PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 27 – Chap. 9 in F & W

Introduction to hydrodynamics

1. Motivation for topic
2. Newton's laws for fluids
3. Conservation relations

23	Wed, 10/16/2024	Chap. 7	Sturm-Liouville equations	#16
	Fri, 10/18/2024	Fall Break		
24	Mon, 10/21/2024	Chap. 7	Laplace transforms and complex functions	#17
25	Wed, 10/23/2024	Chap. 7	Complex integration	#18
26	Fri, 10/25/2024	Chap. 8	Wave motion in 2 dimensional membranes	#19
27	Mon, 10/28/2024	Chap. 9	Motion in 3 dimensional ideal fluids	#20
28	Wed, 10/30/2024	Chap. 9	Motion in 3 dimensional ideal fluids	
29	Fri, 11/01/2024	Chap. 9	Ideal gas fluids	

PHY 711 -- Assignment #20

Assigned: 10/28/2024 Due: 11/04/2024

Read Chapter 9 in **Fetter & Walecka**.

1. A tank having an area of 100 m^2 is open to the atmosphere and contains 1000 m^3 of water. It has a spigot at it bottom which has a height of 1 m above a drain in the floor. A hose having a diameter of 1 cm is used to empty the water from the tank when the spigot is opened. Using Bernoulli's analysis, estimate the time it takes to empty the tank via the floor drain.

Now is a good time to start thinking about your projects --

November 2024

S	M	T	W	T	F	S
27	28	29	30	31	1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27 Thanksgiving	28 Thanksgiving	29 Thanksgiving	30
1	2	3	4	5	6	7

December 2024

S	M	T	W	T	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	1	2	3	4
5	6	7	8	9	10	11

Presentations?

~ 2 days needed

Presentations?

Last day of class

Take home final

Presentation expectations

- Prepare with powerpoint (or equivalent) for ~ 20 minutes and expect ~ 5 minutes for discussion (2 presentations per day)
- To accommodate all students, we will need 2 days….
- Details listed on webpage http://users.wfu.edu/natalie/f23phy711/info/computational.html

Hydrodynamic analysis

Motivation

1. Natural progression from strings, membranes, fluids; description of 1, 2, and 3 dimensional continua
2. Interesting and technologically important phenomena associated with fluids

Plan

1. Newton's laws within fluids (leaving out dissipative effects for now)
2. Continuity equation
3. Stress tensor
4. Energy relations
5. Bernoulli's theorem
6. Various examples
7. Sound waves

Newton's equations for fluids

Use Euler formulation; following "particles" of fluid

$$\text{Variables:} \quad \text{Density} \quad \rho(x,y,z,t)$$

$$\text{Pressure} \quad p(x,y,z,t)$$

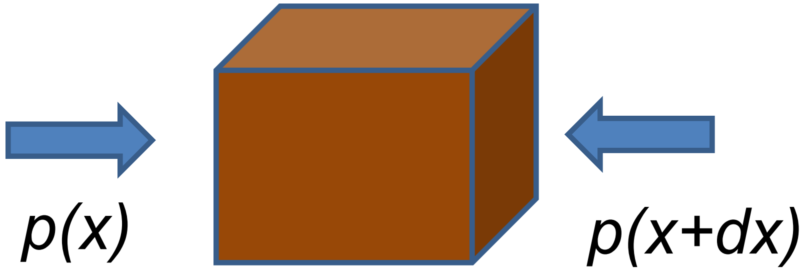
$$\text{Velocity} \quad \mathbf{v}(x,y,z,t)$$

$$m\mathbf{a} = \mathbf{F}$$

$$m \rightarrow \rho dV$$

$$\mathbf{a} \rightarrow \frac{d\mathbf{v}}{dt}$$

$$\mathbf{F} \rightarrow \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$



$$F_{pressure}\Big|_x = \left(-p(x+dx,y,z)+p(x,y,z)\right)dydz$$

$$= \frac{\left(-p(x+dx,y,z)+p(x,y,z)\right)}{dx}dxdydz$$

$$= -\frac{\partial p}{\partial x}dV$$

Newton's equations for fluids -- continued

$$m\mathbf{a} = \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$

$$m = \rho dV$$

$$\rho dV \frac{d\mathbf{v}}{dt} = \mathbf{f}_{applied}\rho dV - (\nabla p)dV$$

$$\mathbf{f}_{\text{applied}} = \frac{\mathbf{F}_{\text{applied}}}{m}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho\mathbf{f}_{applied} - \nabla p$$

$$\mathbf{F}_{\text{pressure}} = -\nabla p dV$$

Detailed analysis of acceleration term:

$$\mathbf{v} = \mathbf{v}(x, y, z, t)$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x}v_x + \frac{\partial \mathbf{v}}{\partial y}v_y + \frac{\partial \mathbf{v}}{\partial z}v_z + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}$$

Note that: $\mathbf{v} \equiv v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = v_x\frac{\partial \mathbf{v}}{\partial x} + v_y\frac{\partial \mathbf{v}}{\partial y} + v_z\frac{\partial \mathbf{v}}{\partial z} = \nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{v}\times(\nabla\times\mathbf{v})$$

Some details --

$$\left(\mathbf{v}\cdot\nabla\right)\mathbf{v} = v_x\frac{\partial\mathbf{v}}{\partial x} + v_y\frac{\partial\mathbf{v}}{\partial y} + v_z\frac{\partial\mathbf{v}}{\partial z} = \nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right)$$

$$v_x\frac{\partial\mathbf{v}}{\partial x} = v_x\left(\frac{\partial v_x}{\partial x}\hat{\mathbf{x}} + \frac{\partial v_y}{\partial x}\hat{\mathbf{y}} + \frac{\partial v_z}{\partial x}\hat{\mathbf{z}}\right) \qquad v_y\frac{\partial\mathbf{v}}{\partial y} = v_y\left(\frac{\partial v_x}{\partial y}\hat{\mathbf{x}} + \frac{\partial v_y}{\partial y}\hat{\mathbf{y}} + \frac{\partial v_z}{\partial y}\hat{\mathbf{z}}\right)$$

$$v_z\frac{\partial\mathbf{v}}{\partial z} = v_z\left(\frac{\partial v_x}{\partial z}\hat{\mathbf{x}} + \frac{\partial v_y}{\partial z}\hat{\mathbf{y}} + \frac{\partial v_z}{\partial z}\hat{\mathbf{z}}\right)$$

$$\nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) = v_x\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_x}{\partial y}\hat{\mathbf{y}} + \frac{\partial v_x}{\partial z}\hat{\mathbf{z}}\right) + v_y\left(\frac{\partial v_y}{\partial x}\hat{\mathbf{x}} + \frac{\partial v_y}{\partial y}\hat{\mathbf{y}} + \frac{\partial v_y}{\partial z}\hat{\mathbf{z}}\right) + v_z\left(\frac{\partial v_z}{\partial x}\hat{\mathbf{x}} + \frac{\partial v_z}{\partial y}\hat{\mathbf{y}} + \frac{\partial v_z}{\partial z}\hat{\mathbf{z}}\right)$$

$$\mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \hat{\mathbf{x}}\left(v_y\frac{\partial v_y}{\partial x} + v_z\frac{\partial v_z}{\partial x} - v_y\frac{\partial v_x}{\partial y} - v_z\frac{\partial v_x}{\partial z}\right) + \hat{\mathbf{y}}\left(v_z\frac{\partial v_z}{\partial y} + v_x\frac{\partial v_x}{\partial y} - v_z\frac{\partial v_y}{\partial z} - v_x\frac{\partial v_y}{\partial x}\right)$$

$$+\,\hat{\mathbf{z}}\left(v_x\frac{\partial v_x}{\partial z} + v_z\frac{\partial v_x}{\partial y} - v_x\frac{\partial v_z}{\partial x} - v_y\frac{\partial v_z}{\partial y}\right)$$

Newton's equations for fluids -- continued

$$\rho\frac{d\mathbf{v}}{dt}=\rho\left(\left(\mathbf{v}\cdot\nabla\right)\mathbf{v}+\frac{\partial\mathbf{v}}{\partial t}\right)=\rho\mathbf{f}_{applied}-\nabla p$$

$$\rho\left(\nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)-\mathbf{v}\times\left(\nabla\times\mathbf{v}\right)+\frac{\partial\mathbf{v}}{\partial t}\right)=\rho\mathbf{f}_{applied}-\nabla p$$

$$\frac{\partial\mathbf{v}}{\partial t}+\nabla\left(\tfrac{1}{2}v^2\right)-\mathbf{v}\times\left(\nabla\times\mathbf{v}\right)=\mathbf{f}_{applied}-\frac{\nabla p}{\rho}$$

Solution in the case of irrotational flow.

$$\text{Irrotational flow:} \quad \nabla \times \mathbf{v} = 0$$

$$\nabla \times \mathbf{v} = \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Which of the following vector functions have zero curl?

a. $\mathbf{v} = C\hat{\mathbf{x}}$ (C is a constant)

b. $\mathbf{v} = Cx\hat{\mathbf{x}}$

c. $\mathbf{v} = Cy\hat{\mathbf{x}}$

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t}+\nabla\left(\tfrac{1}{2}v^2\right)-\mathbf{v}\times\left(\nabla\times\mathbf{v}\right)=\mathbf{f}_{applied}-\frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $\left(\nabla\times\mathbf{v}\right)=0$ "irrotational flow"

$$\Rightarrow \mathbf{v}=-\nabla\Phi \quad \Phi \text{ is "velocity potential"}$$

2. $\mathbf{f}_{applied}=-\nabla U$ conservative applied force

3. $\rho=$ (constant) incompressible fluid

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t}+\nabla\left(\tfrac{1}{2}v^2\right)=-\nabla U-\frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Bernoulli's integral of Euler's equation for irrotational and incompressible fluid

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Integrating over space:

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C(t)$$

where $\quad \mathbf{v}=-\nabla\Phi(\mathbf{r},t)=-\nabla\big(\Phi(\mathbf{r},t)+C'(t)\big)$

$$\Rightarrow \frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=0 \qquad \text{Bernoulli's theorem}$$

Examples of Bernoulli's theorem

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial \Phi}{\partial t} = 0$$

Modified form; assuming $\dfrac{\partial \Phi}{\partial t} = 0$

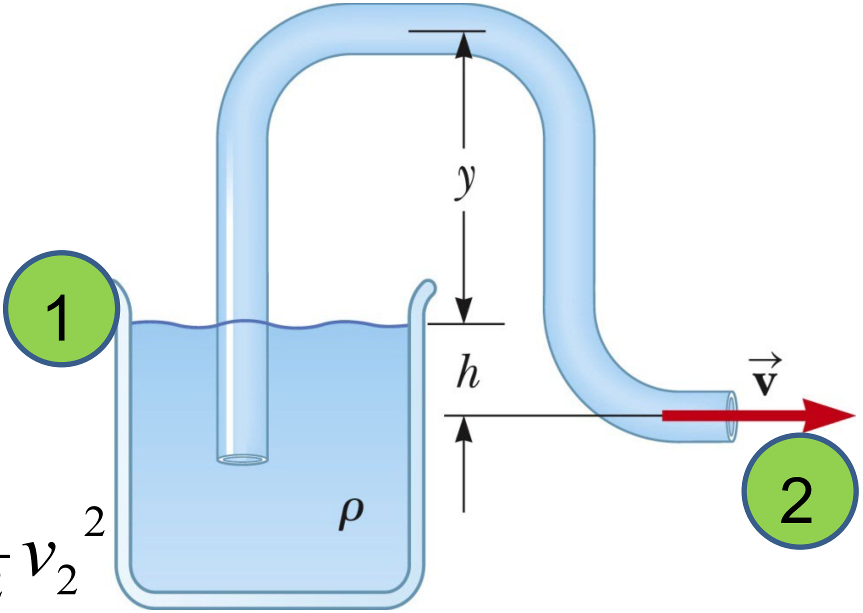
$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$

$$p_1 = p_2 = p_{atm}$$

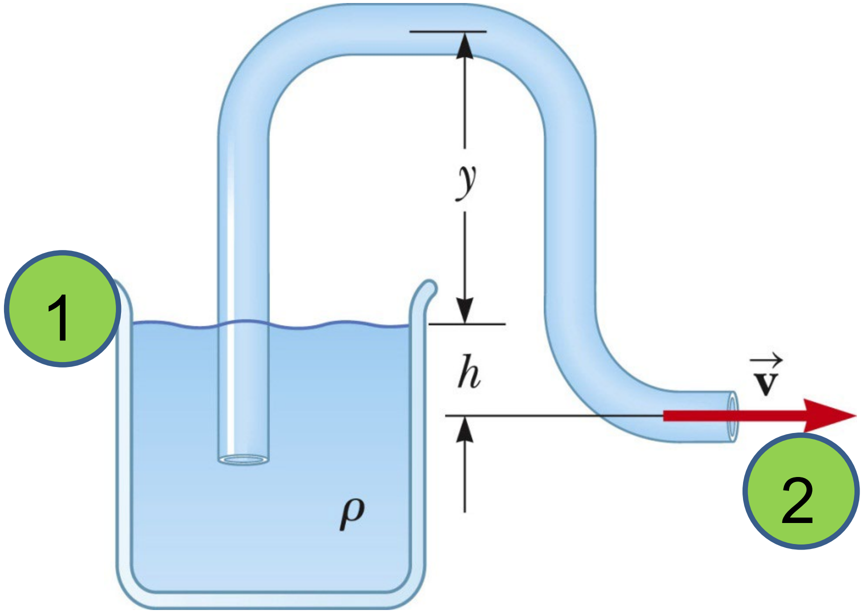
$$U_1 - U_2 = gh$$

$$v_1 \approx 0$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}{v_1}^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}{v_2}^2$$



Examples of Bernoulli's theorem -- continued



$$p_1 = p_2 = p_{atm}$$

$$U_1 - U_2 = gh$$

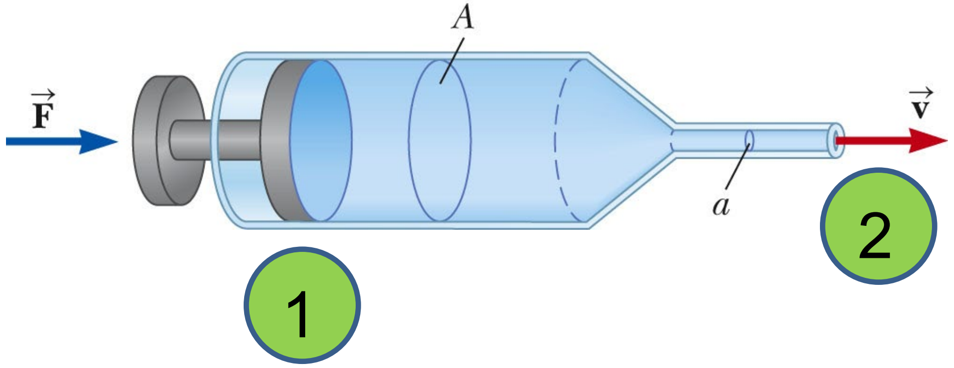
$$v_1 \approx 0$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}v_1^{\,2} = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}v_2^{\,2}$$

$$v_2 \approx \sqrt{2gh}$$

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$



$$p_1 = \frac{F}{A} + p_{atm} \qquad\qquad p_2 = p_{atm}$$

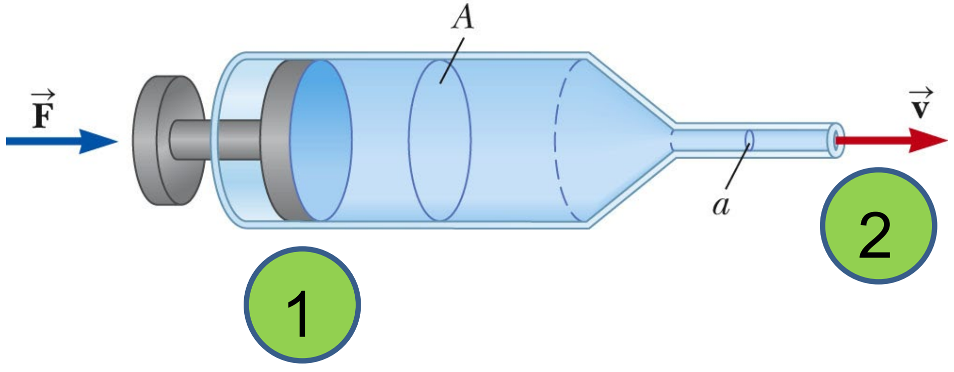
$$U_1 = U_2$$

$$v_1 A = v_2 a \qquad \text{continuity equation}$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}v_1^{\,2} = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}v_2^{\,2}$$

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$



$$\frac{2F}{A} = v_2^{\,2}\left(1 - \left(\frac{a}{A}\right)^2\right)$$

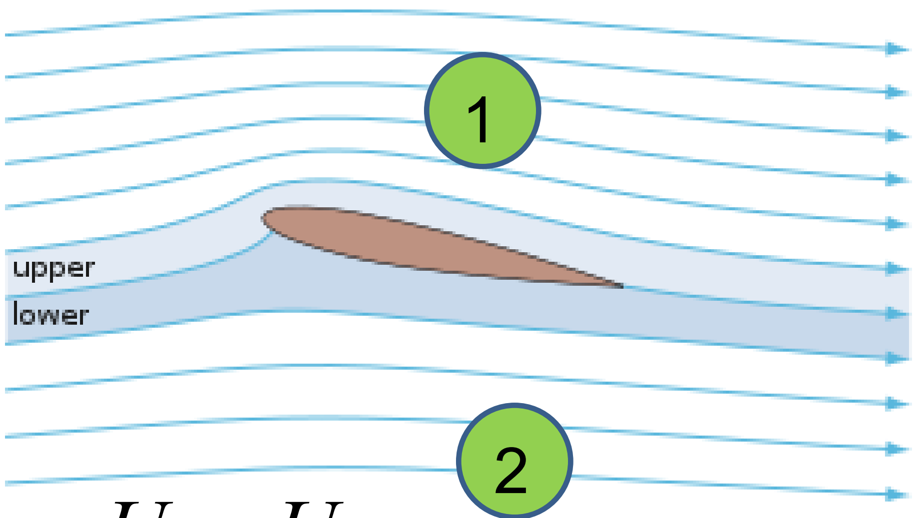
$$v_2 = \sqrt{\frac{2F/A}{1-\left(\frac{a}{A}\right)^2}}$$

Examples of Bernoulli's theorem – continued

Approximate explanation of airplane lift

Cross section view of airplane wing

http://en.wikipedia.org/wiki/Lift_%28force%29



$$U_1 \approx U_2$$

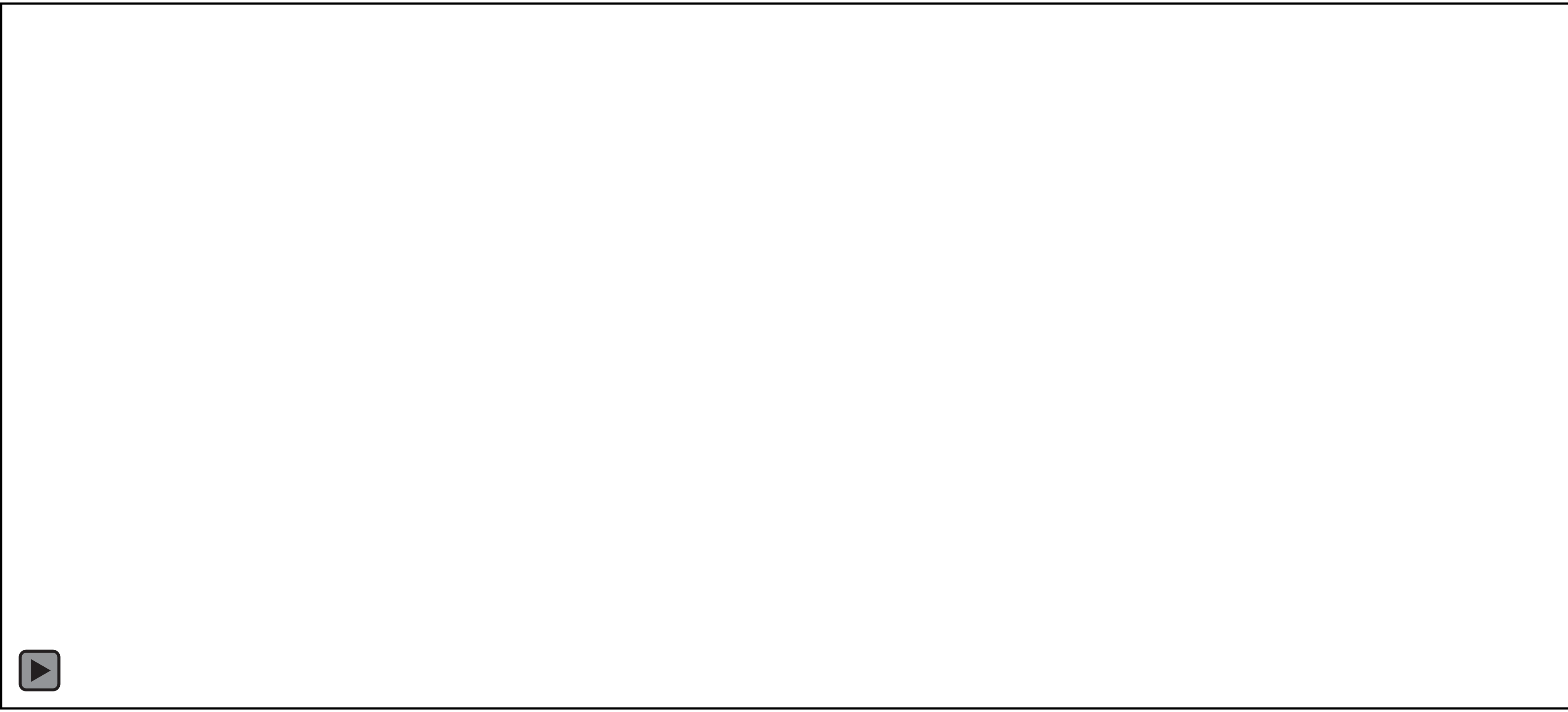
$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2} v_1^{\ 2} = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2} v_2^{\ 2}$$

$$p_2 - p_1 = \tfrac{1}{2}\rho\left(v_1^{\ 2} - v_2^{\ 2}\right)$$

Question -- What aspects do over simplified Bernoulli's equation not include in studying fluid dynamics?

According to a Scientific American article, the conclusion that $v_2>v_1$ because of the shape of the airplane wing is not quite true and because viscosity has an important effect. Numerical modeling reveal a more complicated picture.

https://www.scientificamerican.com/article/no-one-can-explain-why-planes-stay-in-the-air/



At NASA Ames Fluid Mechanics Laboratory, streamlines of dye in a water channel interact with a model airplane. Credit: *Ian Allen* (copied from Scientific American page mentioned above).

Continuity equation connecting fluid density and velocity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \right) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \left(\nabla \cdot \mathbf{v} \right) + \left(\nabla \rho \right) \cdot \mathbf{v} = 0$$

Consider: $$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \left(\nabla \rho \right) \cdot \mathbf{v}$$

$$\Rightarrow \frac{d\rho}{dt} + \rho \left(\nabla \cdot \mathbf{v} \right) = 0$$

alternative form

of continuity equation

Some details on the velocity potential

Continuity equation :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

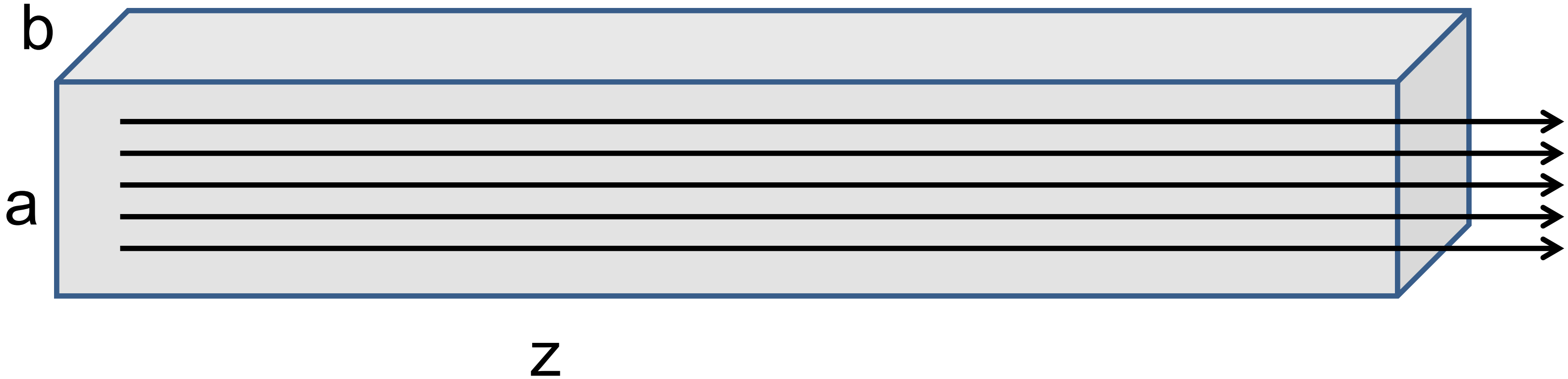
For incompressible fluid : $\rho = (\text{constant})$

$$\Rightarrow \nabla \cdot \mathbf{v} = 0$$

Irrotational flow : $\nabla \times \mathbf{v} = 0 \qquad \Rightarrow \mathbf{v} = -\nabla \Phi$

$$\Rightarrow \nabla^2 \Phi = 0$$

Example – uniform flow



$$\nabla^2\Phi = 0$$

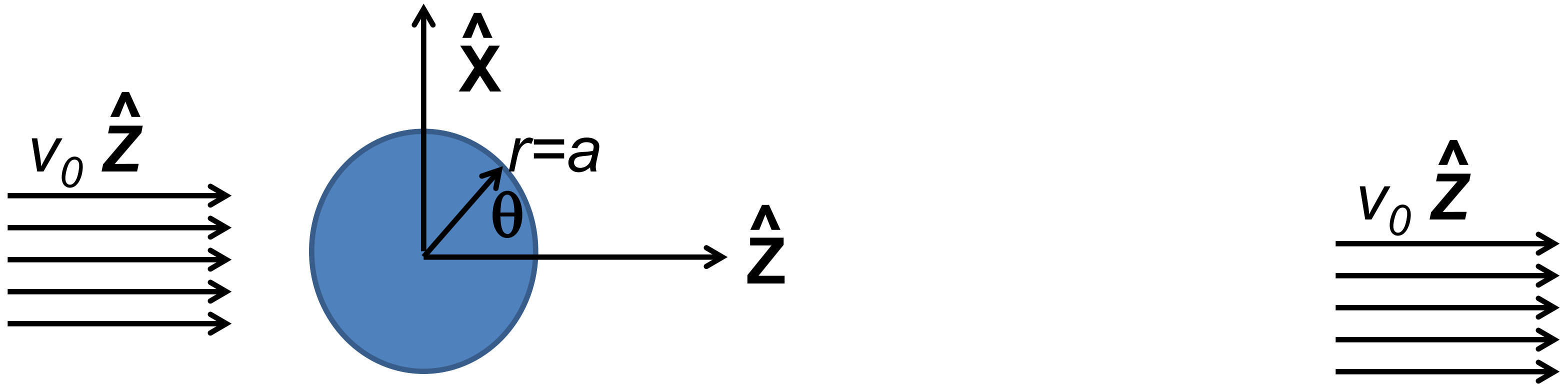
$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

$$\text{Possible solution :}$$

$$\Phi = -v_o z$$

$$\mathbf{v} = -\nabla\Phi = v_o\hat{\mathbf{z}}$$

Example – flow around a long cylinder (oriented in the *Y* direction)



$$\nabla^2\Phi = 0$$

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 0$$

Laplace equation in cylindrical coordinates

($r, \theta,$ defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r, \theta, y) = \Phi(r, \theta)$$

From boundary condition : $v_z(r \to \infty) = v_0$

$$\frac{\partial \Phi}{\partial z}(r \to \infty) = -v_0 \qquad \Rightarrow \Phi(r \to \infty, \theta) = -v_0 r \cos\theta$$

Note that : $\dfrac{\partial^2 \cos\theta}{\partial \theta^2} = -\cos\theta$

Guess form : $\Phi(r, \theta) = f(r)\cos\theta$

Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial f}{\partial r} - \frac{1}{r^2} f = 0$$

$$f(r) = Ar + \frac{B}{r} \qquad \text{where } A, B \text{ are constants}$$

Boundary condition on cylinder surface:

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

$$\frac{df}{dr}(r = a) = 0 = A - \frac{B}{a^2}$$

$$\Rightarrow B = Aa^2$$

Boundary condition at ∞: $\quad \Rightarrow A = -v_0$

$$\Phi(r,\theta) = -v_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$v_r = -\frac{\partial \Phi}{\partial r} = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = v_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \Phi}{\partial \varphi^2}$$

to be continued …