PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 25: Chap. 7 & App. A-D (F&W)

Generalization of the one dimensional wave equation → various mathematical problems and techniques including:

1. **Complex variables**
2. **Contour integrals**
3. **Kramers-Kronig relationships**

Physics Colloquium

- Thursday -

October 24, 2024

4 PM

Olin 101

Imaging Through Unusual Apertures by Advanced Digital Methods and 3D Tracking Comparison of a Refractive vs. a Geometric Phase Plate

The talk will present methods to resolve the challenges faced in classical optics with a full aperture imaging system by utilizing unusual apertures with the method of incoherent digital holography. The development of superresolution imaging techniques that provide similar resolution with a fraction of aperture area as low as 1.4% and 0.48% of the total aperture area using partial and synthetic aperture (SA) imaging will be discussed. We developed a novel single-channel incoherent SA by implementing a single opening instead of two and it solves a core problem of SA with multiple channels in the optical regime will be reviewed. These research attempts to provide an alternative to the large and bulky reflective mirrors or refractive/diffractive lenses commonly used in space-based, or ground-based telescopes. These techniques inspire a new generation of telescopes that can be lightweight, smaller in size, and cost-effective, while at the same time delivering a similar resolution to that from a large SA telescope system with 3D capabilities. The second part of the talk will be on tracking biological objects in three dimensions (3D) for a better understanding of the dynamic behavior of cellular components. It can be achieved by multifocal imaging with diffractive optical elements (DOEs) converting depth (z) information into a modification of the 2D image. A quantitative comparison of the performance of a DOE (3rd-generation refractive phase plate (RPP)) to a 4th-

Dr. Angika Bulbul
Wake Forest University

Reception 3:30
Olin Lobby

Colloquium 4:00

15	Fri, 9/27/2024	Chap. 1	Scattering analysis	#14
16	Mon, 9/30/2024	Chap. 4	Small oscillations near equilibrium	
17	Wed, 10/2/2024	Chap. 1-6	Review	THE-10/3-9/24
18	Fri, 10/4/2024	Chap. 4	Normal mode analysis	THE-10/3-9/24
19	Mon, 10/7/2024	Chap. 4	Normal mode analysis in multiple dimensions	THE-10/3-9/24
20	Wed, 10/9/2024	Chap. 4&7	Normal modes of continuous strings	THE-10/3-9/24
21	Fri, 10/11/2024	Chap. 7	The wave and other partial differential equations	
22	Mon, 10/14/2024	Chap. 7	Sturm-Liouville equations	#15
23	Wed, 10/16/2024	Chap. 7	Sturm-Liouville equations	#16
	Fri, 10/18/2024	Fall Break		
24	Mon, 10/21/2024	Chap. 7	Laplace transforms and complex functions	#17
25	Wed, 10/23/2024	Chap. 7	Complex integration	#18

PHY 711 – Homework # 18

Assigned: 10/23/2024 Due: 10/28/2024

Read Appendix A of **Fetter and Walecka**.

1. Assume that $a > 0$ and $b > 0$; use contour integration methods to evaluate the integral:

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 + b^2} dx.$$

Note that you may use Maple, Mathematica, or other software to evaluate this integral, but full credit will be earned by using the contour integration methods.

Reminder -- Time to think about your presentations due towards then end of the semester -- best lectures of the class – 2/day

Project

The purpose of this assignment is to provide an opportunity for you to study a topic of your choice in greater depth. The general guideline for your choice of project is that it should have something to do with classical mechanics, and there should be some degree of of analytic or numerical computation associated with the project. The completed project will include a short write-up and a presentation to the class. You may design your own project or use one of the following list (which will be updated throughout the term).

- Explain the details of a homework problem that was assigned or one you design, including the basic principles and the solution methods and results.
- Consider a scattering experiment in which you specify the spherically symetric interaction potential V(r). Write a computer program (using your favorite language) to evaluate the scattering cross section for your system. (Depending on your choice, you may wish to present your results either in the the center-of-mass or lab frames of reference.)
- Consider the Foucoult Pendulum. Analyze the equations of motion including both the horizontal and vertical motions. You can either solve the equations exactly or use perturbation theory. Compare the effects of the vertical motion to the effects of air friction.
- Consider a model system of 2 or more interacting particles with appropriate initial conditions, using numerical methods to find out how the system evolves in time and space. For few particles and special initial conditions this approach can be used to explore orbital mechanics. For many particles and random initial conditions, this approach can be used to explore statistical mechanics via molecular dynamics simulations.
- Examine the normal modes of vibration for a model system with 3 or more masses in 2 or 3 dimensions.
- Analyze the soliton equations beyond what was covered in class.

Basic ideas of complex integration --

For an analytic function, its integral over a closed region in the complex plane vanishes:

$$\oint f(z)dz = 0$$

However, consider the integration of a function which has a pole --

Behavior of $f(z) = \dfrac{1}{z^n}$ about the point $z = 0$:

For an integer n, consider

$$\oint \frac{1}{z^n} dz = \int_0^{2\pi} \frac{\rho e^{i\phi} i d\phi}{\rho^n e^{in\phi}} = \rho^{1-n} \int_0^{2\pi} e^{i(1-n)\phi} i d\phi = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

Behavior of $f(z) = \frac{1}{z^n}$ about the point $z = 0$:

For an integer n, consider

$$\oint \frac{1}{z^n} dz = \int_0^{2\pi} \frac{\rho e^{i\phi} i d\phi}{\rho^n e^{in\phi}} = \rho^{1-n} \int_0^{2\pi} e^{i(1-n)\phi} i d\phi = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

This observation helps us to focus on a special kind of singularity called a "pole"

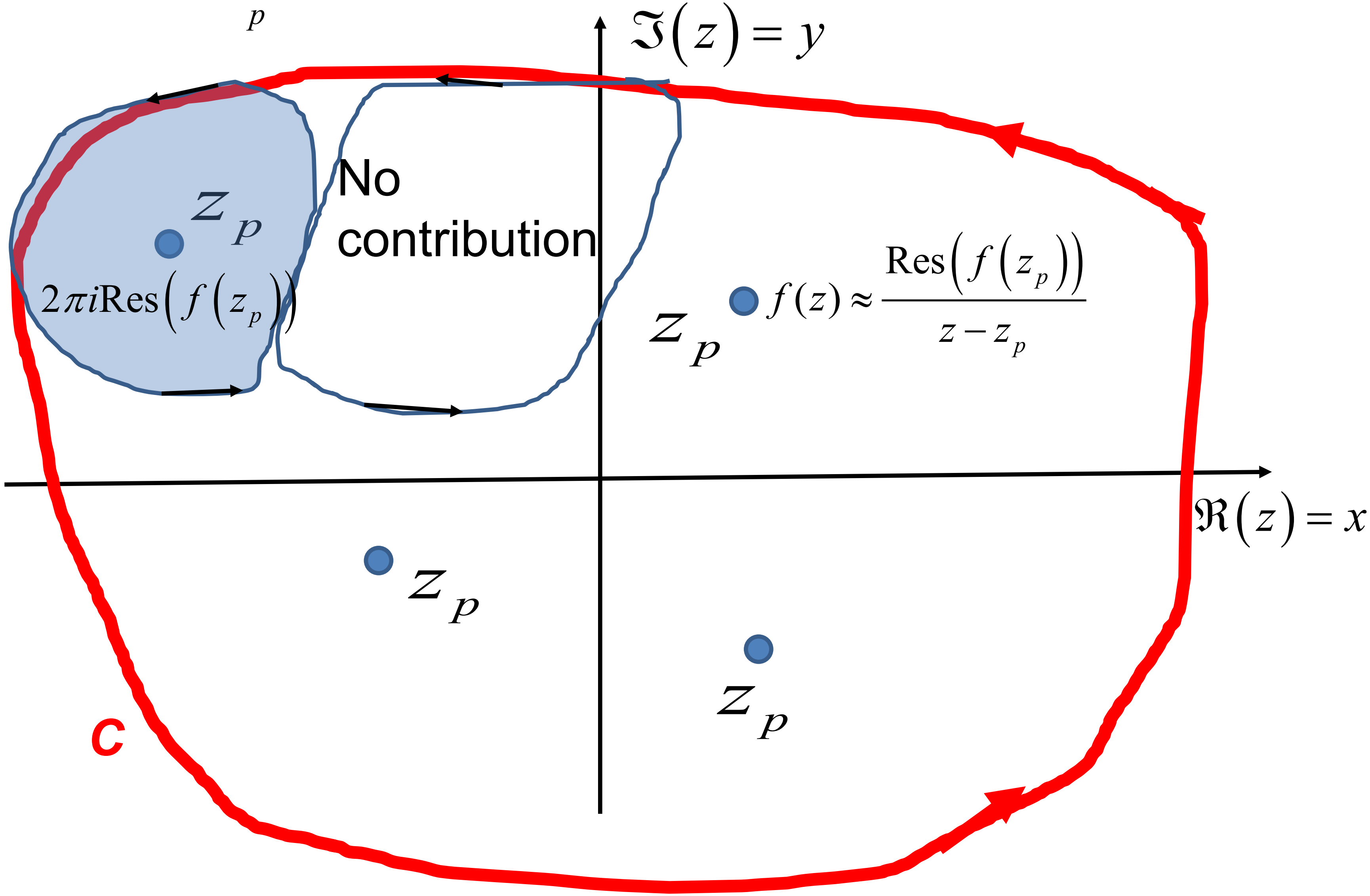
For $f(z)$ in the vicinity of $z = z_p$: $\quad f(z) \approx \frac{g(z_p)}{z - z_p}$

Therefore: $\oint f(z) dz = 0$ or $\oint f(z) dz = g(z_p) \oint \frac{dz}{z - z_p} = 2\pi i g(z_p)$

Integration does not include z_p

Integration does include z_p

$$\oint_C f(z)dz = 2\pi i \sum_p \mathrm{Res}\left(f(z_p)\right)$$



General formula for determining residue:

Suppose that in the neighborhood of z_p, $f(z) \approx \frac{h(z)}{\left(z-z_p\right)^m} \underset{z\to z_p}{\equiv} \frac{\text{Res}\left(f(z_p)\right)}{z-z_p}$

Since $h(z) \equiv \left(z-z_p\right)^m f(z)$ is analytic near z_p, we can make a Taylor exansion

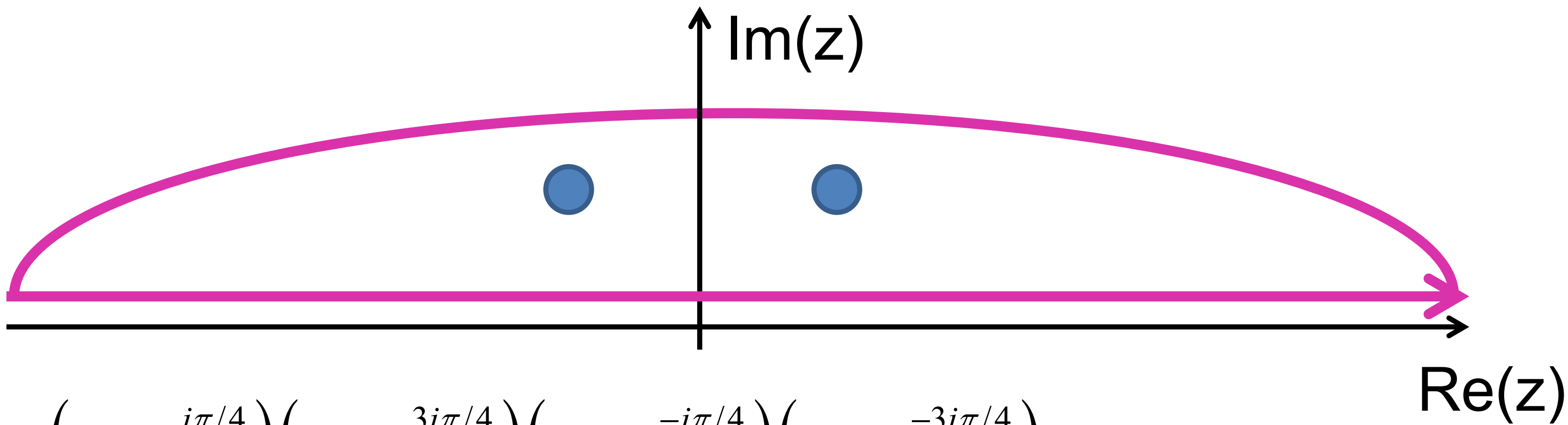
about z_p :

$$h(z) \approx h(z_p) + \left(z-z_p\right)\frac{dh(z_p)}{dz} + \ldots + \frac{\left(z-z_p\right)^{m-1}}{(m-1)!}\frac{d^{m-1}h(z_p)}{dz^{m-1}} + \ldots.$$

$$\Rightarrow \text{Res}\left(f(z_p)\right) = \lim_{z\to z_p}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}\left(\left(z-z_p\right)^m f(z)\right)}{dz^{m-1}}\right\}$$

In the following examples *m=1*

Example: $$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx = \int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx + 0 = \oint \frac{z^2}{1+z^4}\,dz$$



$$1+z^4 = \left(z-e^{i\pi/4}\right)\left(z-e^{3i\pi/4}\right)\left(z-e^{-i\pi/4}\right)\left(z-e^{-3i\pi/4}\right)$$

$$\oint \frac{z^2}{1+z^4}\,dz = 2\pi i\left(\mathrm{Res}\left(z_p = e^{i\pi/4}\right) + \mathrm{Res}\left(z_p = e^{3i\pi/4}\right)\right)$$

Note:
m=1

$$\mathrm{Res}\left(z_p = e^{i\pi/4}\right) = \frac{e^{i\pi/4}}{4i} \qquad \mathrm{Res}\left(z_p = e^{3i\pi/4}\right) = -\frac{e^{3i\pi/4}}{4i}$$

$$\oint \frac{z^2}{1+z^4}\,dz = 2\pi i\left(\frac{e^{i\pi/4}}{4i} - \frac{e^{3i\pi/4}}{4i}\right) = \frac{\pi}{2}\left(\left(\sqrt{\frac{1}{2}} + i\sqrt{\frac{1}{2}}\right) - \left(-\sqrt{\frac{1}{2}} + i\sqrt{\frac{1}{2}}\right)\right) = \frac{\pi}{\sqrt{2}}$$

Some details:

Note that: $e^{i\pi} = -1 = e^{-i\pi}$

$$e^{-3i\pi/4} = e^{i\pi/4 - i\pi} = -e^{i\pi/4}$$

$$f(z) = \frac{z^2}{1+z^4}$$

$$\text{Res}\left(f(z = e^{i\pi/4}\right) = \frac{\left(e^{i\pi/4}\right)^2}{\left(e^{i\pi/4} - e^{3i\pi/4}\right)\left(e^{i\pi/4} - e^{-i\pi/4}\right)\left(e^{i\pi/4} - e^{-3i\pi/4}\right)}$$

$$= \frac{e^{i\pi/2}}{\left(e^{i\pi/4} + e^{-i\pi/4}\right)\left(e^{i\pi/4} - e^{-i\pi/4}\right)\left(e^{i\pi/4} + e^{i\pi/4}\right)}$$

$$= \frac{e^{i\pi/4}}{2\left(i - (-i)\right)} = \frac{e^{i\pi/4}}{4i}$$

Question – Could we have chosen the contour in the lower half plane?

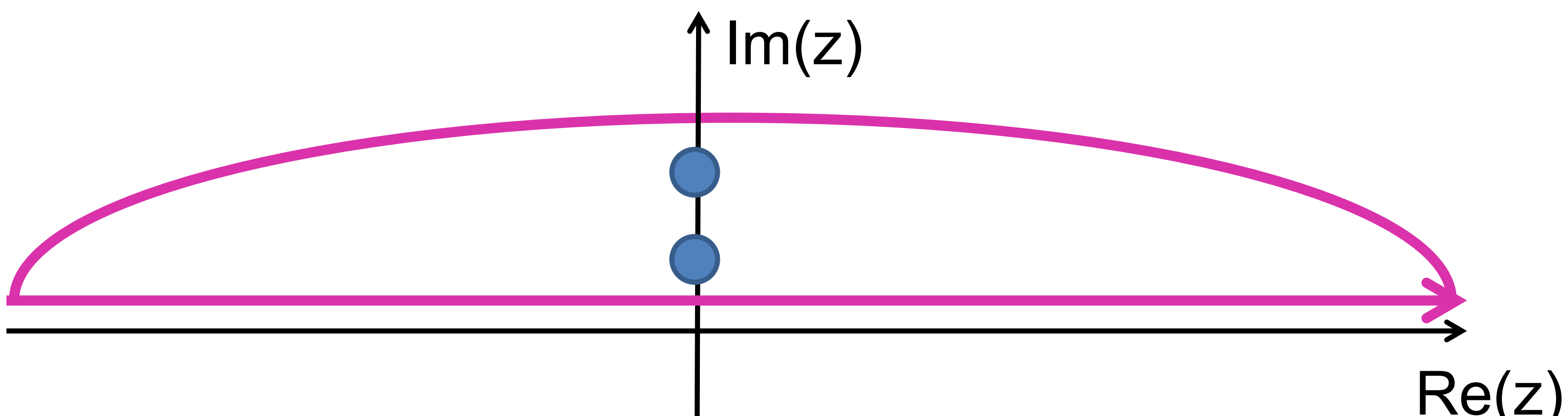
a. Yes b. No

Another example: $I = \int_0^{\infty} \frac{\cos(ax)}{4x^4 + 5x^2 + 1} dx$

$$\int_0^{\infty} \frac{\cos(ax)}{4x^4 + 5x^2 + 1} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{iax}}{4x^4 + 5x^2 + 1} dx = \frac{1}{2}\oint \frac{e^{iaz}}{4z^4 + 5z^2 + 1} dz$$

$$4z^4 + 5z^2 + 1 = 4(z-i)\left(z-\tfrac{i}{2}\right)(z+i)\left(z+\tfrac{i}{2}\right)$$

Note:
m=1



$$I = 2\pi i\left(\mathrm{Res}\left(z_p = i\right) + \mathrm{Res}\left(z_p = \tfrac{i}{2}\right)\right)$$

$$\int_0^\infty \frac{\cos(ax)}{4x^4+5x^2+1}dx = \frac{1}{2}\oint \frac{e^{iaz}}{4z^4+5z^2+1}dz$$

$$= 2\pi i\left(\text{Res}\left(z_p = i\right) + \text{Res}\left(z_p = \tfrac{i}{2}\right)\right)$$

$$= \frac{\pi}{6}\left(-e^{-a} + 2e^{-a/2}\right)$$

Question – Could we have chosen the contour in the lower half plane?

a. Yes

b. No

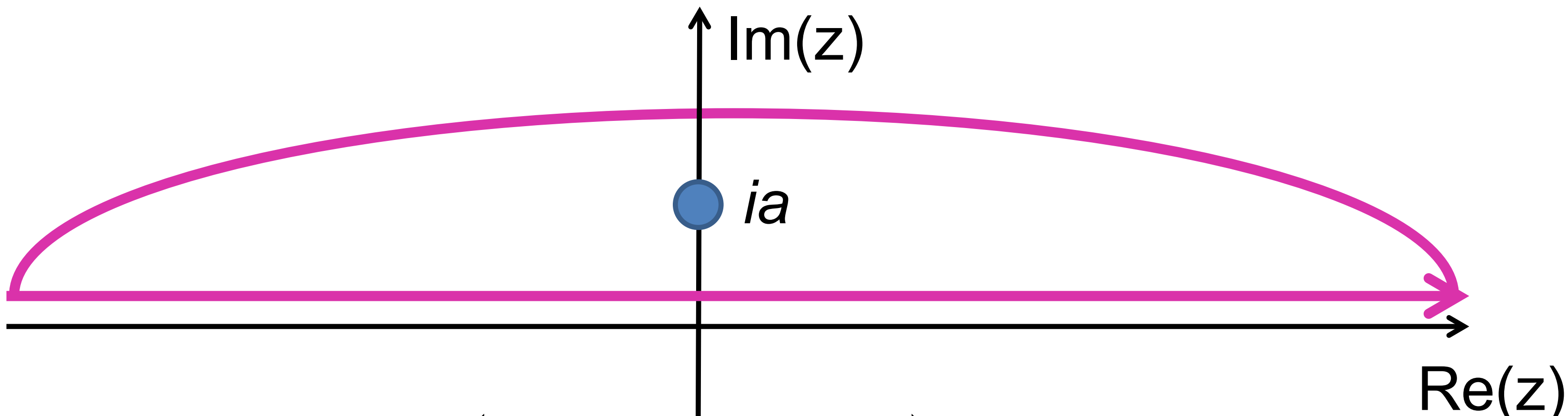
Note that for $a > 0$ and $z_I > 0$

in the lower half plane: $e^{iaz} = e^{iaz_R}e^{az_I}$

Another example: $I = \int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + a^2} dx \quad \text{for } k > 0 \text{ and } a > 0$

$$\int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + a^2} dx = \frac{1}{i} \int_{-\infty}^{\infty} \frac{x e^{ikx}}{x^2 + a^2} dx = \frac{1}{i} \oint \frac{z e^{ikz}}{z^2 + a^2} dz$$

$$z^2 + a^2 = (z - ia)(z + ia)$$



$$I = 2\pi i \left(\text{Res}\left(z_p = ia \right) \right) = \pi e^{-ka}$$

Some details --

$$\int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + a^2} dx = \frac{1}{i} \int_{-\infty}^{\infty} \frac{x e^{ikx}}{x^2 + a^2} dx = \frac{1}{i} \oint \frac{z e^{ikz}}{z^2 + a^2} dz$$

$$z^2 + a^2 = (z - ia)(z + ia)$$

$$\frac{1}{i} \oint \frac{z e^{ikz}}{z^2 + a^2} dz = 2\pi i \frac{1}{i} \lim_{z \to ia} \left((z - ia) \frac{z e^{ikz}}{z^2 + a^2} \right)$$

$$= 2\pi i \frac{1}{i} \frac{ia e^{-ka}}{2ia} = \pi e^{-ka}$$

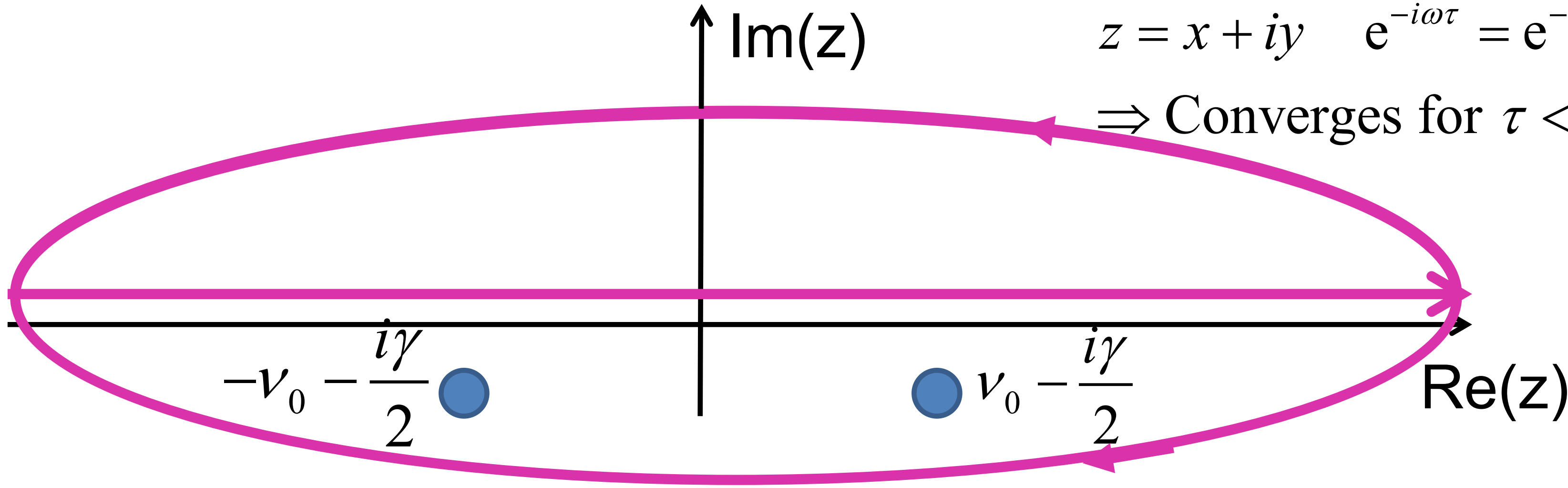
From the Drude model of dielectric response --

$$G(\tau)=\frac{\omega_p^2}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega_0^2-\omega^2-i\gamma\omega}$$ where ω_p, ω_0, and γ are positive constants

Upper hemisphere:

$z = x + iy \quad e^{-i\omega\tau} = e^{-ix\tau + y\tau}$

$\Rightarrow$ Converges for $\tau < 0$



Lower hemisphere:

$z = x - iy \quad e^{-i\omega\tau} = e^{-ix\tau - y\tau}$

$\Rightarrow$ Converges for $\tau > 0$

$$\nu_0 \equiv \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

From the Drude model of dielectric response -- continued --

$$G(\tau) = \frac{\omega_p^2}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega_0^2 - \omega^2 - i\gamma\omega} \quad \text{where } \omega_p,\ \omega_0,\ \text{and } \gamma \text{ are positive constants}$$

$$G(\tau) = \omega_p^2 \begin{cases} 0 & \text{for } \tau < 0 \\ e^{-\gamma\tau/2} \dfrac{\sin \nu_0 \tau}{\nu_0} & \text{for } \tau > 0 \end{cases}$$

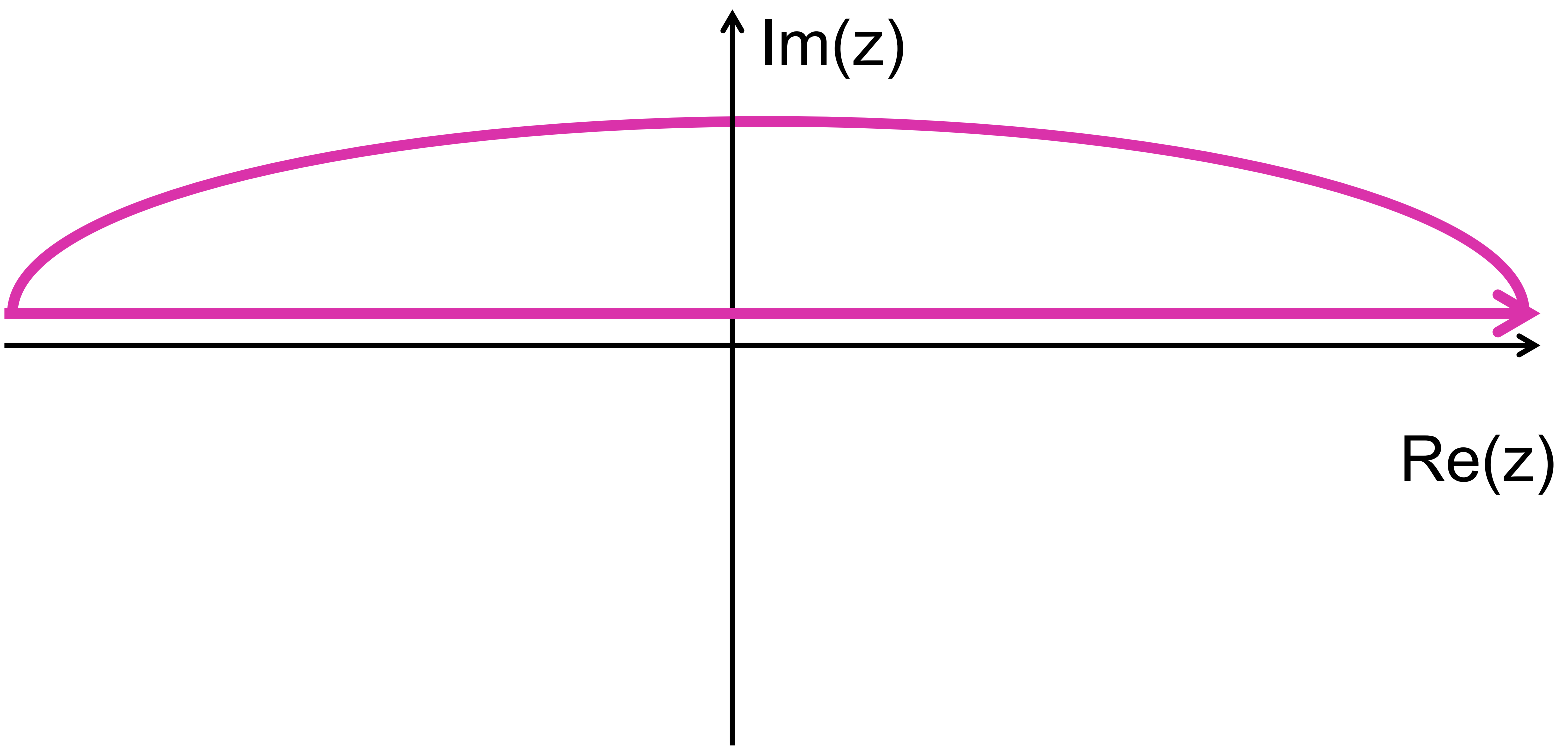
Cauchy integral theorem for analytic function *f(z)*:

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} dz'.$$

Example

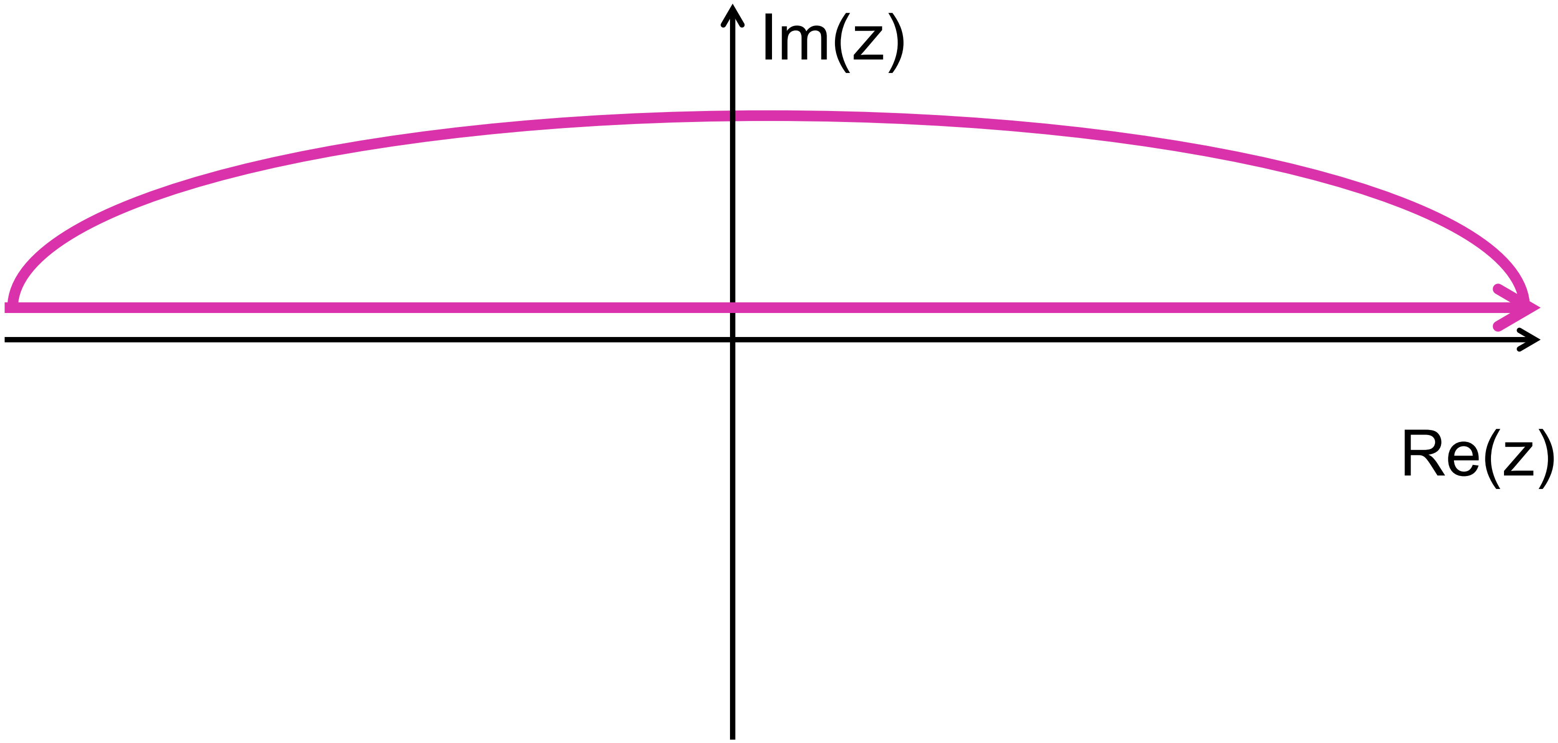
$$\text{Suppose } f(|z| \to \infty) = 0 \text{ and for } z = x:$$

$$f(x) = a(x) + ib(x)$$



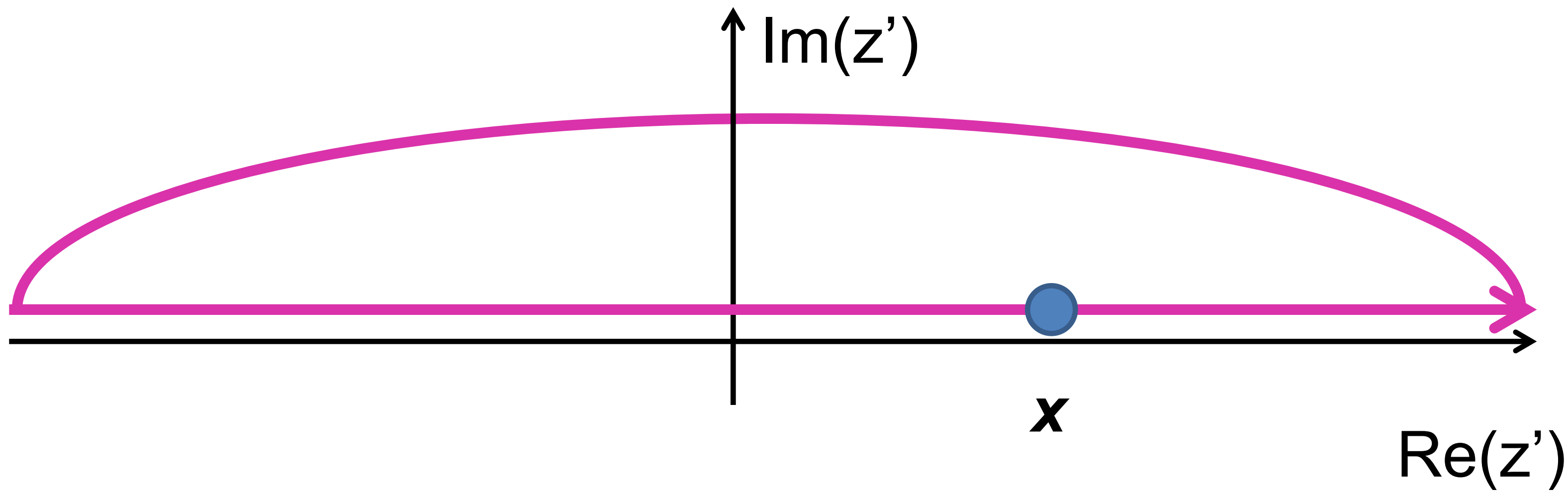
Example -- continued

$$f(z)=\frac{1}{2\pi i}\oint\frac{f(z')}{z'-z}dz' \quad \text{where} \quad f(x)=a(x)+ib(x)$$



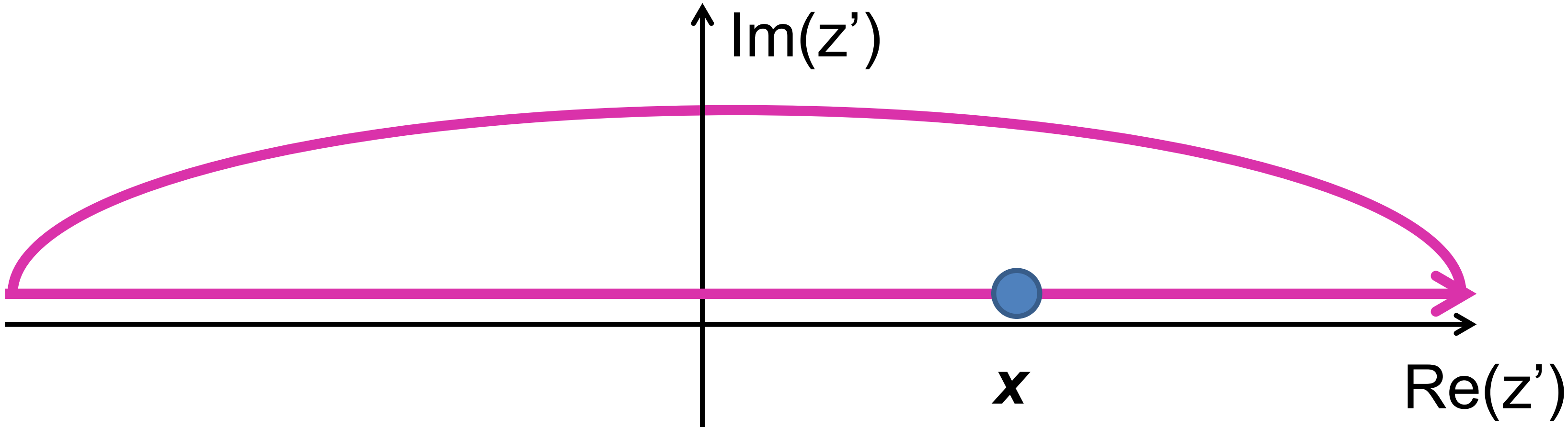
$$a(x)+ib(x)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{a(x')+ib(x')}{x'-x}dx'+0$$

Example -- continued



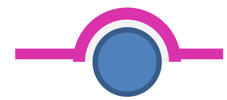
$$\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx' = \int_{-\infty}^{x-\varepsilon}\frac{f(x')}{x'-x}dx' + \int_{x+\varepsilon}^{\infty}\frac{f(x')}{x'-x}dx' + \int_{x-\varepsilon}^{x+\varepsilon}\frac{f(x')}{x'-x}dx'$$

$$= P\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx' + i\pi\, f(x)$$



let $\quad u = x' - x$

let $\quad x \rightarrow x + i\eta$

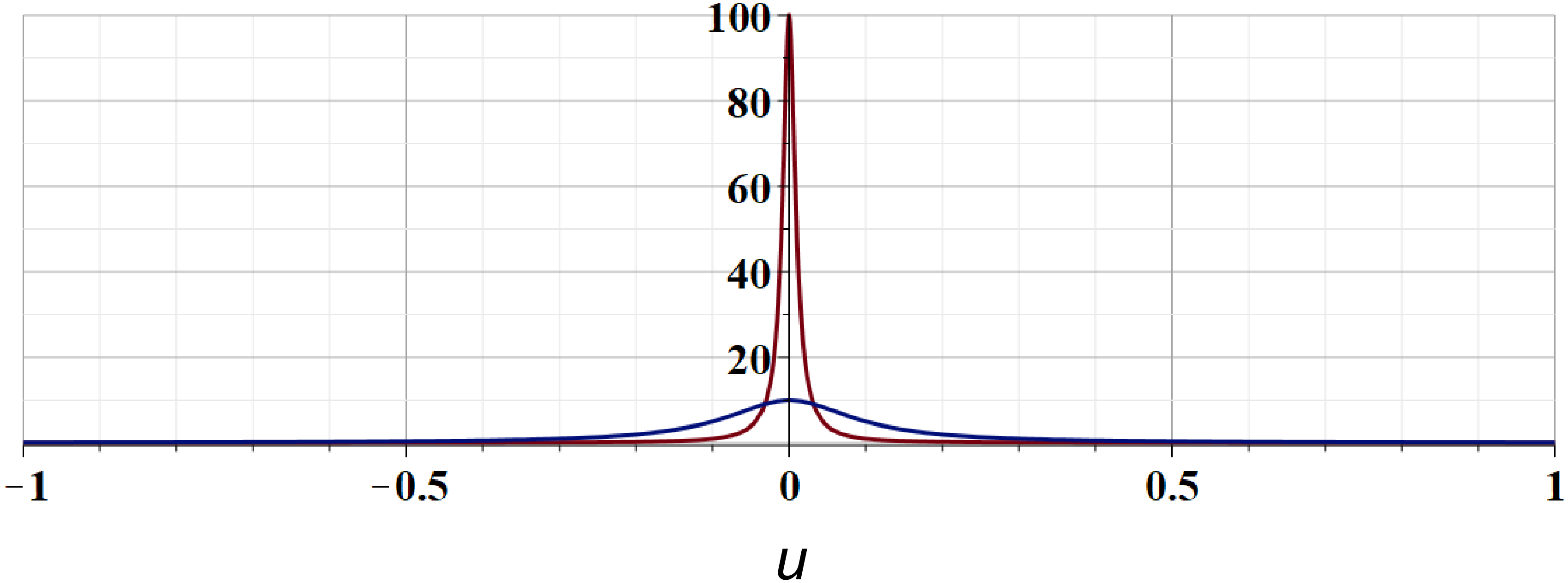


$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{f(x')}{x'-x} dx' \approx f(x) \lim_{\eta\to 0} \int_{-\varepsilon}^{\varepsilon} \frac{1}{u - i\eta} du = f(x) \lim_{\eta\to 0} \int_{-\varepsilon}^{\varepsilon} \frac{u + i\eta}{u^2 + \eta^2} du$$

$$= i\pi f(x) \qquad \text{since} \qquad \lim_{\eta\to 0} \frac{i\eta}{u^2+\eta^2} \approx i\pi\delta(u)$$

More details --

$$\lim_{\eta\to 0}\frac{\eta}{u^2+\eta^2}\approx \pi\delta(u)$$



Example -- continued

$$f(x)=\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx'=\int_{-\infty}^{x-\varepsilon}\frac{f(x')}{x'-x}dx'+\int_{x+\varepsilon}^{\infty}\frac{f(x')}{x'-x}dx'+\int_{x-\varepsilon}^{x+\varepsilon}\frac{f(x')}{x'-x}dx'$$

$$=P\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx'+i\pi f(x)$$

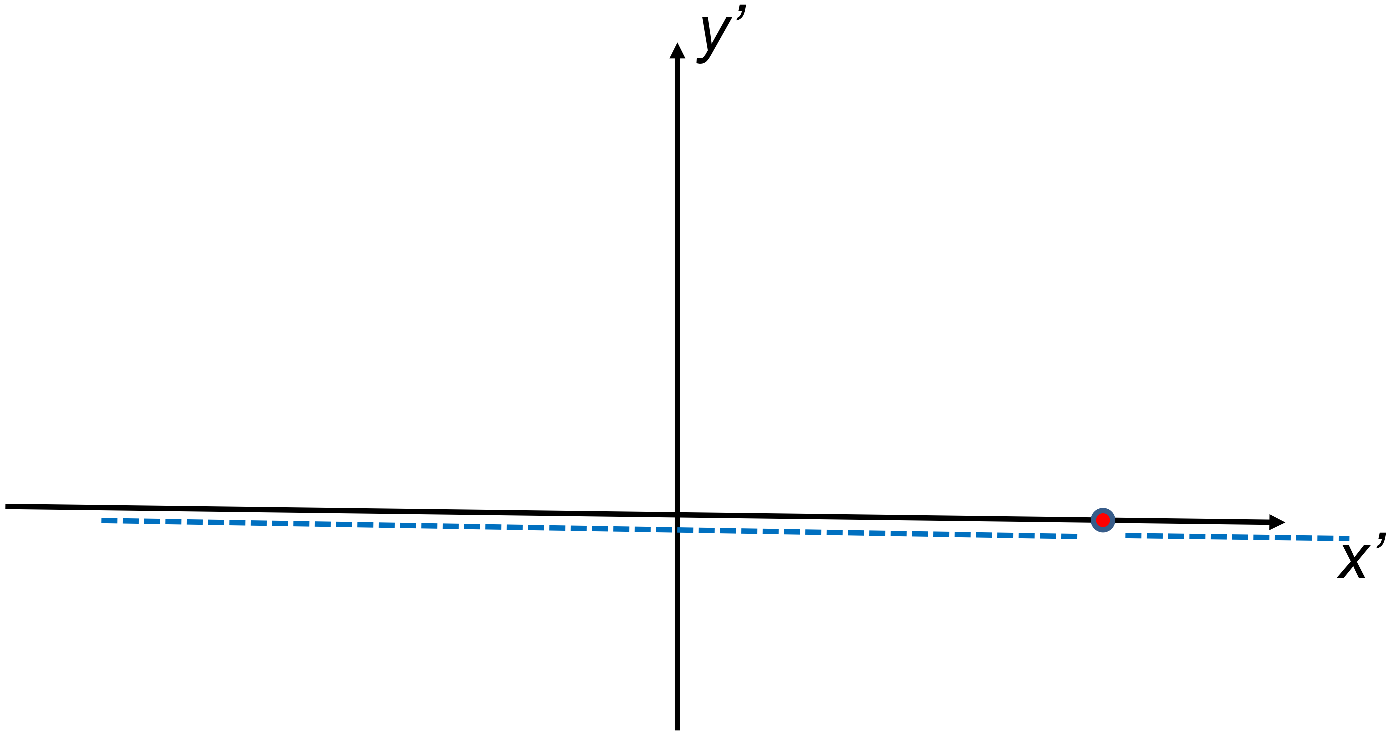
$$a(x)+ib(x)=\frac{P}{2\pi i}\int_{-\infty}^{\infty}\frac{a(x')+ib(x')}{x'-x}dx'+\frac{\pi i}{2\pi i}\left(a(x)+ib(x)\right)$$

$$\Rightarrow a(x)=\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' \qquad b(x)=-\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{a(x')}{x'-x}dx'$$

Kramers-Kronig relationships

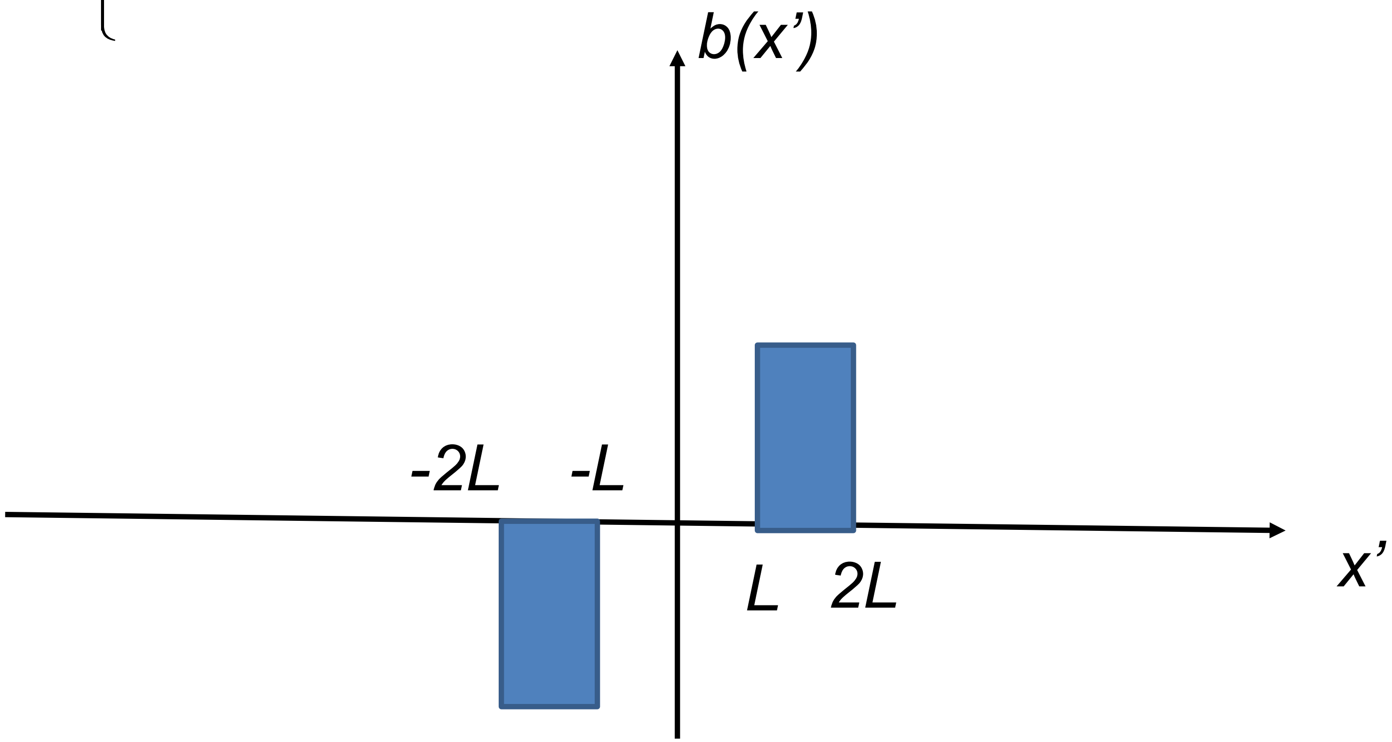
Comment on evaluating principal parts integrals

$$a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' = \lim_{\epsilon \to 0}\left(\frac{1}{\pi}\int_{-\infty}^{x-\epsilon}\frac{b(x')}{x'-x}dx' + \frac{1}{\pi}\int_{x+\epsilon}^{\infty}\frac{b(x')}{x'-x}dx'\right)$$



Example:

$$b(x') = \begin{cases} 0 & \text{for } x' < -2L, \quad -L < x' < L, \quad x' > 2L \\ B_0 & \text{for } \quad L < x' < 2L \\ -B_0 & \text{for } \quad -2L < x' < -L \end{cases}$$



Example:

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$$a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{b(x')}{x'-x} dx' = \lim_{\epsilon \to 0}\left(\frac{1}{\pi}\int_{-\infty}^{x-\epsilon} \frac{b(x')}{x'-x} dx' + \frac{1}{\pi}\int_{x+\epsilon}^{\infty} \frac{b(x')}{x'-x} dx' \right)$$

For $x < -2L$ or $x > 2L$ $\quad -L < x < L:$

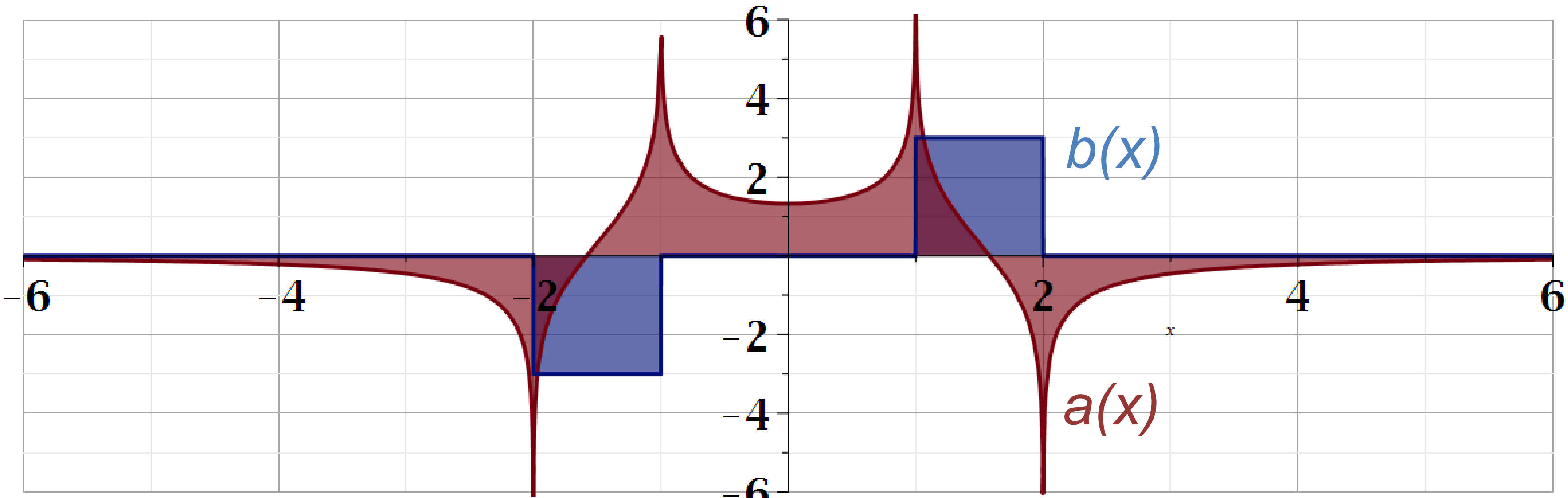
$$a(x) = \frac{-B_0}{\pi}\int_{-2L}^{-L} \frac{dx'}{x'-x} + \frac{B_0}{\pi}\int_{L}^{2L} \frac{dx'}{x'-x}$$

$$= \frac{-B_0}{\pi}\ln\left(\left|\frac{x+L}{x+2L}\right|\right) + \frac{B_0}{\pi}\ln\left(\left|\frac{x-2L}{x-L}\right|\right) = \frac{B_0}{\pi}\ln\left(\left|\frac{x^2-4L^2}{x^2-L^2}\right|\right)$$

$$a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' = \lim_{\epsilon\to 0}\left(\frac{1}{\pi}\int_{-\infty}^{x-\epsilon}\frac{b(x')}{x'-x}dx' + \frac{1}{\pi}\int_{x+\epsilon}^{\infty}\frac{b(x')}{x'-x}dx'\right)$$

For our example:

$$a(x) = \frac{B_0}{\pi}\ln\left(\left|\frac{4L^2 - x^2}{L^2 - x^2}\right|\right)$$



Summary

For a function $f(x)$, analytic along the real line:

$$f(x) = \Re\left(f(x)\right) + i\Im\left(f(x)\right) = a\left(x\right) + ib\left(x\right)$$

$$\Rightarrow a\left(x\right) = \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b\left(x'\right)}{x'-x}dx' \qquad b\left(x\right) = -\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{a\left(x'\right)}{x'-x}dx'$$

Example:

$$f(x) = \frac{1}{x+i} \qquad a(x) = \frac{x}{x^2+1} \qquad b(x) = -\frac{1}{x^2+1}$$

Check:

$$\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b\left(x'\right)}{x'-x}dx' = -\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{\left(x'-x\right)\left(x'^2+1\right)}dx' \overset{?}{=} \frac{x}{x^2+1} = a(x)$$

Continued:

$$\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' = -\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{(x'-x)(x'^2+1)}dx'$$

$$= -\frac{P}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{(x'-x)(x'^2+1)} - \frac{1}{(x'-x)(x^2+1)}\right)dx' - \frac{1}{(x^2+1)}\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'-x}dx'$$

$$= -\frac{P}{\pi}\int_{-\infty}^{\infty}\left(\frac{x^2-x'^2}{(x'-x)(x'^2+1)(x^2+1)}\right)dx' - \frac{1}{(x^2+1)}\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'-x}dx'$$

$$= \frac{P}{\pi}\int_{-\infty}^{\infty}\left(\frac{x+x'}{(x'^2+1)(x^2+1)}\right)dx' - \frac{1}{(x^2+1)}\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'-x}dx'$$

Note that: $\int_{x+\epsilon}^{X}\frac{1}{x'-x}dx' = \ln(X-x) - \ln(\epsilon) = \ln\left(\frac{X-x}{\epsilon}\right)$

$$\int_{-X}^{x-\epsilon}\frac{1}{x'-x}dx' = -\ln(-X-x) + \ln(-\epsilon) = -\ln\left(\frac{X+x}{\epsilon}\right)$$

$$\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'-x}dx' = \lim_{X\to\infty}\ln\left(\frac{X-x}{X+x}\right) = 0 \qquad \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'^2+1}dx' = 1 \qquad \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{x'}{x'^2+1}dx' = 0$$

$$\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' = \frac{x}{x^2+1} = a(x)$$

$$a(\omega) = \frac{\omega - 10}{(\omega - 10)^2 + 1} \qquad b(\omega) = \frac{1}{(\omega - 10)^2 + 1}$$