# PHY 711 Classical Mechanics and Mathematical Methods
# 10-10:50 AM MWF in Olin 103
# Notes for Lecture 26 – Chap. 8 (F & W)

**Motions of elastic membranes**

1. Review of standing waves on a string
2. Standing waves on a two dimensional membrane.
3. Boundary value problems

| | | | | |
|---|---|---|---|---|
| 19 | Mon, 10/7/2024 | Chap. 4 | Normal mode analysis in multiple dimensions | THE-10/3-9/24 |
| 20 | Wed, 10/9/2024 | Chap. 4&7 | Normal modes of continuous strings | THE-10/3-9/24 |
| 21 | Fri, 10/11/2024 | Chap. 7 | The wave and other partial differential equations | |
| 22 | Mon, 10/14/2024 | Chap. 7 | Sturm-Liouville equations | #15 |
| 23 | Wed, 10/16/2024 | Chap. 7 | Sturm-Liouville equations | #16 |
| | Fri, 10/18/2024 | Fall Break | | |
| 24 | Mon, 10/21/2024 | Chap. 7 | Laplace transforms and complex functions | #17 |
| 25 | Wed, 10/23/2024 | Chap. 7 | Complex integration | #18 |
| 26 | Fri, 10/25/2024 | Chap. 8 | Wave motion in 2 dimensional membranes | #19 |
| 27 | Mon, 10/28/2024 | Chap. 9 | Motion in 3 dimensional ideal fluids | |
| 28 | Wed, 10/30/2024 | Chap. 9 | Motion in 3 dimensional ideal fluids | |
| 29 | Fri, 11/01/2024 | Chap. 9 | Ideal gas fluids | |

# PHY 711 -- Assignment #19

Assigned: 10/25/2024   Due: 10/28/2024

Read Chapter 8 in **Fetter & Walecka**.

1. In the last few slides of Lecture 19, we considered an annular membrane stretched between the circular radii $a \leq r \leq b$. Find a wave form $f(r)$ for the case that $a=0.5$ and $b=1$, assuming angular uniformity ($m=0$) and that $f(a)=f(b)=0$.

Elastic media in two or more dimensions --

Review of wave equation in one-dimension – here $\mu(x,t)$ can describe either a longitudinal or transverse wave.

Traveling wave solutions --

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

satisfies the wave equation.

Initial value problem: $\mu(x,0) = \phi(x)$ and $\frac{\partial \mu}{\partial t}(x,0) = \psi(x)$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

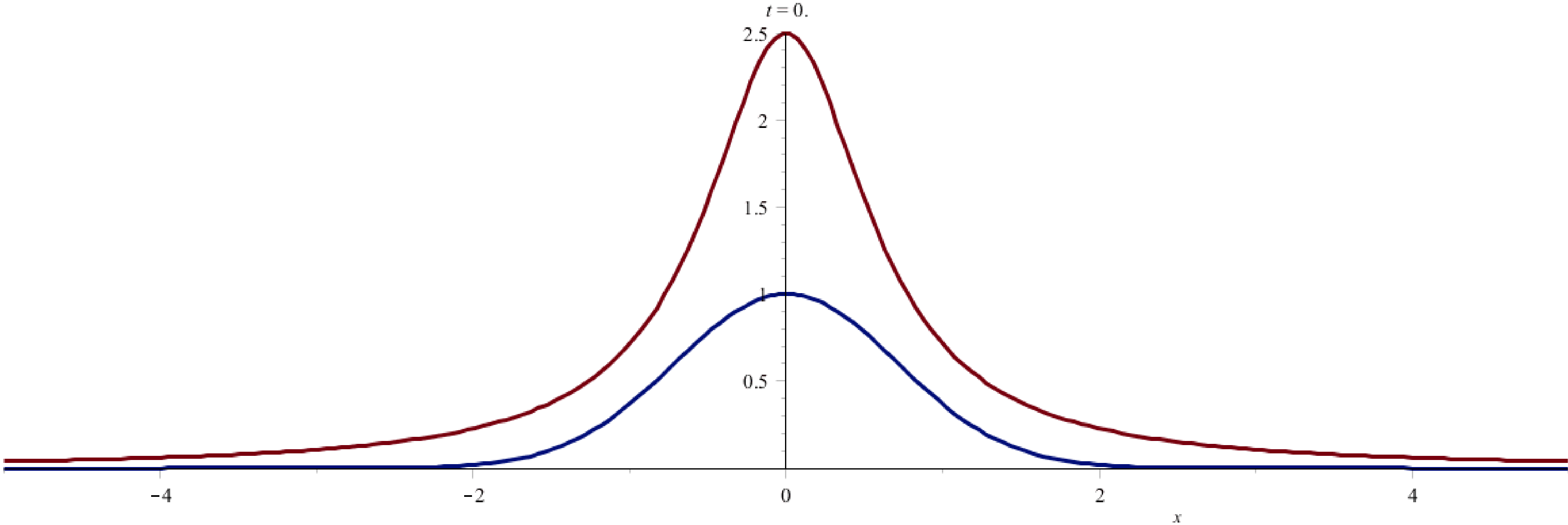
For each $x$, find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$g(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\big(\phi(x-ct) + \phi(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

Example with $\psi(x) = 0$ and $\phi(x) = \frac{1}{x^2 + 0.4}$




Example with $\psi(x) = 0$ and $\phi(x) = e^{-x^2}$

Standing wave solutions of wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

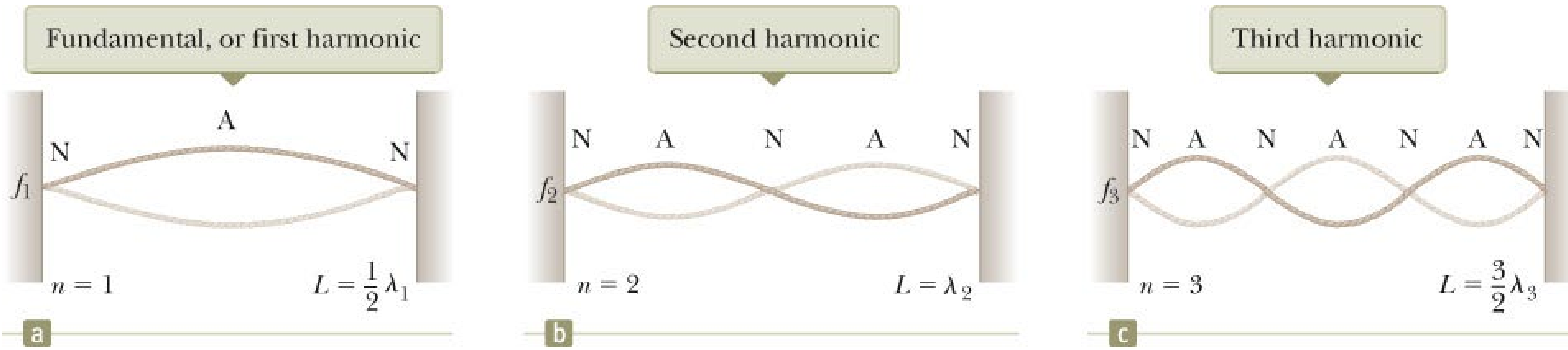
with $\mu(0,t) = \mu(L,t) = 0.$

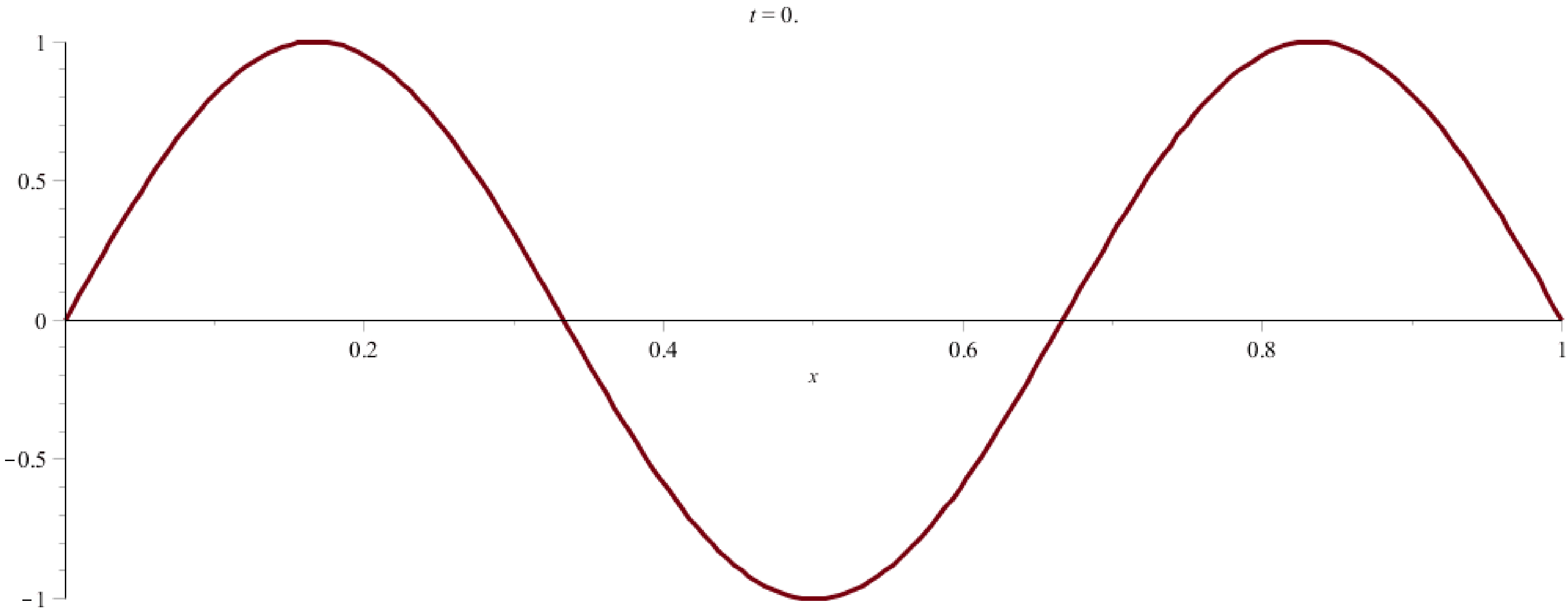
Assume: $\mu(x,t) = \Re\left(e^{-i\omega t}\rho(x)\right)$

where $\frac{d^2\rho(x)}{dx^2} + k^2\rho(x) = 0$ $\qquad k = \frac{\omega}{c}$

$$\rho_\nu(x) = A\sin\left(\frac{\nu\pi x}{L}\right) \qquad \nu = 1, 2, 3, 4....$$

$$k_\nu = \frac{\nu\pi}{L} \qquad \omega_\nu = ck_\nu$$


Fundamental, or first harmonic
Second harmonic
Third harmonic
N A N
$f_1$
$n = 1$
$L = \frac{1}{2}\lambda_1$
a
N A N A N
$f_2$
$n = 2$
$L = \lambda_2$
b
N A N A N A N
$f_3$
$n = 3$
$L = \frac{3}{2}\lambda_3$
c


t = 0.
1
0.5
0
−0.5
−1
0.2
0.4
0.6
0.8
1
x

Wave motion on a two-dimensional surface – elastic membrane (transverse wave; linear regime).

Two-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

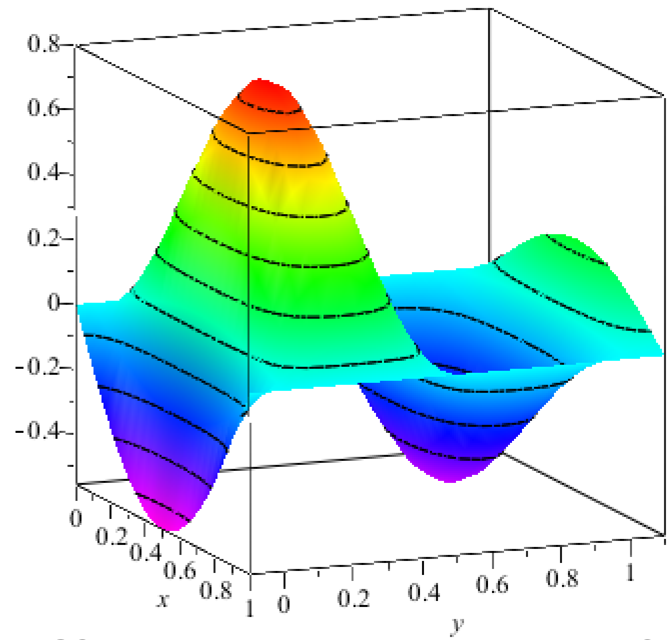
Standing wave solutions:

$$u(x,y,t) = \Re\left(e^{-i\omega t}\rho(x,y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x,y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

Note that here we are visualizing transverse waves. Longitudinal waves can also exist.

$\rho(x,y)$



In this case, we have mapped the one dimensional elastic string to a two dimensional elastic membrane

$$\frac{\partial^2}{\partial x^2} \rightarrow \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{(in Cartesian coordinates)}$$

Lagrangian density : $\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right)$

$$L = \int \mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) dxdy$$

Hamilton's principle :

$$\delta \int_{t_1}^{t_2} L dt = 0$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial t)} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial x)} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial y)} = 0$$

Lagrangian density for elastic membrane with constant $\sigma$ and $\tau$ :

$$\mathcal{L}\left(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial t};x,y,t\right)=\frac{1}{2}\sigma\left(\frac{\partial u}{\partial t}\right)^2-\frac{1}{2}\tau\left(\nabla u\right)^2$$

$$\frac{\partial\mathcal{L}}{\partial u}-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\left(\partial u/\partial t\right)}-\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\left(\partial u/\partial x\right)}-\frac{\partial}{\partial y}\frac{\partial\mathcal{L}}{\partial\left(\partial u/\partial y\right)}=0$$

$$\frac{\partial^2 u}{\partial t^2}-c^2\nabla^2 u=0 \qquad \text{where } c^2=\frac{\tau}{\sigma}$$

Two-dimensional wave equation:

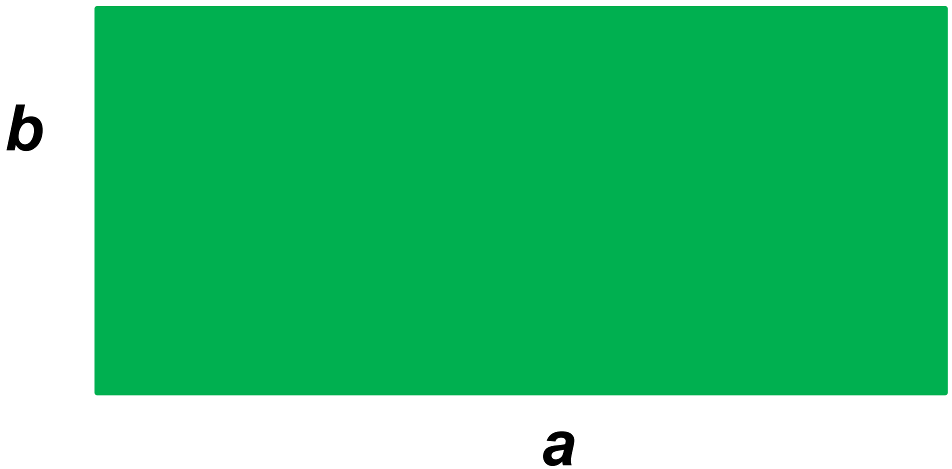
$$\frac{\partial^2 u}{\partial t^2}-c^2\nabla^2 u=0 \qquad \text{where } c^2=\frac{\tau}{\sigma}$$

Standing wave solutions:

$$u(x,y,t)=\Re\left(e^{-i\omega t}\rho(x,y)\right)$$

$$\left(\nabla^2+k^2\right)\rho(x,y)=0 \qquad \text{where } k=\frac{\omega}{c}$$

Consider a rectangular boundary:




Clamped boundary conditions:

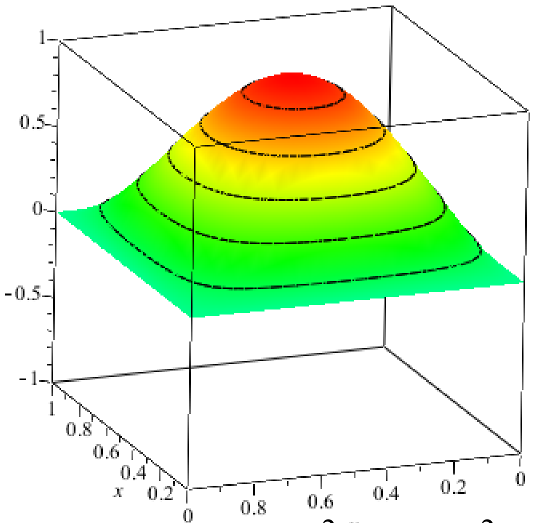
$$\rho(0,y)=\rho(a,y)=\rho(x,0)=\rho(x,b)=0$$

$$\Rightarrow \rho_{mn}(x,y)=A\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

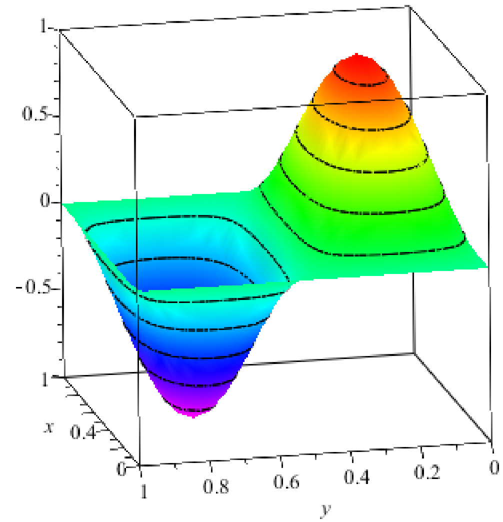
$$k_{mn}^{\ 2}=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn}=ck_{mn}$$

$$\left(\nabla^2+k^2\right)\rho(x,y)=0$$

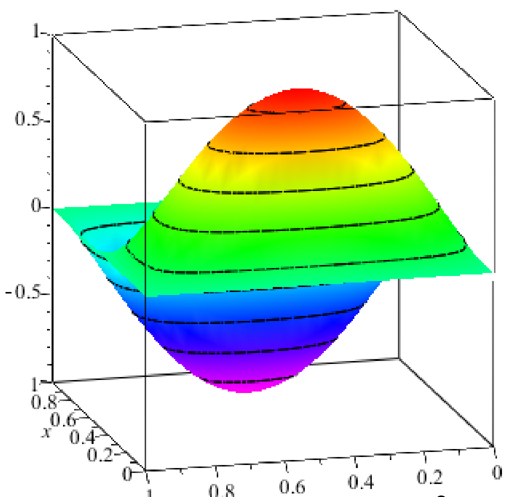
$$\text{where } k=\frac{\omega}{c}$$



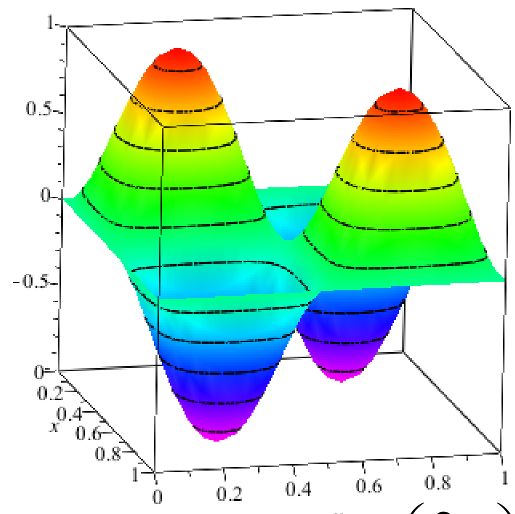

$$k_{11}^{\;2} = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$




$$k_{12}^{\;2} = \left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$$




$$k_{21}^{\;2} = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$




$$k_{22}^{\;2} = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$$

More general boundary conditions:

$\tau \nabla u\big|_b = \kappa u\big|_b$ represents bounded side constrained with spring

$\tau \nabla u\big|_b = 0$ represents "free" side

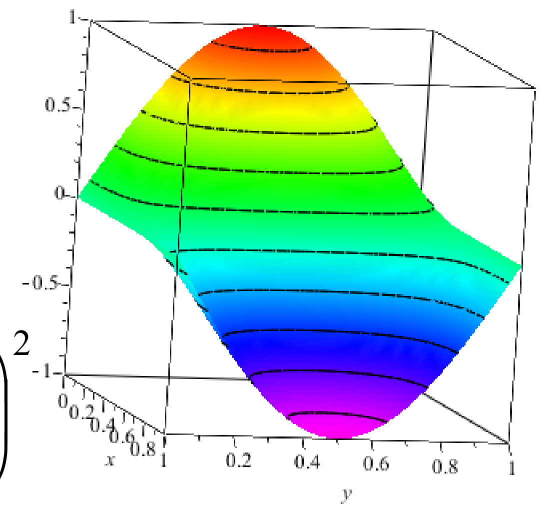
Mixed boundary conditions :

$$\rho(x,0) = \rho(x,b) = \frac{\partial \rho(0,y)}{\partial x} = \frac{\partial \rho(a,y)}{\partial x} = 0$$

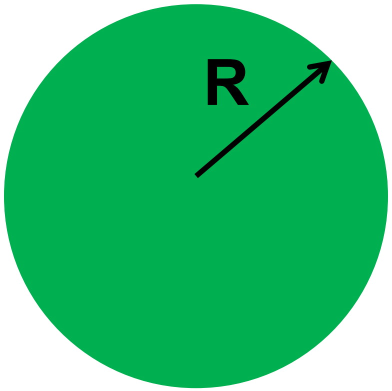
$$\Rightarrow \rho_{mn}(x,y) = A\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}{}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn} = ck_{mn}$$

$$k_{11}{}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



# Consider a circular boundary:




Clamped boundary conditions for $\rho(r,\varphi)$:

$\rho(R,\varphi)=0$

$$\left(\nabla^2+k^2\right)\rho(r,\phi)=0 \qquad \text{where } k=\frac{\omega}{c}$$

In cylindrical coordinate system

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

Assume: $\rho(r,\varphi)=f(r)\Phi(\varphi)$

Let: $\Phi(\varphi)=e^{im\varphi}$

Note: $\Phi(\varphi)=\Phi(\varphi+2\pi)$

$\Rightarrow m=\text{integer}$

# Consider circular boundary -- continued

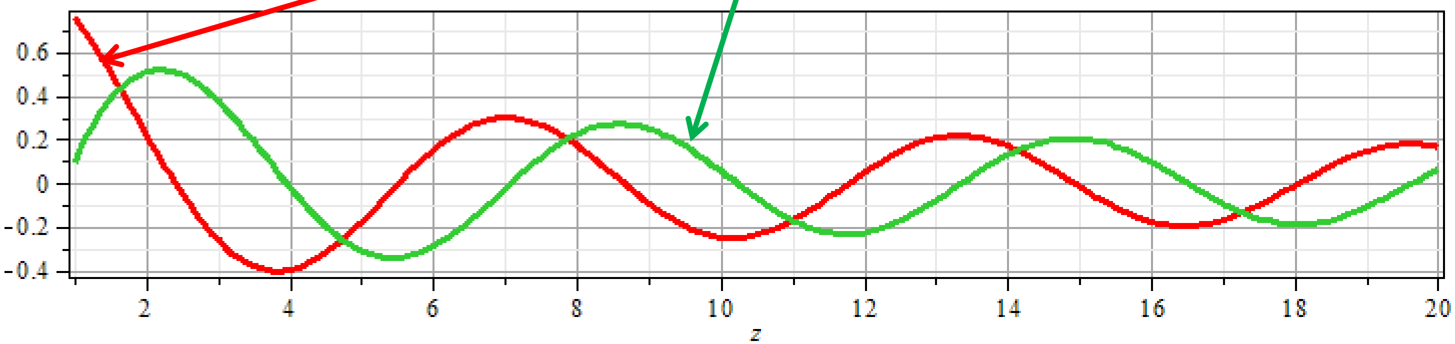
Differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$\Rightarrow$ Bessel equation of integer order with transcendential solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$ also called $Y_m(z)$



# Some properties of Bessel functions

Asending series: $J_m(z) = \left(\frac{z}{2}\right)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+m)!} \left(\frac{z}{2}\right)^{2j}$

Recursion relations: $J_{m-1}(z) + J_{m+1}(z) = \frac{2m}{z} J_m(z)$

$$J_{m-1}(z) - J_{m+1}(z) = 2\frac{dJ_m(z)}{dz}$$

Asymptotic form: $J_m(z) \xrightarrow[z>>1]{} \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right)$

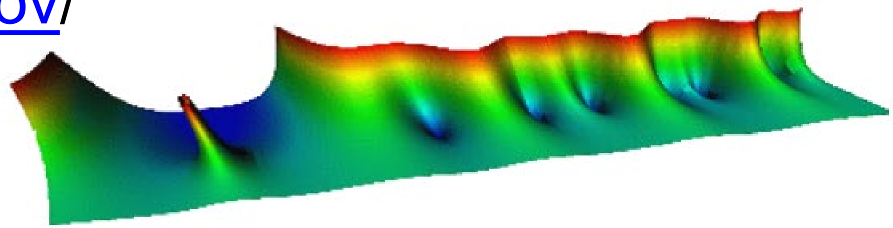
Zeros of Bessel functions $J_m(z_{mn}) = 0$

$m = 0: \quad z_{0n} = 2.406,\ 5.520,\ 8.654, \ldots$

$m = 1: \quad z_{1n} = 3.832,\ 7.016,\ 10.173, \ldots$

$m = 2: \quad z_{2n} = 5.136,\ 8.417,\ 11.620, \ldots$

http://dlmf.nist.gov/



# NIST Digital Library of Mathematical Functions

***Project News***

2014-08-29 *DLMF Update; Version 1.0.9*
2014-04-25 *DLMF Update; Version 1.0.8; errata & improved MathML*
2014-03-21 *DLMF Update; Version 1.0.7; New Features improve Math & 3D Graphics*
2013-08-16 *Bille C. Carlson, DLMF Author, dies at age 89*
*More news*


Foreword
Preface
Mathematical Introduction
1 Algebraic and Analytic Methods
2 Asymptotic Approximations
3 Numerical Methods
4 Elementary Functions
5 Gamma Function
6 Exponential, Logarithmic, Sine, and Cosine Integrals
7 Error Functions, Dawson's and Fresnel Integrals
8 Incomplete Gamma and Related Functions
9 Airy and Related Functions
10 Bessel Functions
11 Struve and Related Functions
12 Parabolic Cylinder Functions
19 Elliptic Integrals
20 Theta Functions
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26 Combinatorial Analysis
27 Functions of Number Theory
28 Mathieu Functions and Hill's Equation
29 Lamé Functions
30 Spheroidal Wave Functions
31 Heun Functions
32 Painlevé Transcendents
33 Coulomb Functions

# Series expansions of Bessel and Neumann functions

$$J_\nu(z) = \left(\tfrac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\tfrac{1}{4}z^2\right)^k}{k!\,\Gamma(\nu+k+1)}.$$

$$Y_n(z) = -\frac{\left(\tfrac{1}{2}z\right)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\tfrac{1}{4}z^2\right)^k + \frac{2}{\pi}\ln\left(\tfrac{1}{2}z\right) J_n(z)$$

$$-\frac{\left(\tfrac{1}{2}z\right)^{n}}{\pi} \sum_{k=0}^{\infty} \left(\psi(k+1) + \psi(n+k+1)\right) \frac{\left(-\tfrac{1}{4}z^2\right)^k}{k!\,(n+k)!},$$

# Some properties of Bessel functions -- continued

Note: It is possible to prove the following

identity for the functions $J_m\left(\frac{z_{mn}}{R}r\right)$:

$$\int_0^R J_m\left(\frac{z_{mn}}{R}r\right)J_m\left(\frac{z_{mn'}}{R}r\right)rdr=\frac{R^2}{2}\left(J_{m+1}\left(z_{mn}\right)\right)^2\delta_{nn'}$$

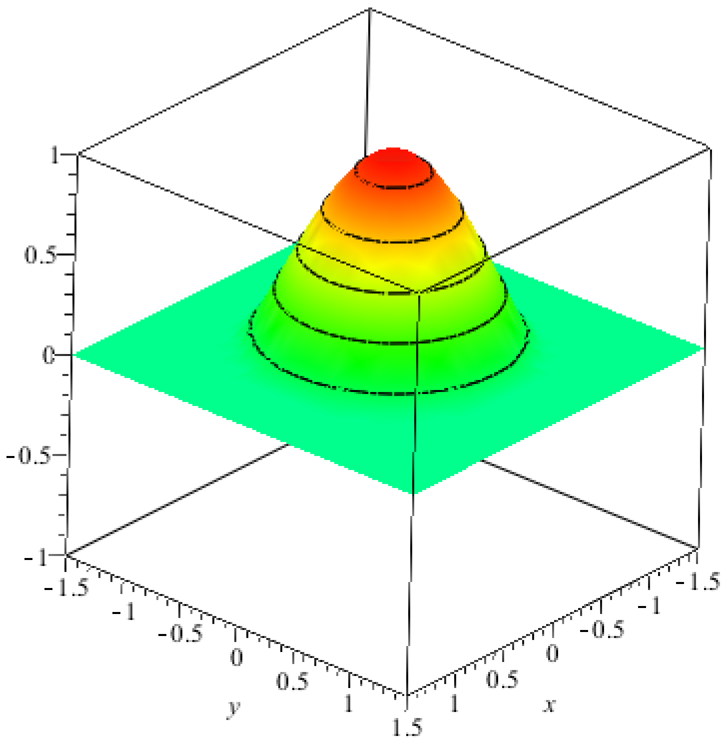
Returning to differential equation for radial function:

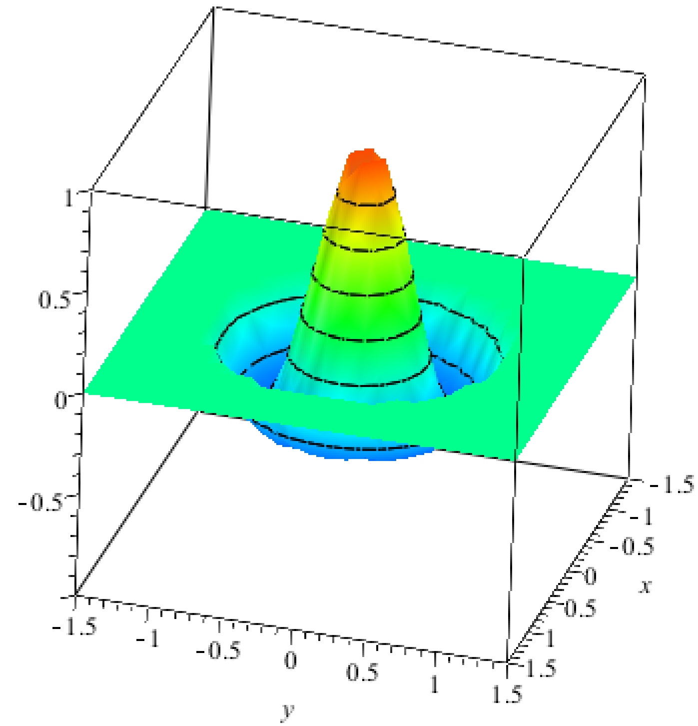
$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$$\Rightarrow\quad f_{mn}(r)=AJ_m\left(\frac{z_{mn}}{R}r\right);\qquad k_{mn}=\frac{z_{mn}}{R}\quad\text{where }\ J(z_{mn})=0$$

$$\rho_{01}(r,\varphi) = f_{01}(r) = AJ_0\left(\frac{z_{01}}{R}r\right)$$

$$\rho_{02}(r,\varphi) = f_{02}(r) = AJ_0\left(\frac{z_{02}}{R}r\right)$$



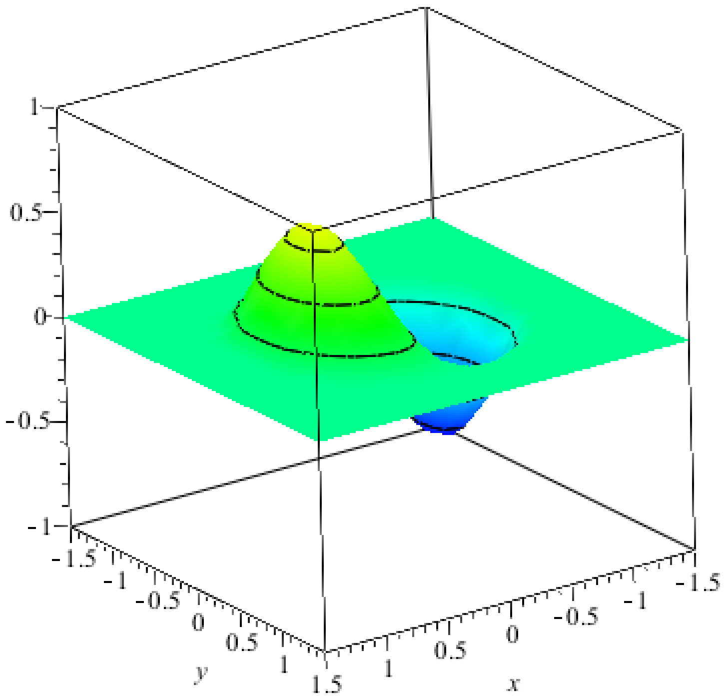




$$k_{01} = \frac{2.406}{R}$$

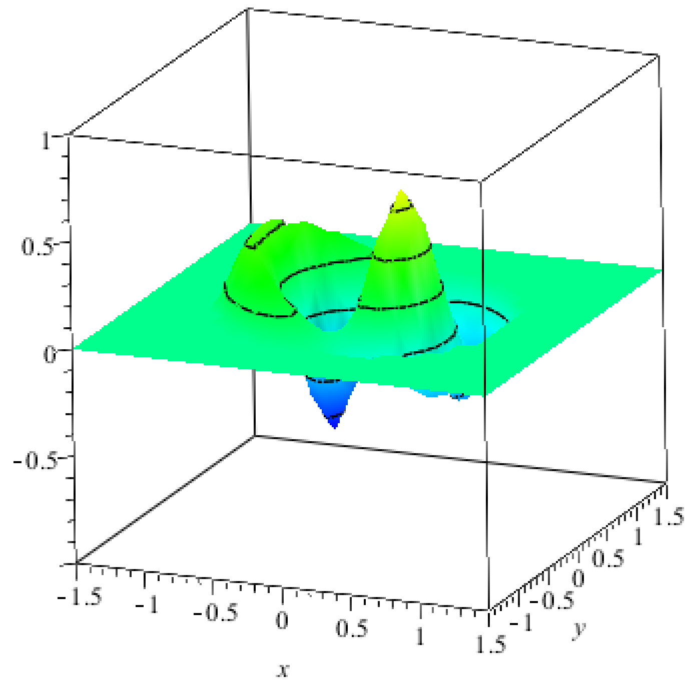
$$k_{02} = \frac{5.520}{R}$$

$$\rho_{11}(r,\varphi) = f_{11}(r)\cos(\varphi)$$
$$= AJ_1\left(\frac{z_{11}}{R}r\right)\cos(\varphi)$$

$$\rho_{12}(r,\varphi) = f_{12}(r)\cos(\varphi)$$
$$= AJ_1\left(\frac{z_{12}}{R}r\right)\cos(\varphi)$$




$$k_{11} = \frac{3.832}{R}$$




$$k_{12} = \frac{7.016}{R}$$

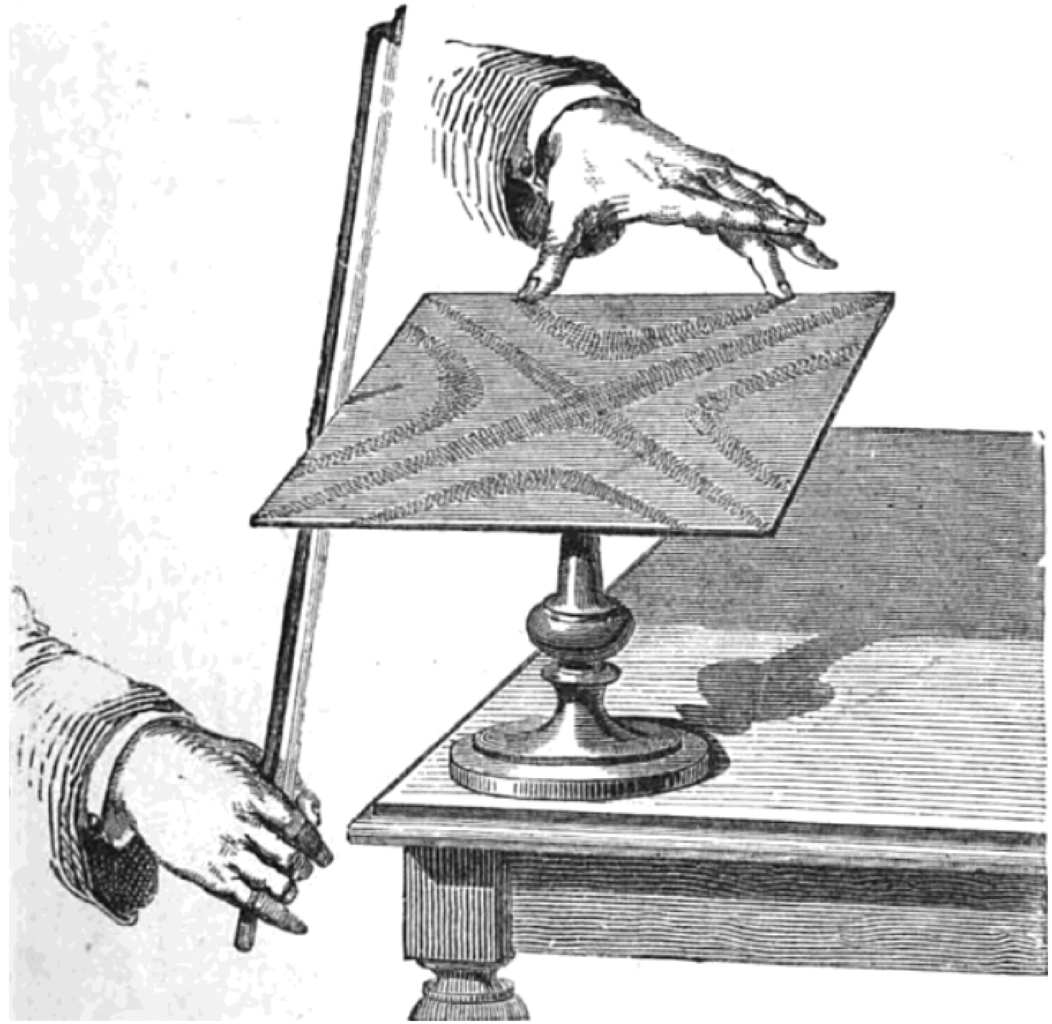
## Ernst Chladni

Ernst Chladni

| | |
|---|---|
| **Born** | 30 November 1756<br>Wittenberg, Electorate of Saxony<br>in the Holy Roman Empire |
| **Died** | 3 April 1827 (aged 70)<br>Breslau, Province of Silesia in the Kingdom of Prussia, a part of the German Confederation |
| **Nationality** | German |
| **Known for** | Study of acoustics<br>Chladni plates and figures<br>Estimating the speed of sound<br>Chladni's law<br>Theory of meteorites' origins |

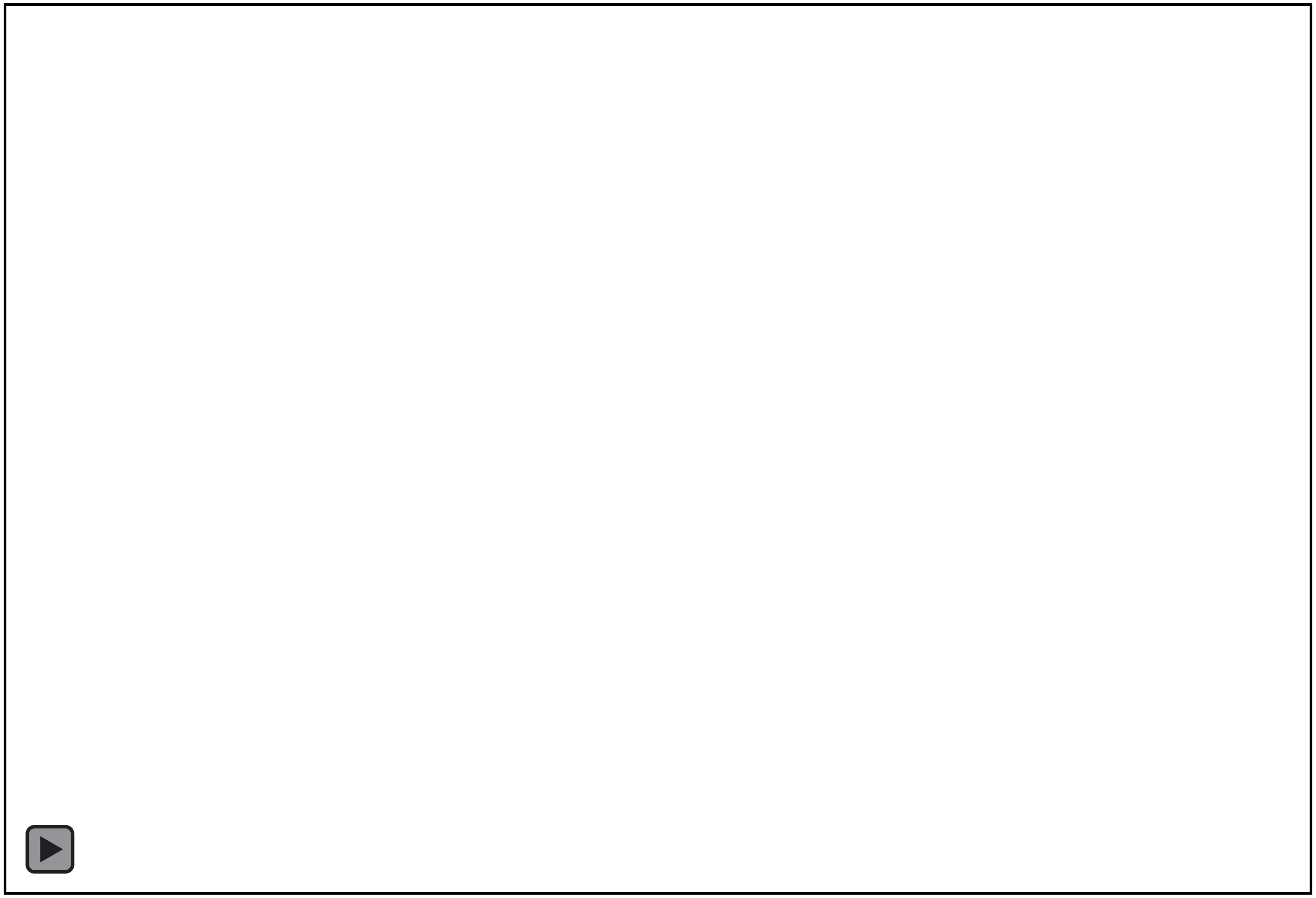
**Scientific career**

| | |
|---|---|
| **Fields** | Physics |

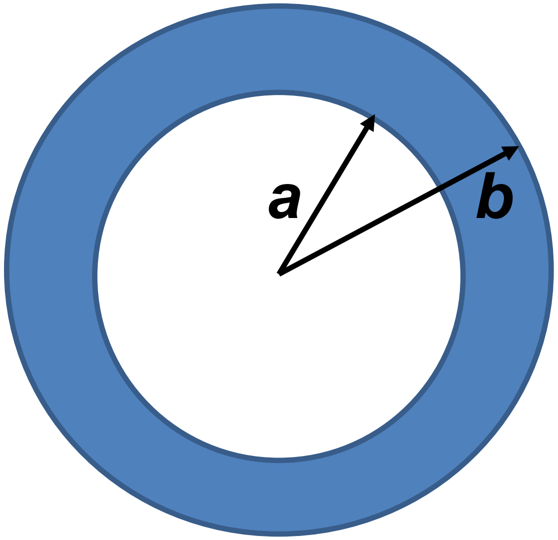


# Demonstration with motor in the middle – (PASCO)

http://www.physics.wfu.edu/resources/education-resources/demo-videos/waves/



# More complicated geometry – annular membrane




In cylindrical coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

Assume: $\rho(r,\varphi) = f(r)\Phi(\varphi)$

Let: $\Phi(\varphi) = e^{im\varphi}$

Note: $\Phi(\varphi) = \Phi(\varphi + 2\pi)$

$$\Rightarrow m = \text{integer}$$

# Consider circular boundary -- continued

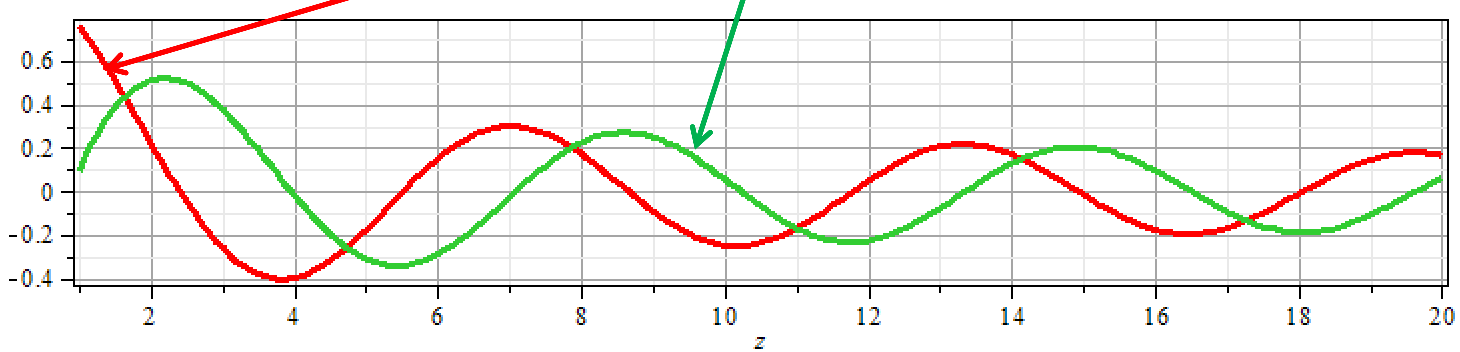
Differential equation for radial function :

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

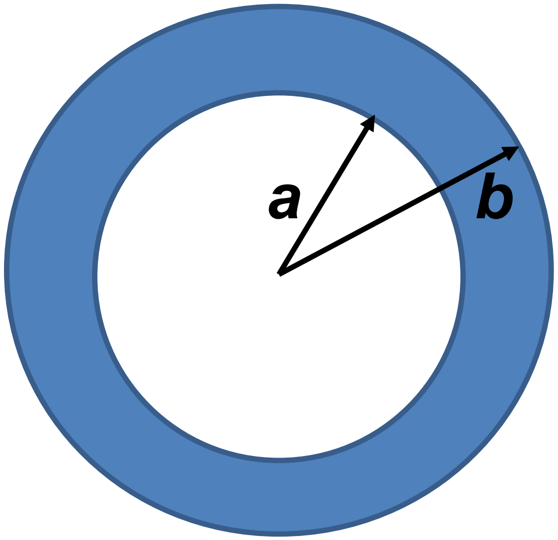
$\Rightarrow$ Bessel equation of integer order with transcendential solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$



# Normal modes of an annular membrane -- continued

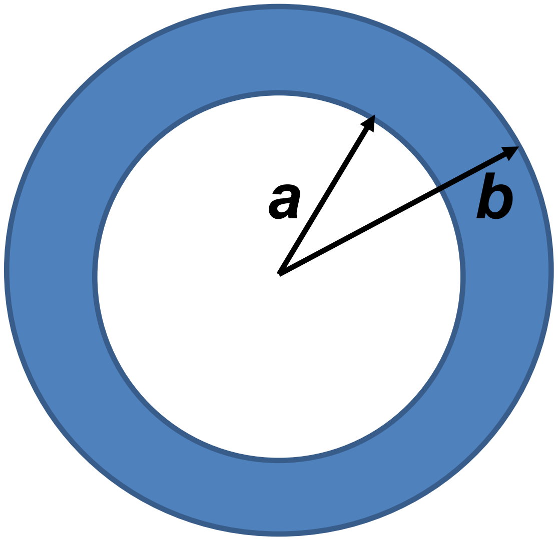



Differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

General form of radial function: $f(r)=AJ_m(kr)+BN_m(kr)$

# Normal modes of an annular membrane -- continued




Boundary conditions:

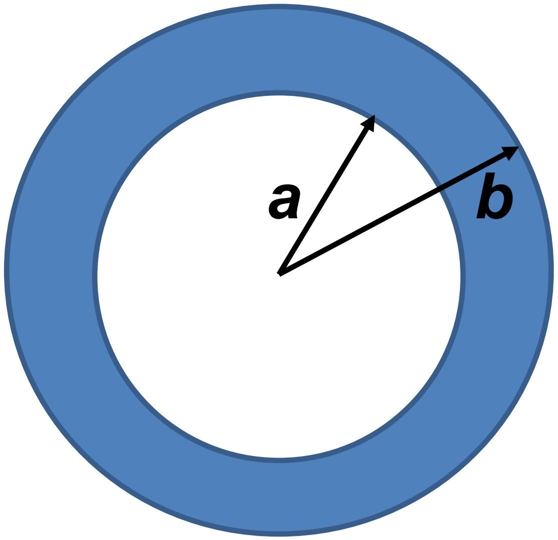
$f(a) = 0 \qquad f(b) = 0$

$$AJ_m(ka) + BN_m(ka) = 0$$
$$AJ_m(kb) + BN_m(kb) = 0$$

$\Rightarrow$ 2 equations and 2 unknowns -- $k$ and $\frac{B}{A}$

$$\frac{B}{A} = \frac{-J_m(ka)}{N_m(ka)} = \frac{-J_m(kb)}{N_m(kb)}$$ (transcendental equation for $k$)

# Normal modes of an annular membrane -- continued




Boundary conditions:

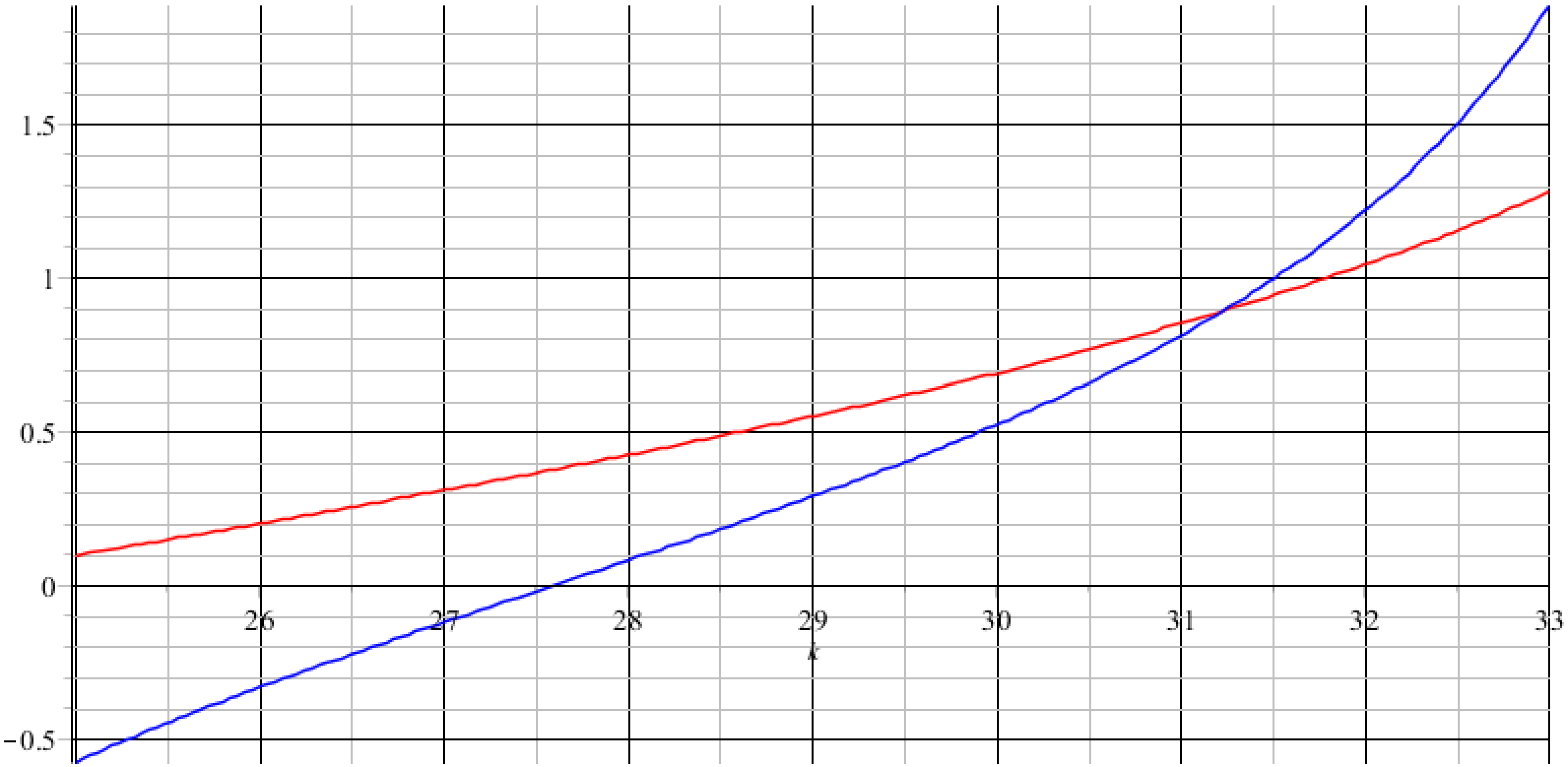
$f(a) = 0 \qquad f(b) = 0$

$$\frac{B}{A} = \frac{-J_m(ka)}{N_m(ka)} = \frac{-J_m(kb)}{N_m(kb)}$$ -- in terms of solution $k_{mn}$ :

$$f(r) = A\left( J_m(k_{mn}r) - \frac{J_m(k_{mn}a)}{N_m(k_{mn}a)} N_m(k_{mn}r) \right)$$

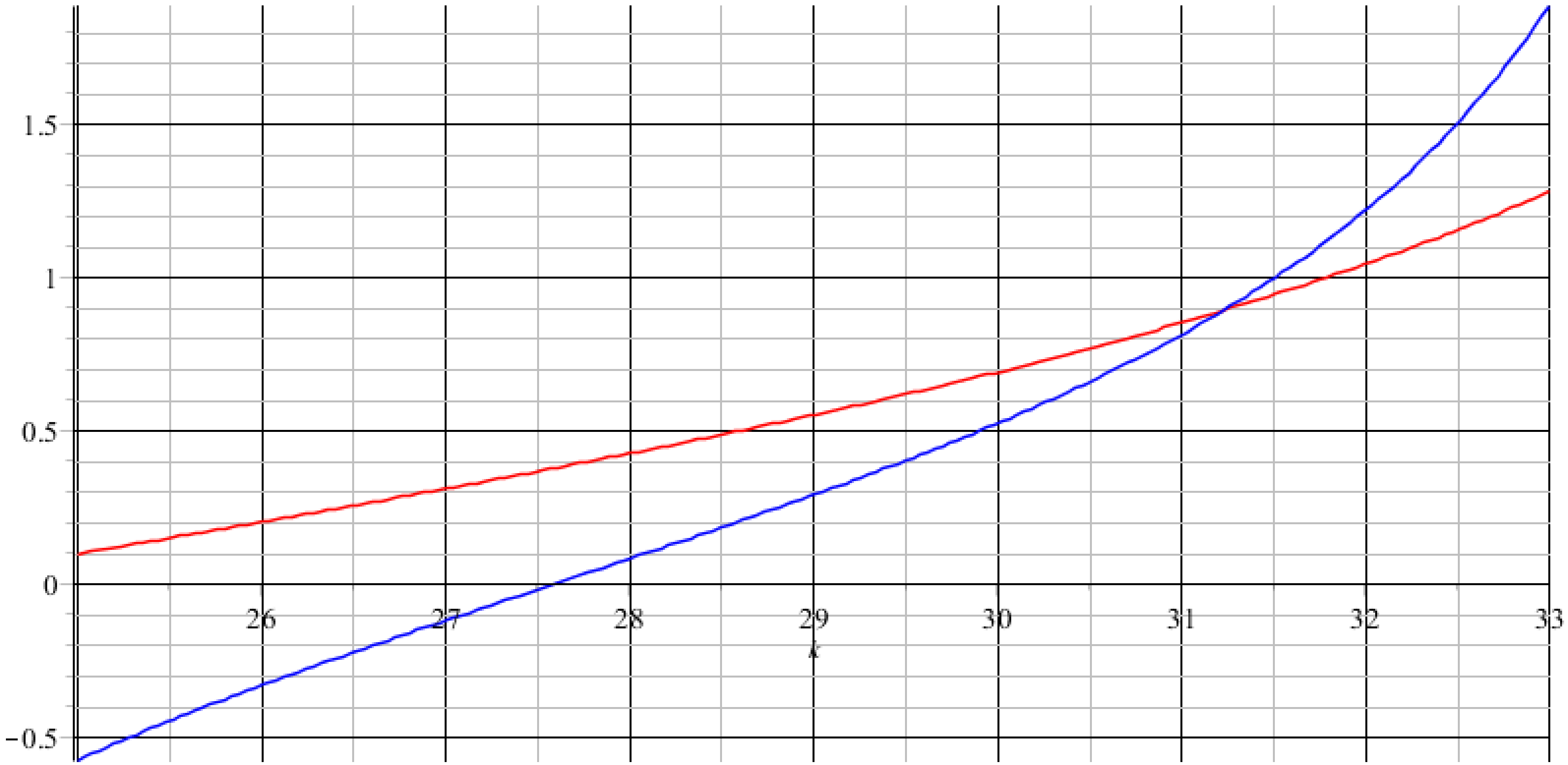
# Analysis for *m=0* and a=0.1, b=0.2:

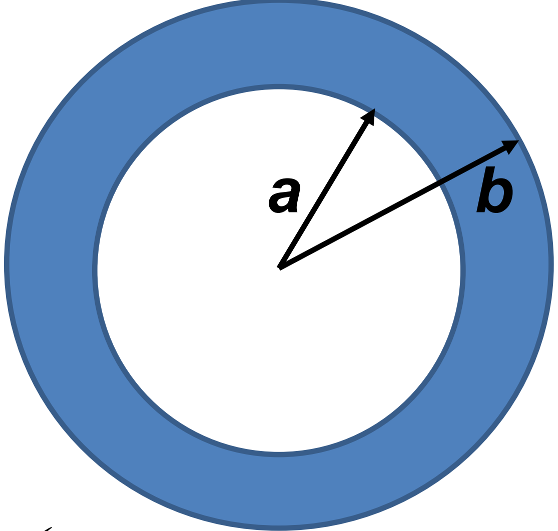
$$> plot\left(\left\{\frac{-\text{BesselJ}(0, 0.1\cdot k)}{\text{BesselY}(0, 0.1\cdot k)}, \frac{-\text{BesselJ}(0, 0.2\cdot k)}{\text{BesselY}(0, 0.2\cdot k)}\right\}, k=25..33, color=[red, blue]\right);$$



$$> fsolve\left( \frac{-\mathrm{BesselJ}(0, 0.1\cdot k)}{\mathrm{BesselY}(0, 0.1\cdot k)} = \frac{-\mathrm{BesselJ}(0, 0.2\cdot k)}{\mathrm{BesselY}(0, 0.2\cdot k)}, k, 30 ..33 \right);$$

31.23030920






$$f(r) = A\left( J_m(k_{mn}r) - \frac{J_m(k_{mn}a)}{N_m(k_{mn}a)} N_m(k_{mn}r) \right) \qquad k_{01} = 31.230309$$