PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 28 -- Chap. 9 in F & W

Introduction to hydrodynamics

1. Newton's laws for fluids and the continuity equation
2. Irrotational and incompressible fluids
3. Irrotational and isentropic fluids
4. Approximate solutions in the linear limit – next time

Physics Colloquium

4 PM
- Thursday -
Olin 101
October 31, 2024

From stable trapping to turbulence: Nonlinear dynamics in gravitational systems

Turbulence is a universal feature in many nonlinear systems, with broad applications across physics, from classical fluid mechanics to gravitational dynamics. Understanding the dynamics of spacetimes with stable trapping—regions where energy can be confined for long times—is crucial for exploring complex gravitational systems. In this talk, I will describe the behavior of nonlinear scalar waves on a simple model spacetime that admits such trapping. While linear waves exhibit slow decay, we show that nonlinear waves lead to turbulent behavior due to energy cascading to higher modes. Our results offer insights into the stability of exotic spacetimes in general relativity, potentially ruling out certain types of instability.

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23	Wed, 10/16/2024	Chap. 7	Sturm-Liouville equations	#16
	Fri, 10/18/2024	Fall Break		
24	Mon, 10/21/2024	Chap. 7	Laplace transforms and complex functions	#17
25	Wed, 10/23/2024	Chap. 7	Complex integration	#18
26	Fri, 10/25/2024	Chap. 8	Wave motion in 2 dimensional membranes	#19
27	Mon, 10/28/2024	Chap. 9	Motion in 3 dimensional ideal fluids	#20
28	Wed, 10/30/2024	Chap. 9	Motion in 3 dimensional ideal fluids	#21
29	Fri, 11/01/2024	Chap. 9	Ideal gas fluids	

PHY 711 -- Assignment #21

Assigned: 10/30/2024 Due: 11/04/2024

Continue reading Chapter 9 in **Fetter & Walecka**.

1. Consider the example discussed in Lecture 27 & 28, concerning the flow of an incompressible fluid in the **z** direction in the presence of a stationary cylindrical log oriented in the **y** direction. For this homework problem, the log is replaced by a stationary sphere. Find the velocity potential for this case, using the center of the sphere as the origin of the coordinate system and spherical polar coordinates.

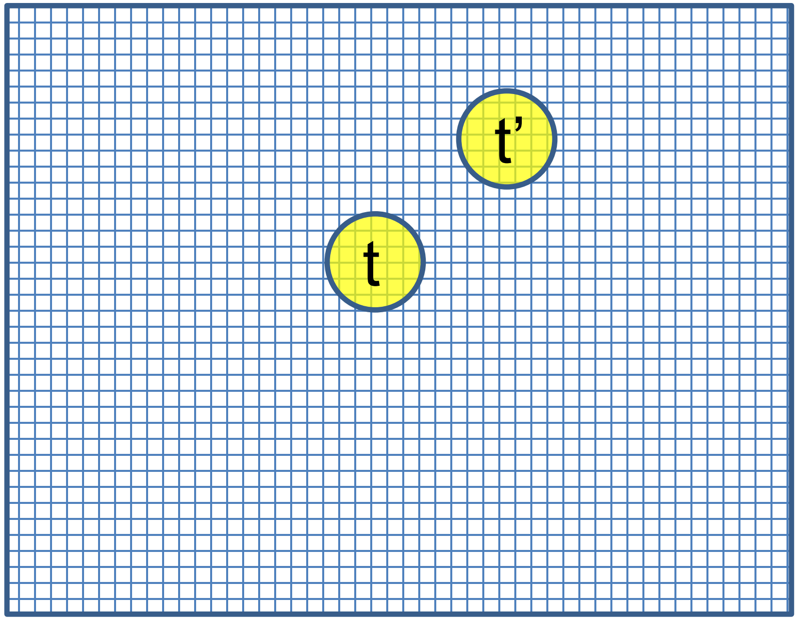
Newton's equations for fluids

Use **Euler** formulation; properties described in terms of stationary spatial grid

$$\text{Variables:} \quad \text{Density} \quad \rho(x,y,z,t)$$

$$\text{Pressure} \quad p(x,y,z,t)$$

$$\text{Velocity} \quad \mathbf{v}(x,y,z,t)$$



$$\text{Particle at } t: \quad \mathbf{r}, t$$

$$\text{Particle at } t': \quad \mathbf{r} + \mathbf{v}\,\delta t, t'$$

$$t' = t + \delta t$$

Euler analysis -- continued

Particle at t : $\quad \mathbf{r},t$

Particle at t': $\quad \mathbf{r}+\mathbf{v}\delta t, t' \qquad$ where $\quad t' = t + \delta t$

For $\; f(\mathbf{r},t)$:

$$\frac{df}{dt} = \lim_{\delta t \to 0}\left(\frac{f(\mathbf{r},t') - f(\mathbf{r},t)}{\delta t} + \frac{f(\mathbf{r}+\mathbf{v}\delta t, t) - f(\mathbf{r},t)}{\delta t} \right)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v}\cdot\nabla) f$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f$$

For $f \to v_x$ $\quad \frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + (\mathbf{v} \cdot \nabla) v_x$

For $f \to v_y$ $\quad \frac{dv_y}{dt} = \frac{\partial v_y}{\partial t} + (\mathbf{v} \cdot \nabla) v_y$

For $f \to v_z$ $\quad \frac{dv_z}{dt} = \frac{\partial v_z}{\partial t} + (\mathbf{v} \cdot \nabla) v_z$

In vector form $\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v}$

Note that $(\mathbf{v} \cdot \nabla) \mathbf{v} = \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right) \left(v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}} \right)$

$$\mathbf{v} = \mathbf{v}(x, y, z, t)$$

In vector form $\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}$

Note that $(\mathbf{v}\cdot\nabla)\mathbf{v} = \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right)\left(v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}} \right)$

$$= \frac{1}{2}\nabla|\mathbf{v}|^2 - \mathbf{v}\times(\nabla\times\mathbf{v})$$

For example, applying this analysis to Newton's equation of motion for fluids:

$$m\mathbf{a} = \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$

$$m = \rho dV$$

$$\rho dV \frac{d\mathbf{v}}{dt} = \mathbf{f}_{applied}\,\rho dV - (\nabla p) dV$$

$$\mathbf{f}_{\text{applied}} = \frac{\mathbf{F}_{\text{applied}}}{m}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{applied} - \nabla p$$

$$\mathbf{F}_{\text{pressure}} = -\nabla p dV$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = \rho \mathbf{f}_{applied} - \nabla p$$

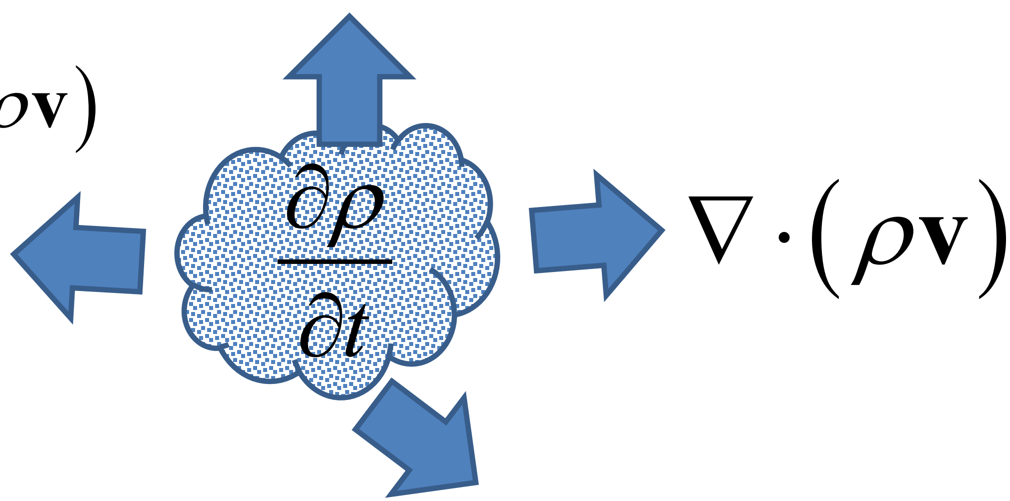
Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

The notion of the continuity is a common feature of continuous closed systems. Here we assume that there are no mechanisms for creation or destruction of the fluid.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})$$



Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \right) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \left(\nabla \cdot \mathbf{v} \right) + \left(\nabla \rho \right) \cdot \mathbf{v} = 0$$

velocity potential

For incompressible fluid: $\rho = \left(\text{constant} \right)$

$$\Rightarrow \nabla \cdot \mathbf{v} = 0$$

Irrotational flow: $\nabla \times \mathbf{v} = 0 \qquad \Rightarrow \mathbf{v} = -\nabla \Phi$

For irrotational flow of an incompressible fluid: $\nabla^2 \Phi = 0$

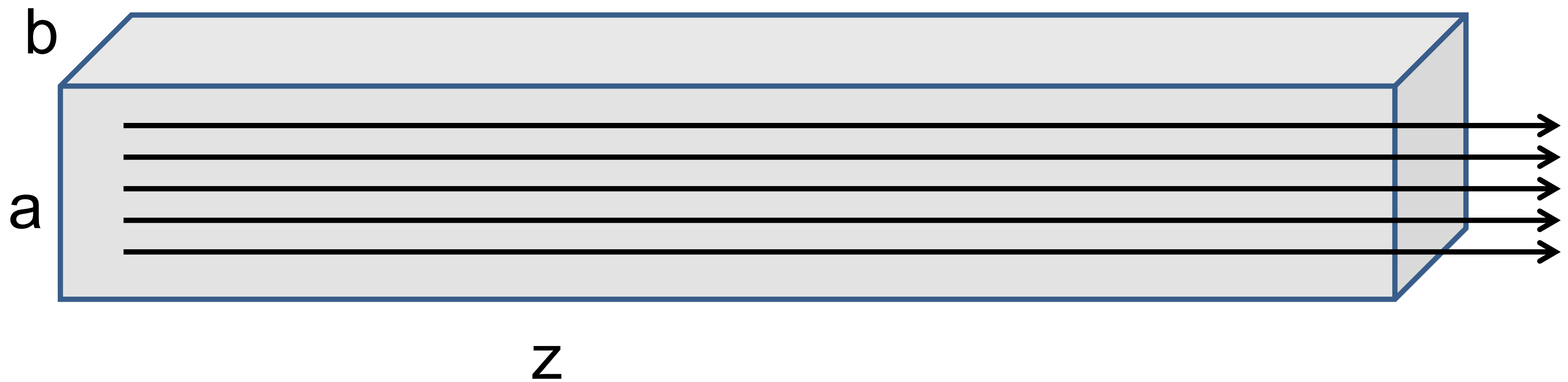
Checking --

Why does $\nabla \times \mathbf{v} = 0$ imply that $\mathbf{v} = -\nabla\Phi$?

Consider: $\nabla\Phi = \frac{\partial\Phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\Phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\Phi}{\partial z}\hat{\mathbf{z}}$

$\nabla \times \left(\nabla\Phi\right)\Big|_x = \frac{\partial^2\Phi}{\partial y \partial z} - \frac{\partial^2\Phi}{\partial z \partial y} = 0$ Similar results for other directions.

Example of irrotational flow of an incompressible fluid – uniform flow



$$\nabla^2\Phi = 0$$

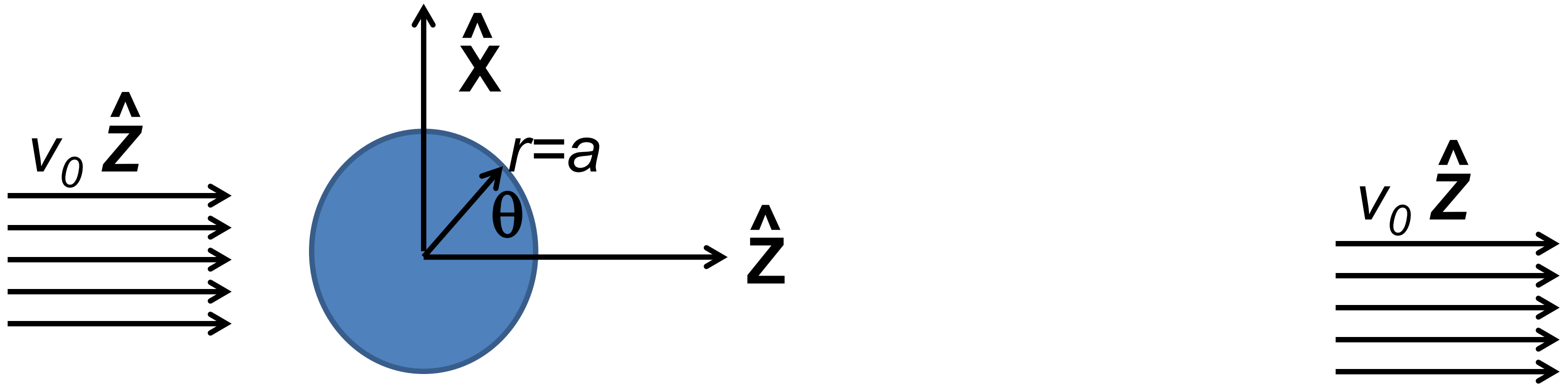
$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

Possible solution :

$$\Phi = -v_o z$$

$$\mathbf{v} = -\nabla\Phi = v_o\hat{\mathbf{z}}$$

Example – flow around a long cylinder (oriented in the $\boldsymbol{Y}$ direction)



$$\nabla^2\Phi = 0$$

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

Laplace equation in cylindrical coordinates

($r, \theta,$ defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r,\theta,y) = \Phi(r,\theta)$$

From boundary condition : $v_z(r \to \infty) = v_0$

$$\frac{\partial \Phi}{\partial z}(r \to \infty) = -v_0 \qquad \Rightarrow \Phi(r \to \infty, \theta) = -v_0 r \cos\theta$$

Note that : $\dfrac{\partial^2 \cos\theta}{\partial \theta^2} = -\cos\theta$

Guess form : $\Phi(r,\theta) = f(r)\cos\theta$

Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial f}{\partial r} - \frac{1}{r^2} f = 0$$

$$f(r) = Ar + \frac{B}{r}$$ where A, B are constants

Boundary condition on cylinder surface:

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

$$\frac{df}{dr}(r = a) = 0 = A - \frac{B}{a^2}$$

$$\Rightarrow B = Aa^2$$

Boundary condition as $r \rightarrow \infty$: $\Rightarrow A = -v_0$

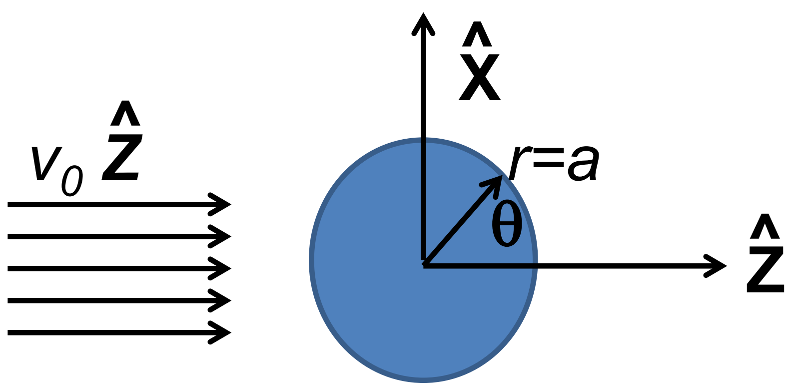
$$\Phi(r,\theta) = -v_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

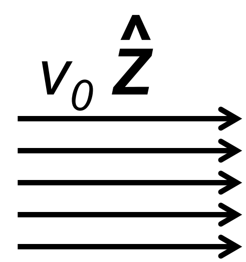
$$v_r = -\frac{\partial \Phi}{\partial r} = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -v_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

For $r \to \infty$

$$\mathbf{v} \to v_0 \cos\theta \hat{\mathbf{r}} - v_0 \sin\theta \hat{\boldsymbol{\theta}} = v_0 \hat{\mathbf{z}}$$





Now consider the case of your homework problem --

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

Spherical system continued:

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

In terms of spherical harmonic functions:

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y_{lm}(\theta,\phi) = -l(l+1)Y_{lm}(\theta,\phi)$$

In our case:

$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$\Phi(r,\theta,\phi) = f(r)Y_{lm}(\theta,\phi)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - \frac{l(l+1)}{r^2}f = 0$$

(Continue analysis for homework)

Summary -- Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla \times \mathbf{v}) = 0$ "irrotational flow"

$$\Rightarrow \mathbf{v} = -\nabla \Phi$$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho = (\text{constant})$ incompressible fluid

$$\frac{\partial(-\nabla\Phi)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial \Phi}{\partial t}\right) = 0$$

For incompressible fluid

Bernoulli's integral of Euler's equation for constant ρ

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Integrating over space:

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C(t)$$

where $\quad \mathbf{v}=-\nabla\Phi(\mathbf{r},t)=-\nabla\big(\Phi(\mathbf{r},t)+C(t)\big)$

It is convenient to modify $\Phi(\mathbf{r},t)\rightarrow\Phi(\mathbf{r},t)+\int^{t}C(t')dt'$

$$\Rightarrow\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=0 \qquad \text{Bernoulli's theorem}$$

Extension of these ideas to some compressible fluids – now assuming conditions of constant entropy (no heat transfer).

Under what circumstances can there be no heat transfer?

Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $\left(\nabla\times\mathbf{v}\right) = 0$ "irrotational flow"

$$\Rightarrow \mathbf{v} = -\nabla\Phi$$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho \neq$ (constant) isentropic fluid

A little thermodynamics

Comment about sign convention

First law of thermodynamics: $dE_{\text{int}} = dQ - dW$

For isentropic conditions: $dQ = 0$

$$dE_{\text{int}} = -dW = -pdV$$

Solution of Euler's equation for fluids – isentropic (continued)

$$dE_{\text{int}} = -dW = -pdV$$

In terms of mass density: $\rho = \dfrac{M}{V}$

For fixed M and variable V: $d\rho = -\dfrac{M}{V^2}dV$

$$dV = -\frac{M}{\rho^2}d\rho$$

In terms in intensive variables: Let $E_{\text{int}} = M\varepsilon$

$$dE_{\text{int}} = Md\varepsilon = -dW = -pdV = M\frac{p}{\rho^2}d\rho$$

$$d\varepsilon = \frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

$$\text{Consider:} \quad \nabla \varepsilon = \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} \nabla \rho = \frac{p}{\rho^2} \nabla \rho$$

$$\text{Rearranging:} \quad \nabla\left(\varepsilon + \frac{p}{\rho}\right) = \frac{\nabla p}{\rho}$$

Is this useful?

a. Yes

b. No

Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

$$\frac{\nabla p}{\rho} = \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

Now suppose we have the additional conditions:

$$\nabla\times\mathbf{v} = 0 \qquad \mathbf{v} = -\nabla\Phi \qquad \mathbf{f}_{applied} = -\nabla U$$

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Summary of Bernoulli's results for irrotational fluids

For incompressible fluid

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

For isentropic fluid with internal energy density ε

$$\nabla\left(\varepsilon+\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Here ε is the internal energy of the fluid per unit mass. For an ideal gas fluid, it has a relatively simple form.

For isentropic fluid with internal energy density ε

$$\nabla\left(\varepsilon+\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Here ε is the internal energy of the fluid per unit mass. In order to continue, we need to know the form of

$$\varepsilon(\rho,s)$$

Where *s* denotes entropy per unit mass.

Example – ideal gas fluid

Equation of state for ideal gas:

$$pV = NkT \qquad\qquad N = \frac{M}{M_0}$$

$$p = \frac{M}{V}\frac{k}{M_0}T = \rho\frac{k}{M_0}T$$

$$k = 1.38\times 10^{-23}\, J/K$$

$M_0 =$ average mass of each molecule

Internal energy for ideal gas: in terms of f (degrees of freedom)

$$E = \frac{f}{2} NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

In terms of specific heat ratio: $\gamma \equiv \frac{C_p}{C_V}$

$$dE = dQ - dW$$

$$C_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{f}{2}\frac{Mk}{M_0}$$

$$C_p = \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = \frac{f}{2}\frac{Mk}{M_0} + \frac{Mk}{M_0}$$

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{f}{2}+1}{\frac{f}{2}} \qquad \Rightarrow \frac{f}{2} = \frac{1}{\gamma - 1}$$

Digression

Internal energy for ideal gas: $\quad f \equiv$ "degrees of freedom"

$$E = \frac{f}{2} NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

$$\frac{f}{2} = \frac{1}{\gamma - 1} \quad \Rightarrow \quad E = \frac{1}{\gamma-1} NkT \qquad \varepsilon = \frac{1}{\gamma-1}\frac{k}{M_0}T = \frac{1}{\gamma-1}\frac{p}{\rho}$$

	f	γ
Spherical atom	3	1.66667
Diatomic molecule	5	1.40000

Internal energy for ideal gas :

$$E = \frac{1}{\gamma - 1} NkT = M\varepsilon \qquad \varepsilon = \frac{1}{\gamma - 1}\frac{k}{M_0}T = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

Internal energy for ideal gas under isentropic conditions:

$$d\varepsilon = -\frac{p}{M}dV = \frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial \varepsilon}{\partial \rho}\right)_s = \frac{p}{\rho^2} = \frac{\partial}{\partial \rho}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right)_s$$

$$\frac{p}{\rho^2} = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{1}{(\gamma - 1)\rho} - \frac{p}{(\gamma - 1)\rho^2} \qquad \Rightarrow \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

Next time, we will use these results to analyze the motion of an isentropic and irrotational ideal gas.

$$\nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial \Phi}{\partial t}\right) = 0$$