



# **PHY 711 Classical Mechanics and Mathematical Methods**

**10-10:50 AM MWF in Olin103**

## **Lecture notes for Lecture 3 Chapter 3.17 of F&W**

**More about the calculus of variations**

- 1. Review examples – Area of lamp shade**
- 2. Brachistochrone problem**
- 3. Calculus of variation with constraints**



# Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/26/2024		Introduction and overview	<a href="#">#1</a>
2	Wed, 8/28/2024	Chap. 3(17)	Calculus of variation	<a href="#">#2</a>
3	Fri, 8/30/2024	Chap. 3(17)	Calculus of variation	<a href="#">#3</a>
4	Mon, 9/02/2024	Chap. 3	Lagrangian equations of motion	
5	Wed, 9/04/2024	Chap. 3 & 6	Lagrangian equations of motion	
6	Fri, 9/06/2024	Chap. 3 & 6	Lagrangian equations of motion	



Note that Monday is not a holiday for us...

# PHY 711 – Assignment #3

Assigned: 08/30/2024      Due: 09/02/2024

This exercise is designed to illustrate the differences between partial and total derivatives.

1. Consider an arbitrary function of the form  $f = f(q, \dot{q}, t)$ , where it is assumed that  $q = q(t)$  and  $\dot{q} \equiv dq/dt$ .

(a) Write a formal expression for  $\frac{df}{dt}$  in terms of an arbitrary form of  $f = f(q, \dot{q}, t)$  and an arbitrary function  $q(t)$ .

(b) Now suppose that

$$f(q, \dot{q}, t) = q\dot{q}^2 t, \quad \text{where} \quad q(t) = e^{-t/\tau}.$$

Here  $\tau$  is a constant. Evaluate  $df/dt$  as a function of  $t$  using the expression you derived in part (a)..

(c) Now find the expression for  $f$  as an explicit function of  $t$  ( $f(t)$ ) and then take its time derivative directly to check your previous result.

## Summary of the method of calculus of variation --

Consider a family of functions  $y(x)$ , with the end points  $y(x_i) = y_i$  and  $y(x_f) = y_f$  and an integral function

$$L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \int_{x_i}^{x_f} f\left(y(x), \frac{dy}{dx}; x\right) dx.$$

Find the function  $y(x)$  which extremizes  $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$ .

$\delta L = 0 \quad \Rightarrow$  Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left(\frac{\partial f}{\partial (dy/dx)}\right)_{x, y} \right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

Example: Find minimum curve between points --  $y(0) = 0; y(1) = 1$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \Rightarrow \quad f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left(\frac{\partial f}{\partial (dy/dx)}\right)_{x, y} \right] = 0$$

$$\Rightarrow -\frac{d}{dx} \left( \frac{dy/dx}{\sqrt{1 + (dy/dx)^2}} \right) = 0$$

Solution:

$$\left( \frac{dy/dx}{\sqrt{1 + (dy/dx)^2}} \right) = K \quad \frac{dy}{dx} = K' \equiv \frac{K}{\sqrt{1 - K^2}}$$

$$\Rightarrow y(x) = K'x + C$$

$$y(x) = x$$

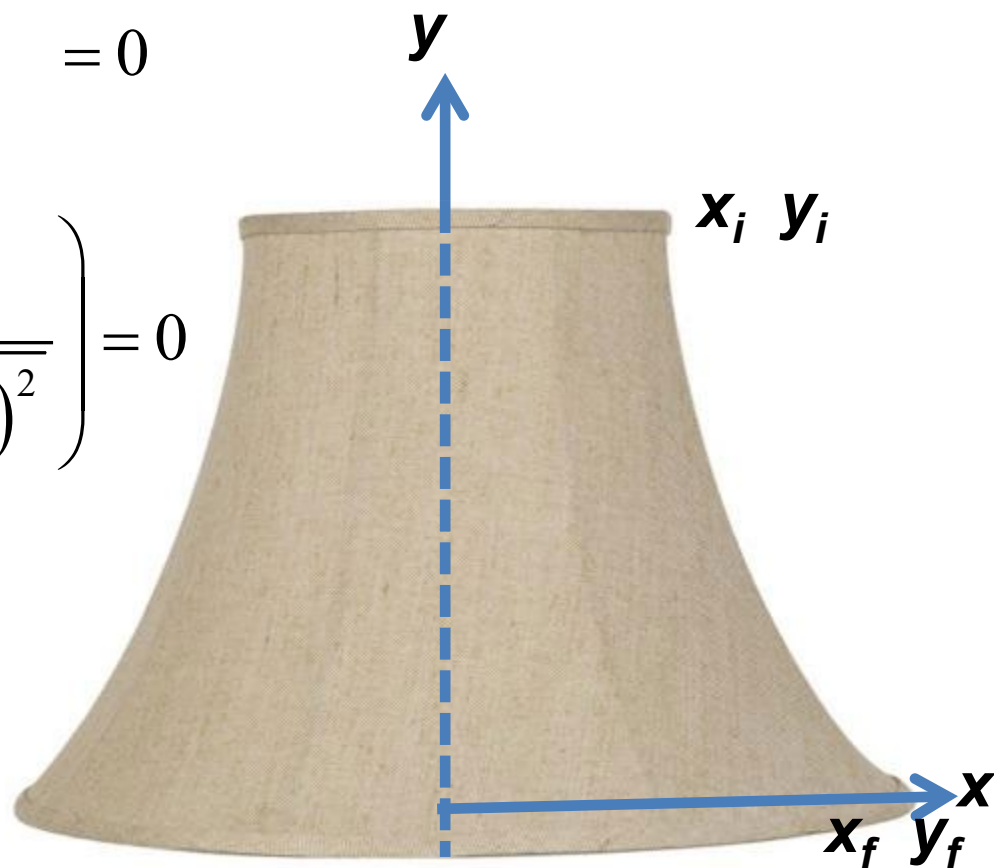


Another example: Lamp shade shape  $y(x)$

$$A = 2\pi \int_{x_i}^{x_f} x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \Rightarrow \quad f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

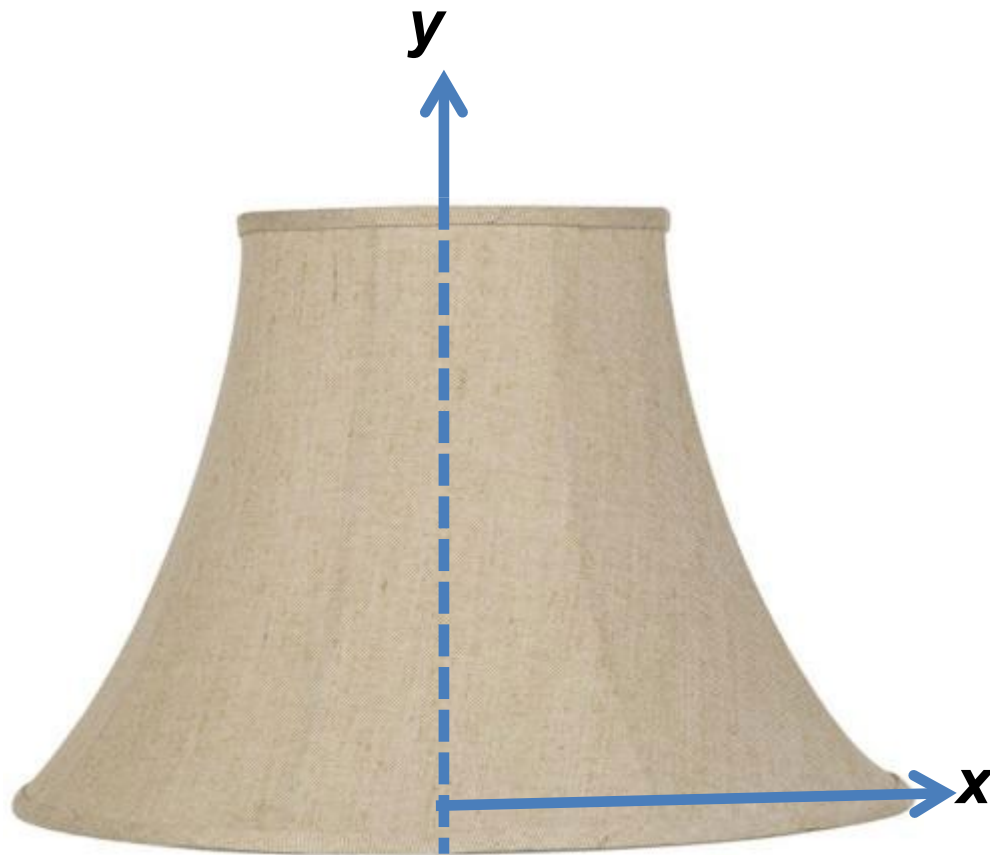
$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left(\frac{\partial f}{\partial (dy/dx)}\right)_{x, y} \right] = 0$$

$$\Rightarrow -\frac{d}{dx} \left( \frac{x dy/dx}{\sqrt{1 + (dy/dx)^2}} \right) = 0$$

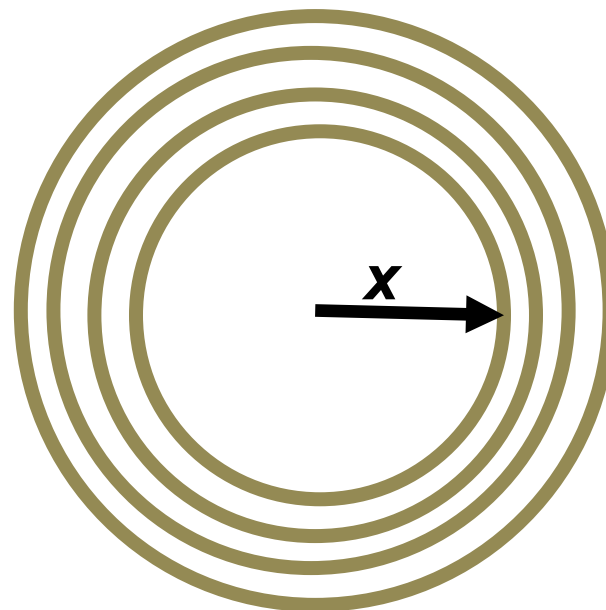




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Top view




Lamp shade area

$$2\pi x dL \text{ where } dL = \sqrt{(dx)^2 + (dy)^2}$$

$$A = 2\pi \int_{x_i y_i}^{x_f y_f} x \sqrt{(dx)^2 + (dy)^2}$$

$$= 2\pi \int_{x_i}^{x_f} x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$




$$-\frac{d}{dx} \left( \frac{xdy/dx}{\sqrt{1+(dy/dx)^2}} \right) = 0$$

$$\frac{xdy/dx}{\sqrt{1+(dy/dx)^2}} = K_1$$

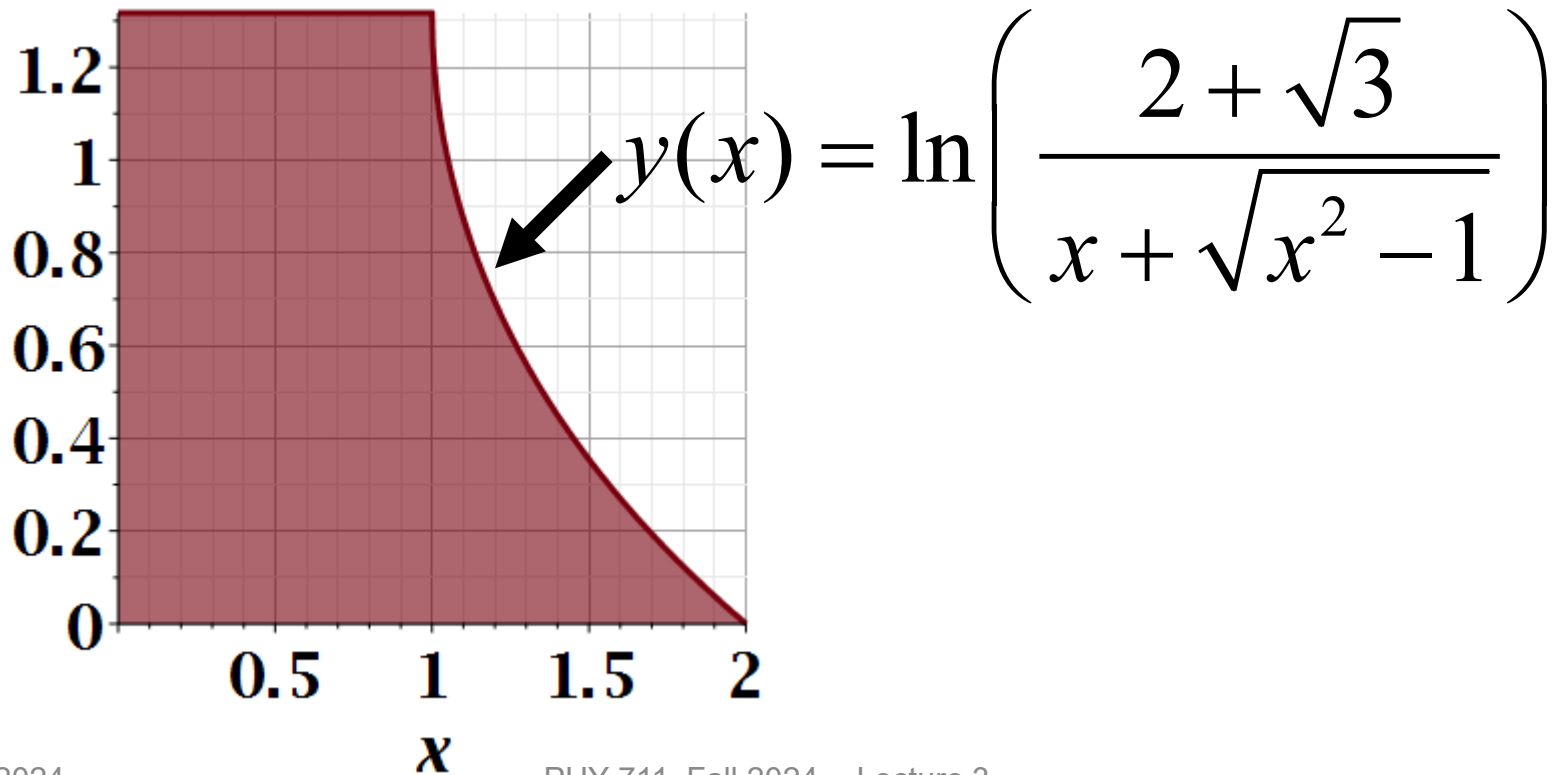
$$\frac{dy}{dx} = \frac{1}{\sqrt{\left(\frac{x}{K_1}\right)^2 - 1}}$$

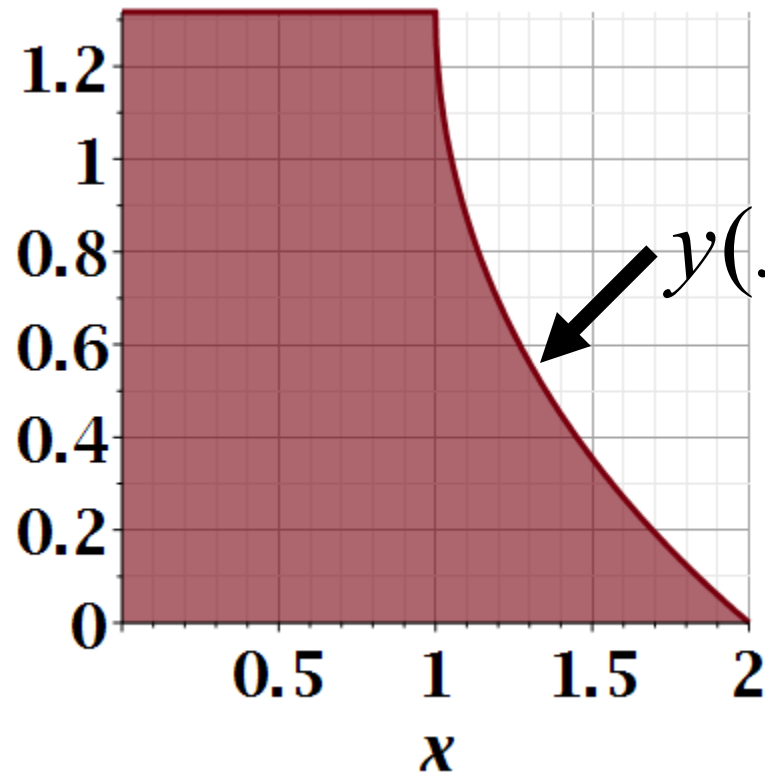
$$\Rightarrow y(x) = K_2 - K_1 \ln \left( \frac{x}{K_1} + \sqrt{\frac{x^2}{K_1^2} - 1} \right)$$

General form of solution --

$$y(x) = K_2 - K_1 \ln \left( \frac{x}{K_1} + \sqrt{\frac{x^2}{K_1^2} - 1} \right)$$

Suppose  $K_1 = 1$  and  $K_2 = \ln(2 + \sqrt{3})$





$$y(x) = \ln \left( \frac{2 + \sqrt{3}}{x + \sqrt{x^2 - 1}} \right)$$

$$A = 2\pi \int_1^2 x \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx = 15.02014144$$

(according to Maple)

Review: for  $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$ ,

a necessary condition to extremize  $\int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx$ :

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left(\frac{\partial f}{\partial (dy/dx)}\right)_{x, y} \right] = 0 \quad \leftarrow \text{Euler-Lagrange equation}$$

Note that for  $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$ ,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right) \frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right) \frac{d}{dx} \frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx} \left(\frac{\partial f}{\partial (dy/dx)}\right)\right) \frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right) \frac{d}{dx} \frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx} \left( f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x}\right) \quad \leftarrow \text{Alternate Euler-Lagrange equation}$$

A few more steps --

Note that for  $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$ ,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right) \frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right) \frac{d}{dx} \frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

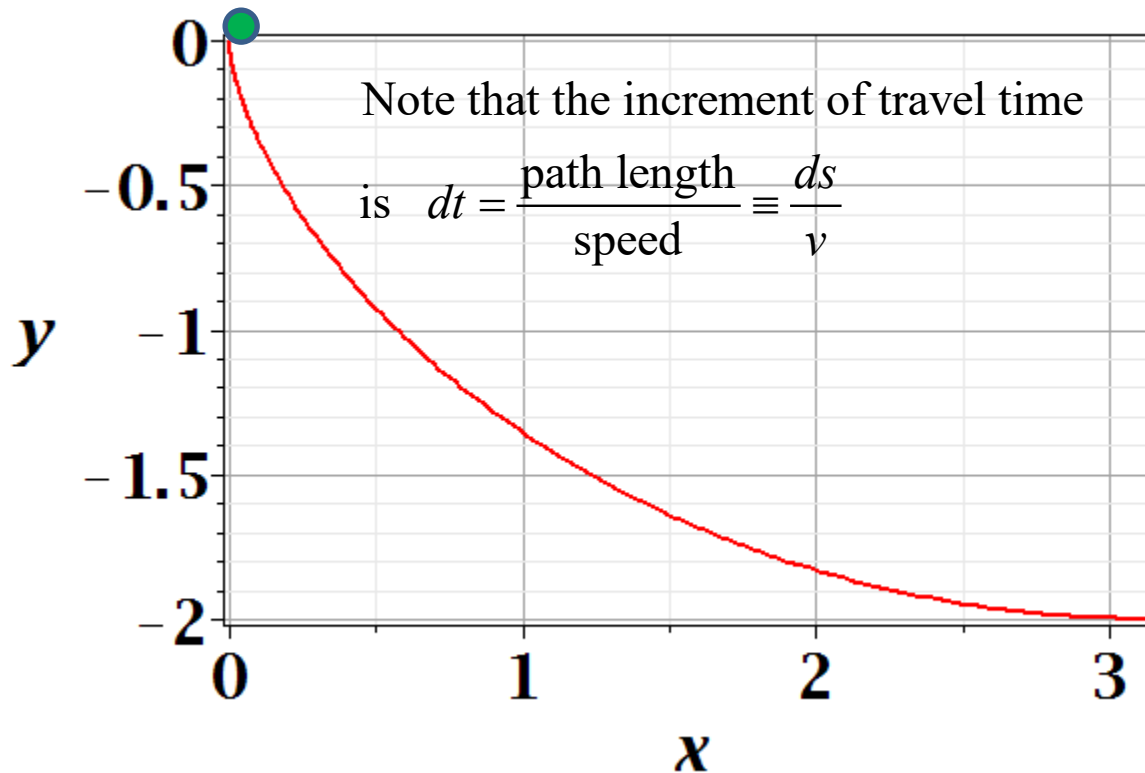
$$= \left(\frac{d}{dx} \left(\frac{\partial f}{\partial(dy/dx)}\right)\right) \frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right) \frac{d}{dx} \frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx} \left(\frac{\partial f}{\partial(dy/dx)} \frac{dy}{dx}\right)\right) + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx} \left(f - \frac{\partial f}{\partial(dy/dx)} \frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

# Brachistochrone problem: (solved by Newton in 1696)

<http://mathworld.wolfram.com/BrachistochroneProblem.html>



A particle of weight  $mg$  travels frictionlessly down a path of shape  $y(x)$ . What is the shape of the path  $y(x)$  that minimizes the travel time from  $y(0)=0$  to  $y(\pi)=-2$ ?

$$E = \frac{1}{2}mv^2 + mgy \quad \text{with } y(t=0) = 0 \text{ and } \dot{y}(t=0) = 0$$

With this choice of initial conditions,  $E = 0$

Note that the increment of travel time

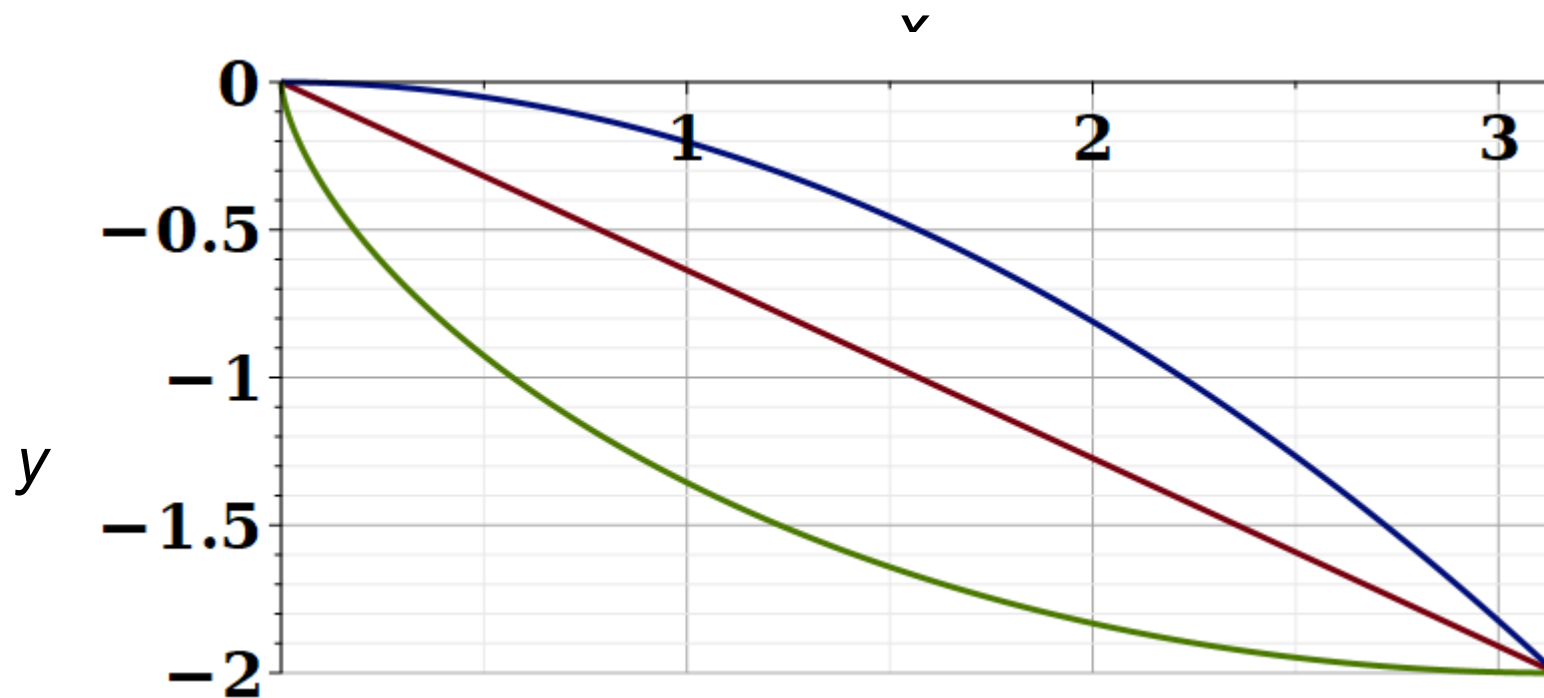
is  $dt = \frac{\text{path length}}{\text{speed}} \equiv \frac{ds}{v}$

Alternatively --

$$v = \frac{ds}{dt} \quad \Rightarrow \quad dt = \frac{ds}{v}$$



Vote for your favorite path



Which gives the shortest time?

- a. Green
- b. Red
- c. Blue



$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}} dx$$

because  $\frac{1}{2}mv^2 = -mgy$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)} \frac{dy}{dx}\right) = 0$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1 + \left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

Note that for the original form of Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x, y}\right] = 0,$$

differential equation is more complicated:

$$-\frac{1}{2}\sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-y^3}} - \frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{-y\left(1 + \left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$



$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-y}}$$


$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)} \frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1 + \left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

$$-y\left(1 + \left(\frac{dy}{dx}\right)^2\right) = K \equiv 2a$$



Question – why this choice?  
 Answer – because the answer will be more beautiful. (Be sure that was not my cleverness.)


$$-y \left( 1 + \left( \frac{dy}{dx} \right)^2 \right) = K \equiv 2a$$

$$\frac{dy}{dx} = -\sqrt{\frac{2a}{-y} - 1}$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y} - 1}} = dx$$

Let  $y = -2a \sin^2 \frac{\theta}{2} = a(\cos \theta - 1)$

$$-\frac{dy}{\sqrt{\frac{2a}{-y} - 1}} = \frac{2a \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta}{\sqrt{\frac{2a}{2a \sin^2 \frac{\theta}{2}} - 1}} = dx$$
$$x = \int_0^{\theta} a(1 - \cos \theta') d\theta' = a(\theta - \sin \theta)$$

Parametric equations for Brachistochrone:

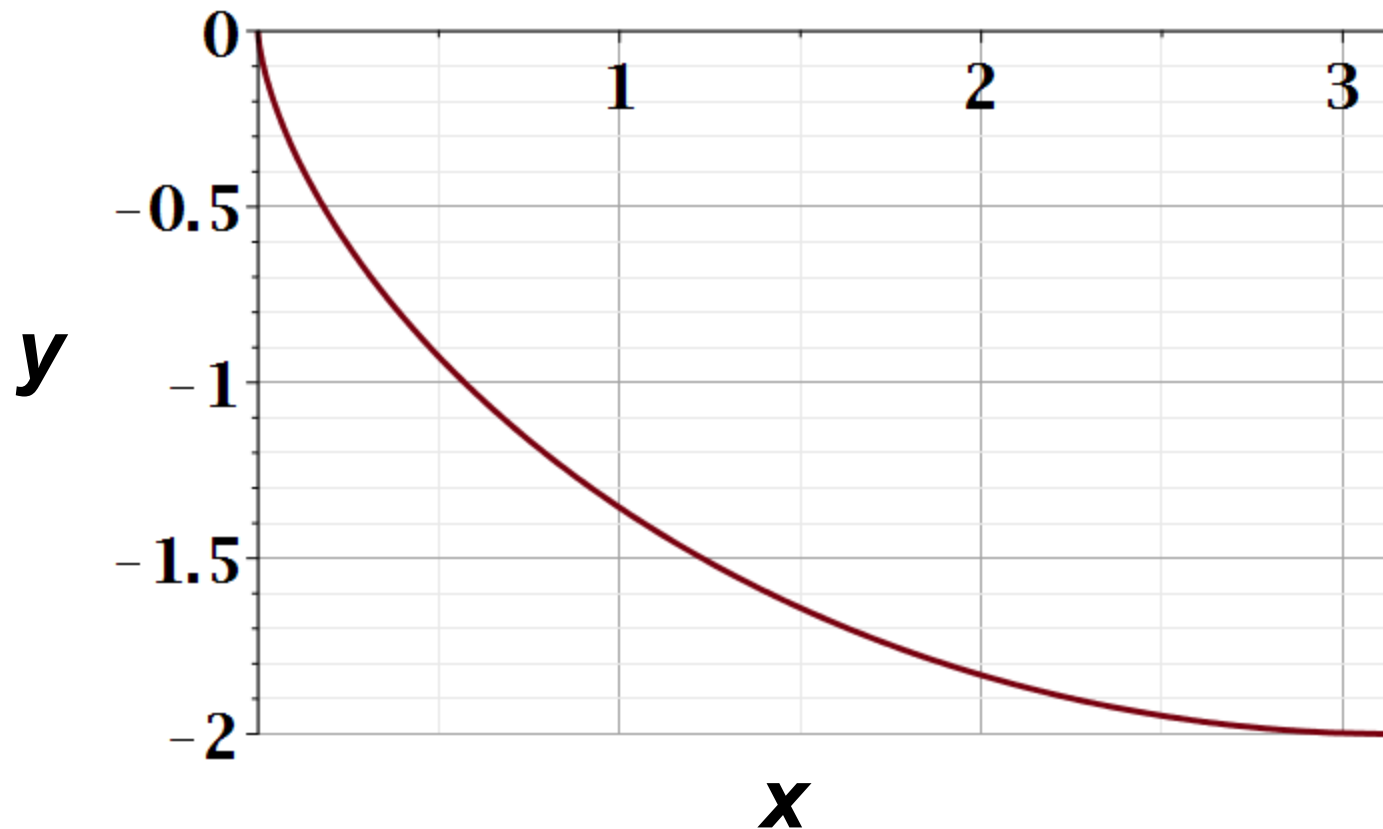
$$x = a(\theta - \sin \theta)$$

$$y = a(\cos \theta - 1)$$



Parametric plot --

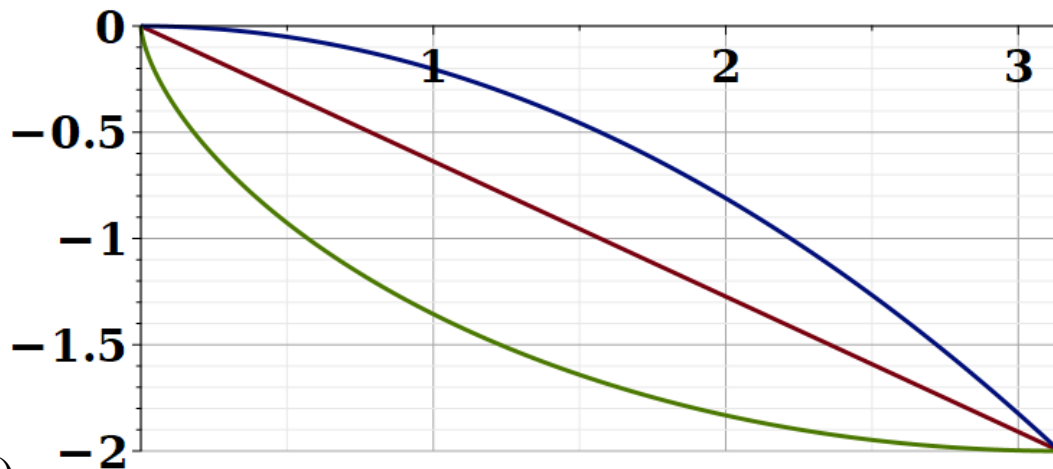
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plot([theta-sin(theta), cos(theta)-1, theta = 0 .. Pi])
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# Checking the results

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}} dx$$

units of  $\frac{1}{\sqrt{(2g)}}$  ;  $(0,0) \rightarrow (\pi, -2)$



**T=4.4429**

$$x = \theta - \sin \theta \quad y = \cos \theta - 1$$

**T=5.2668**

$$y(x) = -2x / \pi$$

**T=infinite**

$$y(x) = -2x^2 / \pi^2$$

## Summary of the method of calculus of variation --

Consider a family of functions  $y(x)$ , with the end points  $y(x_i) = y_i$  and  $y(x_f) = y_f$  and an integral function

$$I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \int_{x_i}^{x_f} f\left(y(x), \frac{dy}{dx}; x\right) dx.$$

Find the function  $y(x)$  which extremizes  $I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$ .

$\delta I = 0 \quad \Rightarrow$  Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left(\frac{\partial f}{\partial (dy/dx)}\right)_{x, y} \right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

Euler-Lagrange equation:

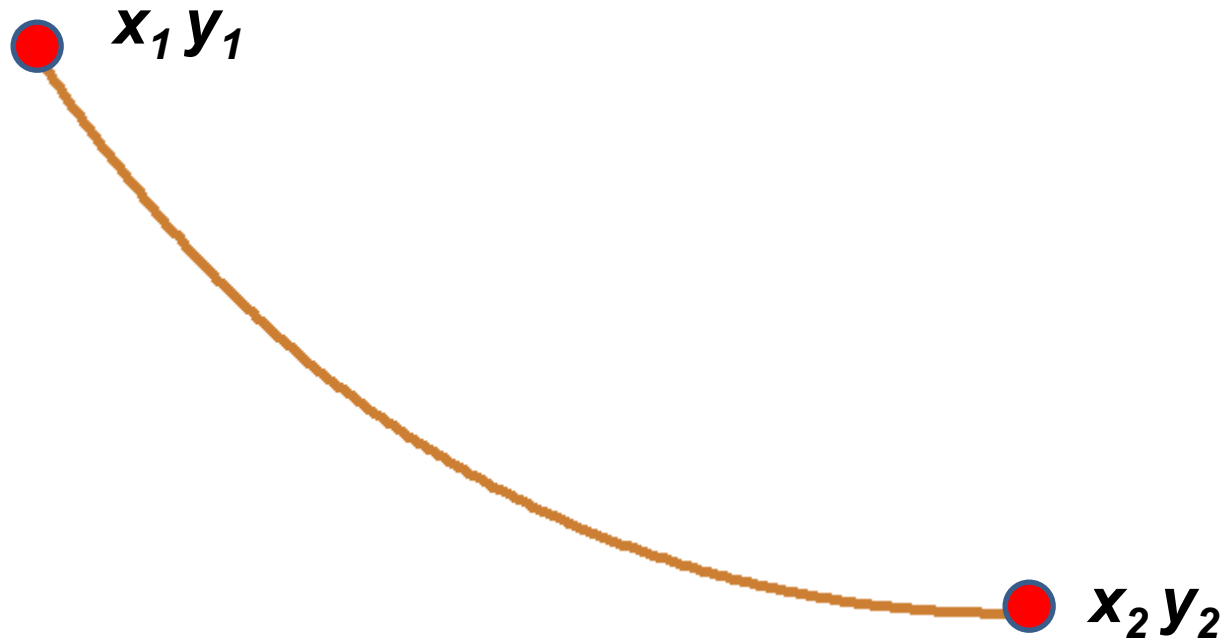
$$\left( \frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \right] = 0$$

Alternate Euler-Lagrange equation:

$$\frac{d}{dx} \left( f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left( \frac{\partial f}{\partial x} \right)$$

Another example optimization problem:

Determine the shape  $y(x)$  of a rope of length  $L$  and mass density  $\rho$  hanging between two points





# Example from internet --



Potential energy of hanging rope :

$$E = \rho g \int_{x_1}^{x_2} y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx$$

Length of rope :

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx$$

Define a composite function to minimize :


$$W \equiv E + \lambda L$$



Lagrange multiplier

$$\delta W = 0 = \delta E + \lambda \delta L \text{ for fixed } \lambda$$

is a very clever mathematical trick to help solve the minimization and constraint at the same time.


$$W = \int_{x_1}^{x_2} (\rho g y + \lambda) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$f\left(\left\{y, \frac{dy}{dx}\right\}\right) = (\rho g y + \lambda) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$


$$\frac{d}{dx} \left( f - \frac{\partial f}{\partial(dy/dx)} \frac{dy}{dx} \right) = \left( \frac{\partial f}{\partial x} \right)$$

$$\Rightarrow (\rho g y + \lambda) \left( \sqrt{1 + \left(\frac{dy}{dx}\right)^2} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right) = K$$

$$(\rho g y + \lambda) \left( \sqrt{1 + \left(\frac{dy}{dx}\right)^2} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right) = K$$

$$(\rho g y + \lambda) \left( \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right) = K$$

$$y(x) = \frac{1}{\rho g} \left( -\lambda + K \cosh \left( \frac{x - a}{K / \rho g} \right) \right)$$


$$y(x) = \frac{1}{\rho g} \left( -\lambda + K \cosh \left( \frac{x - a}{K / \rho g} \right) \right)$$

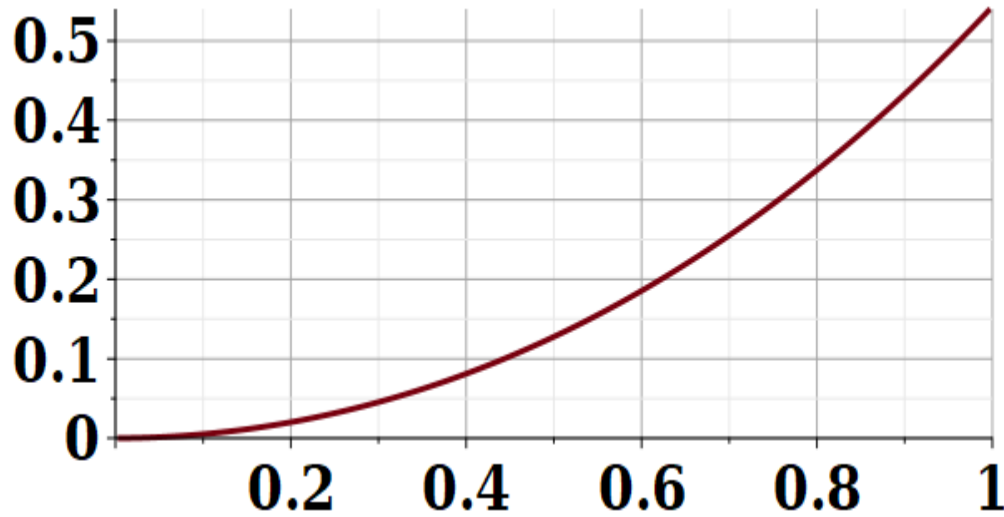
Integration constants:  $K, a, \lambda$

Constraints:  $y(x_1) = y_1$

$$y(x_2) = y_2$$

$$\int_{x_1}^{x_2} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx = L$$

$$y(x) = \frac{1}{\rho g} \left( -\lambda + K \cosh \left( \frac{x-a}{K / \rho g} \right) \right)$$



# Summary of results

For the class of problems where we need to perform an extremization on an integral form:

$$I = \int_{x_i}^{x_f} f \left( \left\{ y(x), \frac{dy}{dx} \right\}, x \right) dx \quad \delta I = 0$$

A necessary condition is the Euler-Lagrange equations:

$$\left( \frac{\partial f}{\partial y} \right) - \frac{d}{dx} \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right) \right] = 0$$

or

$$\frac{d}{dx} \left( f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left( \frac{\partial f}{\partial x} \right)$$