty} dx' \rho(x')$$

Suppose $\rho(x) = \begin{cases} 0 & x \le -a \\ \rho_0 x/a & -a \le x \le a \\ 0 & x \ge a \end{cases}$

$$\Phi(x) = \Phi_0(x) + \begin{cases} 0 & x \le -a \\ \frac{\rho_0}{\epsilon_0 a}\left(\frac{a^3}{3} + \frac{xa^2}{2} - \frac{x^3}{6}\right) & -a \le x \le a \\ \frac{2}{3\epsilon_0}\rho_0 a^2 & x \ge a \end{cases}$$

$$\Phi(x) = \begin{cases} 0 & x \le -a \\ \dfrac{\rho_0}{\epsilon_0 a}\left(\dfrac{a^3}{3} + \dfrac{xa^2}{2} - \dfrac{x^3}{6}\right) & -a \le x \le a \\ \dfrac{2}{3\epsilon_0}\rho_0 a^2 & x \ge a \end{cases}$$