



# **PHY 711 Classical Mechanics and Mathematical Methods**

**10-10:50 AM MWF in Olin103**

**Lecture notes for Lecture 2  
Chapter 3.17 of F&W**

**Introduction to the calculus of variations**

- 1. Mathematical construction**
- 2. Practical use**
- 3. Examples**



# Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/26/2024		Introduction and overview	<a href="#">#1</a>
2	Wed, 8/28/2024	Chap. 3(17)	Calculus of variation	<a href="#">#2</a>
3	Fri, 8/30/2024	Chap. 3(17)	Calculus of variation	
4	Mon, 9/02/2024	Chap. 3	Lagrangian equations of motion	



## PHY 711 -- Assignment #2

Assigned: 8/28/2024 Due: 9/2/2024

Start reading Chapter 3, especially Section 17, in **Fetter & Walecka**.

- Using calculus of variations, find the equation,  $y(x)$ , of the shortest length "curve" which passes through the points  $(x=0, y=0)$  and  $(x=2, y=4)$ . What is the length of this "curve"?

# PHYSICS COLLOQUIUM

THURSDAY

4 PM

AUGUST 29, 2024

Olin 101

Refreshments at 3:30 PM  
Olin Lobby

## Exploring Alternative Origins of Life with Genes and Proteins Designed De Novo

All life on Earth arose from common ancestry. And all living systems contain similar genes, proteins, and metabolisms. Do these commonalities among living systems exist because they diverged from shared ancestry? Alternatively, if living systems arose from different ancestry (or no ancestry), would they contain different genes, proteins, and metabolisms? There are two ways to address this question: (i) Find life on another planet (different ancestry); or (ii) Create novel biomolecules and metabolisms in the laboratory (no ancestry). Our lab does not have the budget for the first option; we'll leave that to NASA. Instead, the Hecht Lab creates vast collections of synthetic genes to encode novel (non-natural) proteins. Many of these novel proteins fold into stable 3-dimensional structures. Moreover, many of them bind biologically relevant metals, metabolites, and cofactors. Most



Your questions –

From **Thomas** – At the end of slide 22 I am confused why you set  $dy/dx=K'$  and why that is a useful trick?

From **Conall** – Calculus of Variation- when trying to minimize  $y(x)$ , can we always utilize the Euler- Lagrange equation...or do these only apply under specific conditions?

For a later lecture --

a. Holonomic Constraints vs non-holonomic constraints. ...holonomic constraints are constraints of motion that can be expressed with an equation that relates coordinates and time? Non-Holonomic constraints also affect motion but are not as easily definable with math?

b. The book states " ...k equations that relate the coordinates and possibly time.." When the author refers to coordinates, they are referring to those that pertain to the position of the particles, correct? I guess I am a little confused about the systems in question here.

More questions –

From **Julia** –

My question is would the calculus of variations method still work for the bead on a string example if there is friction? And what are some real examples of when the calculus of variations method is used?

The “calculus of variation” as a mathematical construction.

Today and Friday, we will focus on the idea and process, before applying it to physical systems next week.



According wikipedia –  
**Joseph-Louis Lagrange** (born **Giuseppe Luigi Lagrangia** or **Giuseppe Ludovico De la Grange Tournier**; 25 January 1736 – 10 April 1813), also reported as **Giuseppe Luigi Lagrange** or **Lagrangia**, was an Italian mathematician and astronomer, later naturalized French. He made significant contributions to the fields of analysis, number theory, and both classical and celestial mechanics.

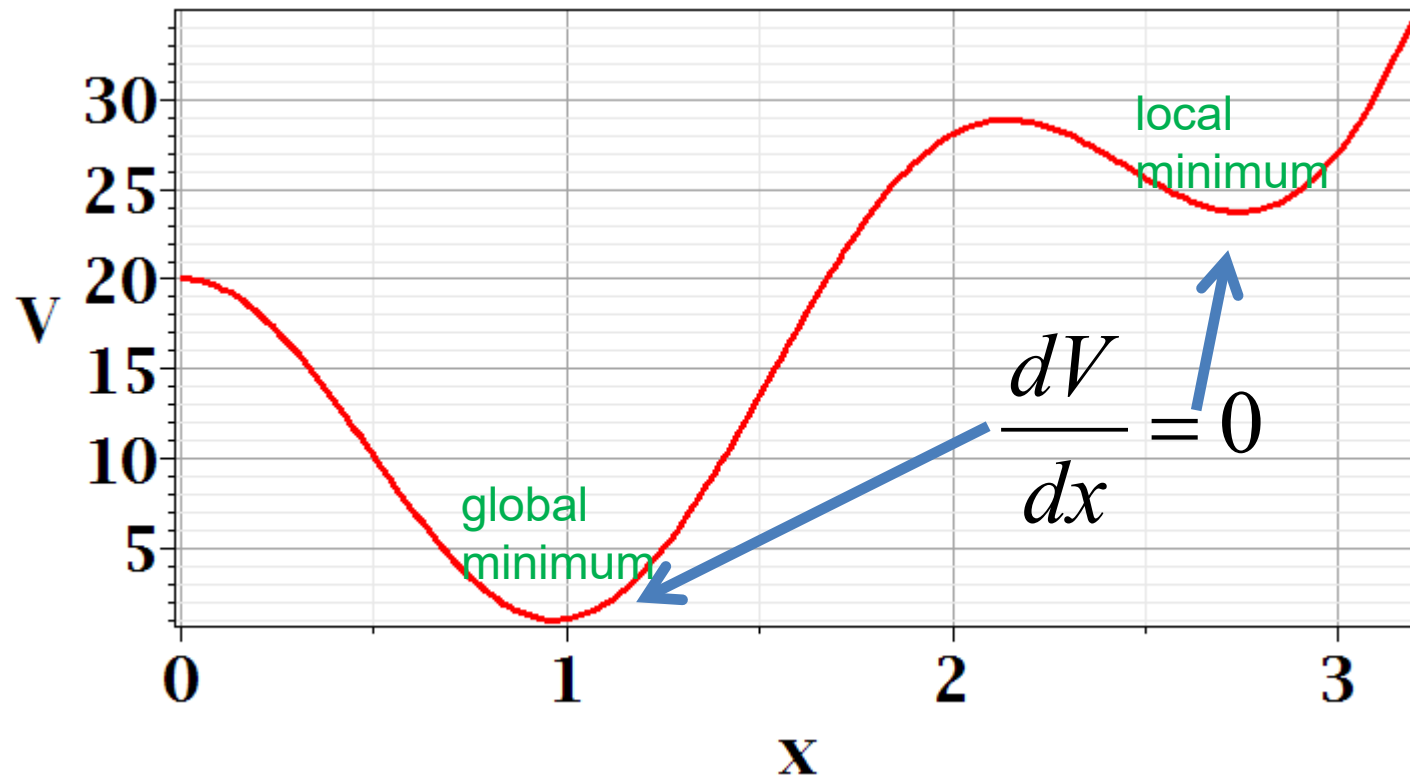


According to Wikipedia – **Leonard Euler** (April 7, 1707-September 18, 1783) Swiss mathematician, physicist, astronomer, geographer, logician and engineer who founded the studies of graph theory and topology and made pioneering and influential discoveries in many other branches of mathematics such as analytic number theory, complex analysis, and infinitesimal calculus. He introduced much of modern mathematical terminology and notation, including the notion of a mathematical function. He is also known for his work in mechanics, fluid dynamics, optics, astronomy and music theory.



In Chapter 3, the notion of Lagrangian dynamics is developed; reformulating Newton's laws in terms of minimization of related functions. In preparation, we need to develop a mathematical tool known as "the calculus of variation".

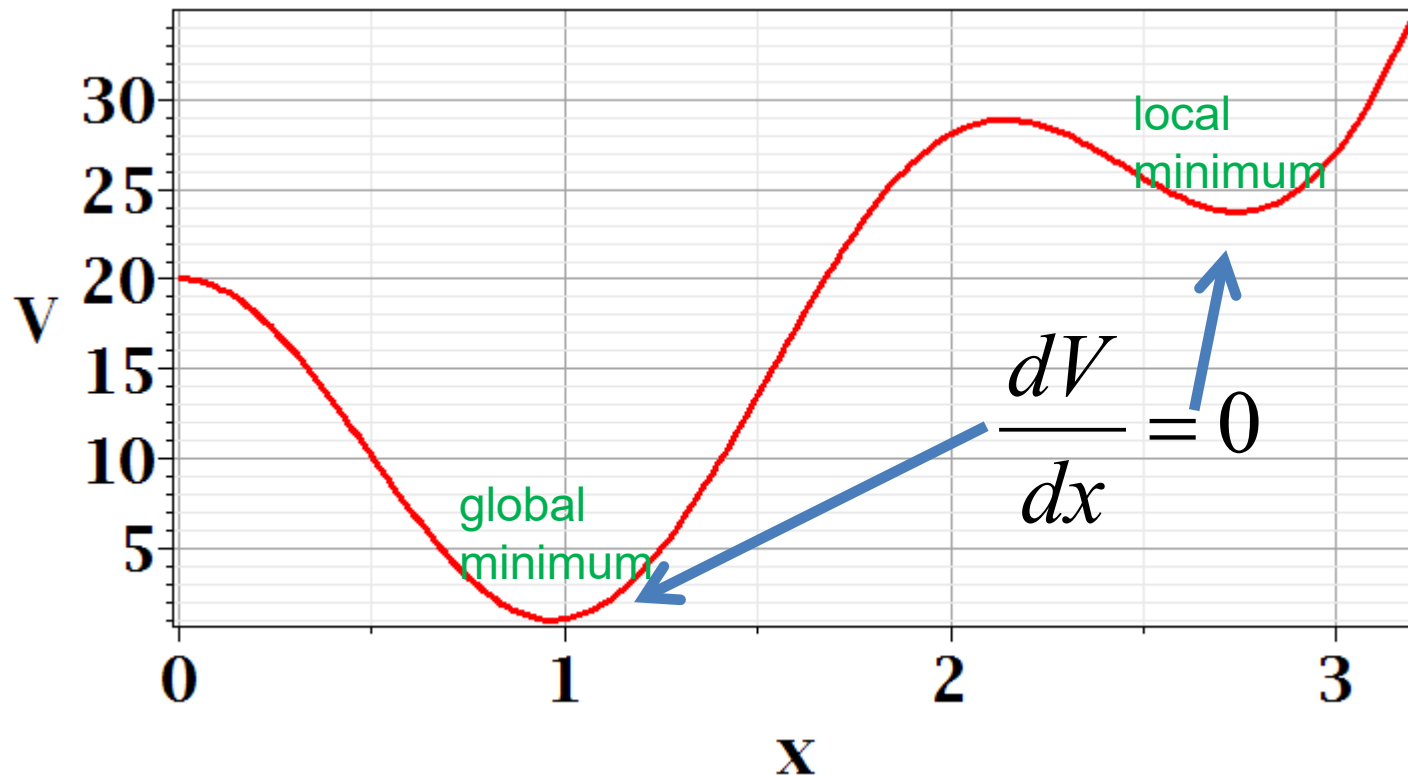
## Minimization of a simple function



## Minimization of a simple function

Given a function  $V(x)$ , find the value(s) of  $x$  for which  $V(x)$  is minimized (or maximized).

Necessary condition :  $\frac{dV}{dx} = 0$





# Functional minimization of an integral relationship

Consider a family of functions  $y(x)$ , with fixed end points

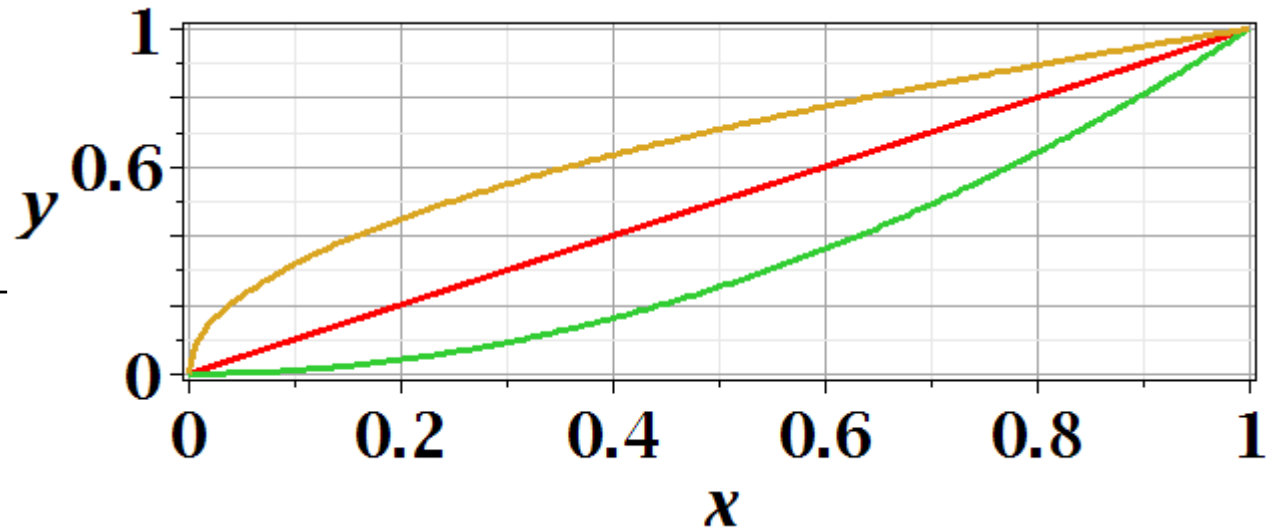
$$y(x_i) = y_i \text{ and } y(x_f) = y_f \text{ and an integral form } L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right).$$


Find the function  $y(x)$  which extremizes  $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right).$

Necessary condition:  $\delta L = 0$

Example:

$$L = \int_{(0,0)}^{1,1} \sqrt{(dx)^2 + (dy)^2}$$





Difference between minimization of a function  $V(x)$  and the minimization in the calculus of variation.

Minimization of a function –  $V(x)$

→ Know  $V(x)$       → Find  $x_0$  such that  $V(x_0)$  is a minimum.

Calculus of variation

For  $x_i \leq x \leq x_f$  want to find a function  $y(x)$

that minimizes an integral that depends on  $y(x)$ .

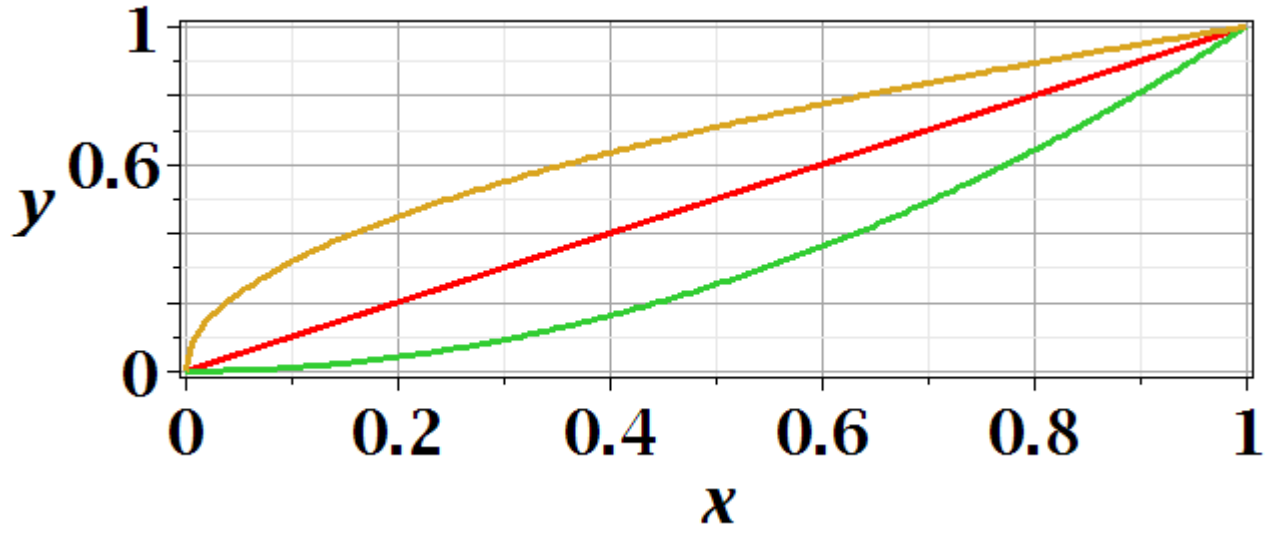
The analysis involves deriving and solving a differential equation for the function  $y(x)$ .



Example:

$$L = \int_{(0,0)}^{(1,1)} \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



Sample functions:

$$y_1(x) = \sqrt{x}$$

$$L = \int_0^1 \sqrt{1 + \frac{1}{4x}} dx = 1.4789$$

$$y_2(x) = x$$

$$L = \int_0^1 \sqrt{1 + 1} dx = \sqrt{2} = 1.4142$$

$$y_3(x) = x^2$$

$$L = \int_0^1 \sqrt{1 + 4x^2} dx = 1.4789$$

# Calculus of variation example for a pure integral function

Find the function  $y(x)$  which extremizes  $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$

where  $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) \equiv \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx.$

Necessary condition :  $\delta L = 0$

At any  $x$ , let  $y(x) \rightarrow y(x) + \delta y(x)$

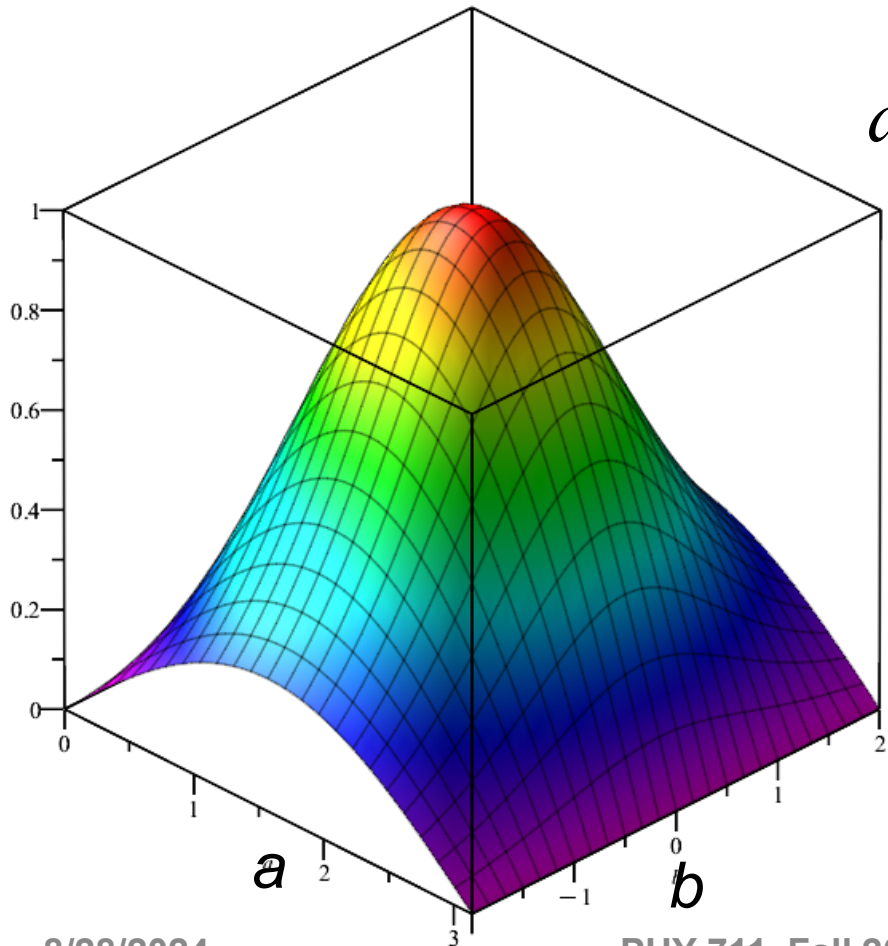
$$\frac{dy(x)}{dx} \rightarrow \frac{dy(x)}{dx} + \delta \frac{dy(x)}{dx}$$

Formally:

$$\delta L = \int_{x_i}^{x_f} \left[ \left( \frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \delta \left( \frac{dy}{dx} \right) \right] \right] dx.$$

Comment on partial derivatives -- function  $f(a, b)$

$$\frac{\partial f}{\partial a} \equiv \lim_{da \rightarrow 0} \left( \frac{f(a + da, b) - f(a, b)}{da} \right) \equiv \left. \frac{\partial f}{\partial a} \right|_b$$



$$df = \left( \frac{\partial f}{\partial a} \right)_b da + \left( \frac{\partial f}{\partial b} \right)_a db$$

## Comment about notation concerning functional dependence and partial derivatives

Suppose  $x, y, z$  represent independent variables that determine a function  $f$  :

We write  $f(x, y, z)$ . A partial derivative with respect to  $x$  implies that we hold  $y, z$  fixed and infinitesimally change  $x$

$$\left(\frac{\partial f}{\partial x}\right)_{y,z} = \lim_{\Delta x \rightarrow 0} \left( \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x} \right)_{y,z}$$



After some derivations, we find

$$\delta L = \int_{x_i}^{x_f} \left[ \left( \frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left( \frac{dy}{dx} \right) \right] \right] dx$$
$$= \int_{x_i}^{x_f} \left[ \left( \frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] \right] \delta y dx = 0 \quad \text{for all } x_i \leq x \leq x_f$$

$$\Rightarrow \left( \frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$



Note that this is a  
“total” derivative



## “Some” derivations --

Consider the term

$$\int_{x_i}^{x_f} \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left( \frac{dy}{dx} \right) \right] dx :$$

If  $y(x)$  is a well-defined function, then  $\delta \left( \frac{dy}{dx} \right) = \frac{d}{dx} \delta y$  \*

$$\int_{x_i}^{x_f} \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left( \frac{dy}{dx} \right) \right] dx = \int_{x_i}^{x_f} \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \frac{d}{dx} \delta y \right] dx$$

$$= \int_{x_i}^{x_f} \left[ \frac{d}{dx} \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] - \frac{d}{dx} \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx$$

Note that the  $\delta y$  notation is meant to imply a general infinitesimal variation of the function  $y(x)$

Clarification -- what is the meaning of the following statement:

$$\delta \left( \frac{dy}{dx} \right) = \frac{d}{dx} \delta y$$

Up to now, the operator  $\delta$  is not well defined and meant to represent

a general infinitesimal difference. Suppose that  $\delta y \equiv \frac{dy}{da}$ , where  $a$

appears in the functional form somehow. For most functional forms

that one can think of,  $\frac{d^2 y(x, a)}{dx da} = \frac{d^2 y(x, a)}{da dx}$ . One can show this to be

the case even for  $y(x, a) = x^a$  where  $\frac{d^2 y(x, a)}{dx da} = \frac{d^2 y(x, a)}{da dx} = x^{a-1} (1 + a \ln(x))$ .

(Note that here we are being imprecise wrt partial and total derivatives.)

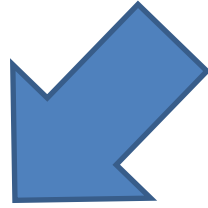
“Some” derivations (continued)--

$$\begin{aligned} & \int_{x_i}^{x_f} \left[ \frac{d}{dx} \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] - \frac{d}{dx} \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx \\ &= \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right]_{x_i}^{x_f} - \int_{x_i}^{x_f} \left[ \frac{d}{dx} \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx \\ &= 0 - \int_{x_i}^{x_f} \left[ \frac{d}{dx} \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx \end{aligned}$$

Euler-Lagrange equation:

$$\Rightarrow \left( \frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

# Clarification – Why does this term go to zero?



$$\begin{aligned} & \int_{x_i}^{x_f} \left[ \frac{d}{dx} \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] - \frac{d}{dx} \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx \\ &= \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right]_{x_i}^{x_f} - \int_{x_i}^{x_f} \left[ \frac{d}{dx} \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx \\ &= 0 - \int_{x_i}^{x_f} \left[ \frac{d}{dx} \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx \end{aligned}$$

Answer --

By construction  $\delta y(x_i) = \delta y(x_f) = 0$

Recap

$$\delta L = \int_{x_i}^{x_f} \left[ \left( \frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \delta \left( \frac{dy}{dx} \right) \right] \right] dx$$

$$= \int_{x_i}^{x_f} \left[ \left( \frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \right] \right] \delta y dx = 0 \quad \text{for all } x_i \leq x \leq x_f$$

$$\Rightarrow \left( \frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

Here we conclude that the integrand has to vanish at every argument in order for the integral to be zero

- a. Necessary?
- b. Overkill?

Example: End points --  $y(0) = 0$ ;  $y(1) = 1$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \Rightarrow \quad f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left(\frac{\partial f}{\partial (dy/dx)}\right)_{x, y} \right] = 0$$

$$\Rightarrow -\frac{d}{dx} \left( \frac{dy/dx}{\sqrt{1 + (dy/dx)^2}} \right) = 0$$

Solution:

$$\left( \frac{dy/dx}{\sqrt{1 + (dy/dx)^2}} \right) = K \quad \frac{dy}{dx} = K' \equiv \frac{K}{\sqrt{1 - K^2}}$$

$$\Rightarrow y(x) = K'x + C$$

$$y(x) = x$$

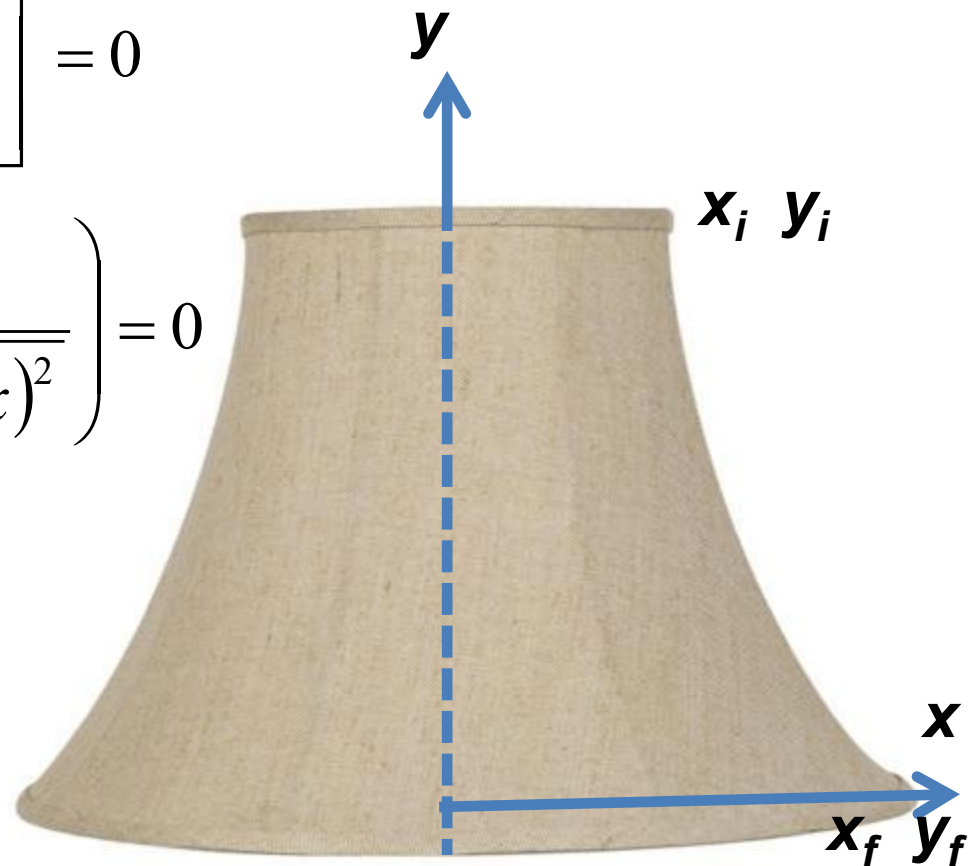



Example: Lamp shade shape  $y(x)$

$$A = 2\pi \int_{x_i}^{x_f} x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \Rightarrow \quad f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left(\frac{\partial f}{\partial (dy/dx)}\right)_{x, y} \right] = 0$$

$$\Rightarrow -\frac{d}{dx} \left( \frac{x dy/dx}{\sqrt{1 + (dy/dx)^2}} \right) = 0$$




$$-\frac{d}{dx} \left( \frac{xdy/dx}{\sqrt{1+(dy/dx)^2}} \right) = 0$$

$$\frac{xdy/dx}{\sqrt{1+(dy/dx)^2}} = K_1$$

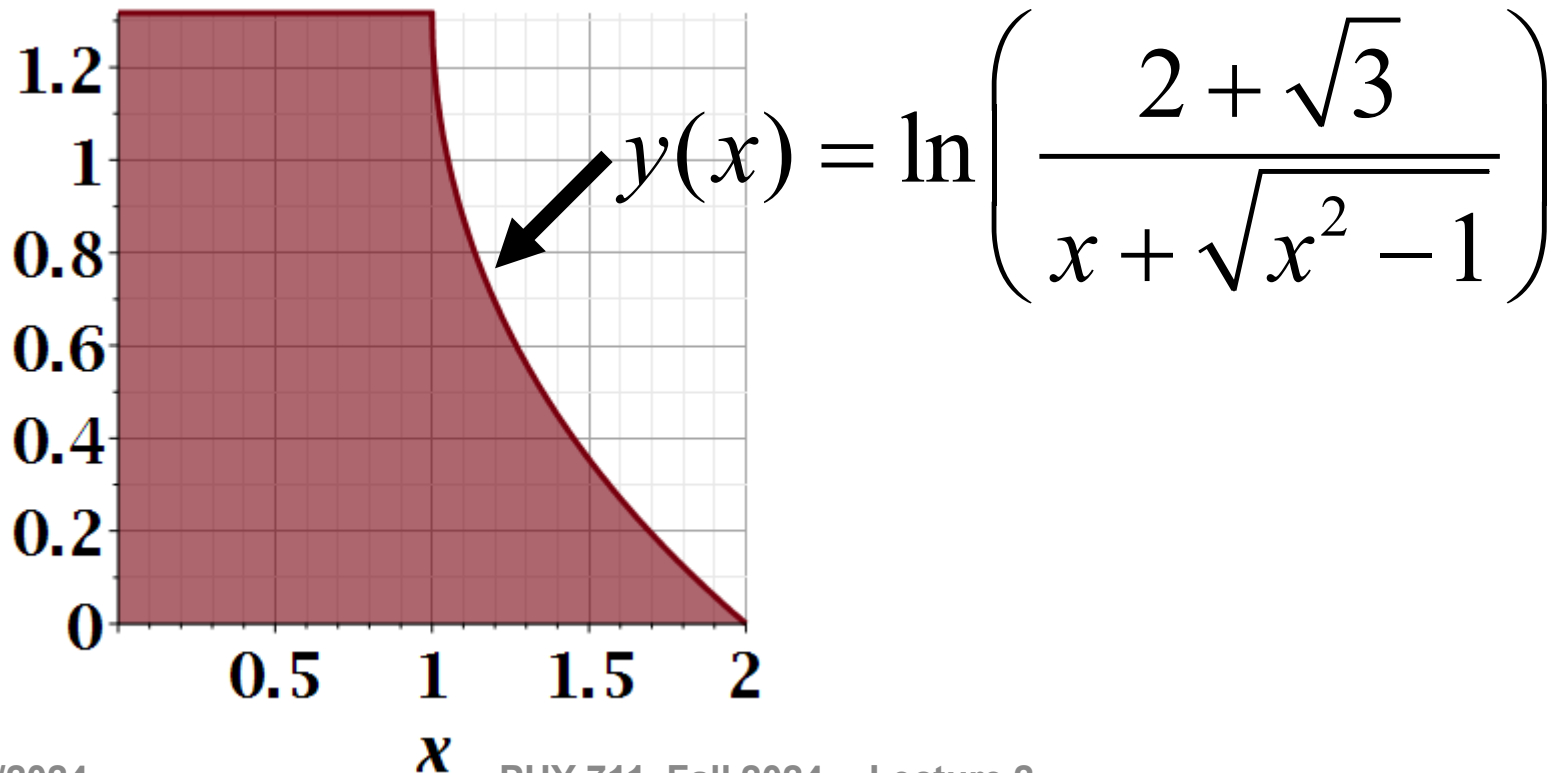
$$\frac{dy}{dx} = \frac{1}{\sqrt{\left(\frac{x}{K_1}\right)^2 - 1}}$$

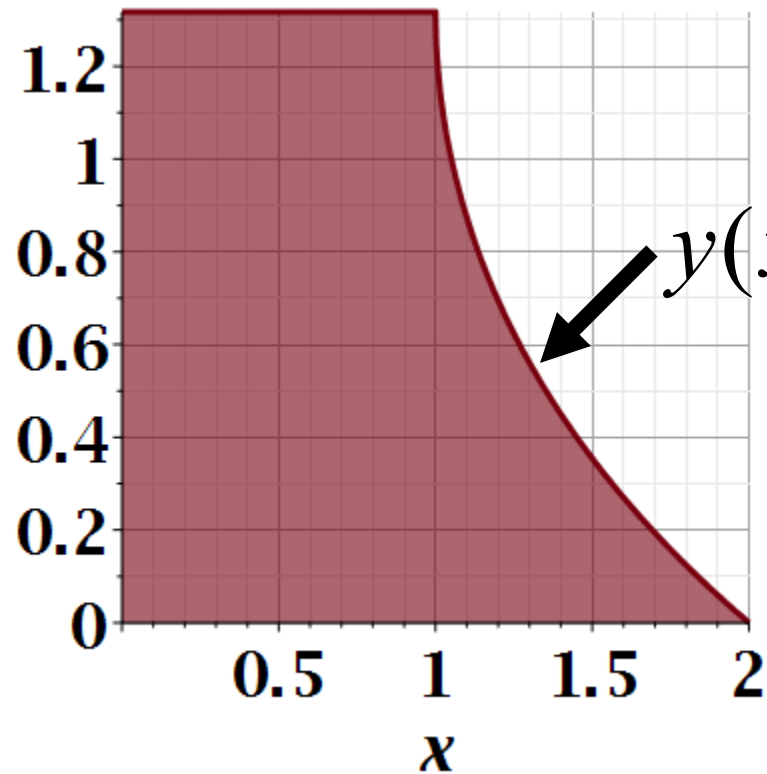
$$\Rightarrow y(x) = K_2 - K_1 \ln \left( \frac{x}{K_1} + \sqrt{\frac{x^2}{K_1^2} - 1} \right)$$

General form of solution --

$$y(x) = K_2 - K_1 \ln \left( \frac{x}{K_1} + \sqrt{\frac{x^2}{K_1^2} - 1} \right)$$

Suppose  $K_1 = 1$  and  $K_2 = 2 + \sqrt{3}$





$$y(x) = \ln \left( \frac{2 + \sqrt{3}}{x + \sqrt{x^2 - 1}} \right)$$

$$A = 2\pi \int_1^2 x \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx = 15.02014144$$

(according to Maple)

Another example:

(Courtesy of F. B. Hildebrand, Methods of Applied Mathematics)

Consider all curves  $y(x)$  with  $y(0) = 0$  and  $y(1) = 1$  that minimize the integral :

$$I = \int_0^1 \left( \left( \frac{dy}{dx} \right)^2 - ay^2 \right) dx \quad \text{for constant } a > 0$$

Euler - Lagrange equation :

$$\frac{d^2 y}{dx^2} + ay = 0$$

$$\Rightarrow y(x) = \frac{\sin(\sqrt{a}x)}{\sin(\sqrt{a})}$$

Review: for  $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$ ,

a necessary condition to extremize  $\int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx$ :

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left(\frac{\partial f}{\partial (dy/dx)}\right)_{x, y} \right] = 0 \quad \leftarrow \text{Euler-Lagrange equation}$$

Note that for  $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$ ,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right) \frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right) \frac{d}{dx} \frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx} \left(\frac{\partial f}{\partial (dy/dx)}\right)\right) \frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right) \frac{d}{dx} \frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx} \left( f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x}\right) \quad \leftarrow \text{Alternate Euler-Lagrange equation}$$