PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 31: Chap. 9 of F&W

Linear and non-linear sound waves

1. **Summary of linear sound phenomena**
2. **Introduction to non-linear effects**
3. **Analysis of instability – shock phenomena**

Physics Colloquium

Olin 101
– Thursday –
4 PM
November 7, 2024

Exploring phase transitions in frustrated magnets using capacitive torque magnetometry

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Geometrically frustrated systems have received a lot of attention because the magneto-crystalline anisotropy combined with magnetic interactions gives rise to numerous unusual noncollinear spin textures [1-3]. In spinel vanadates such noncollinear states lead to multiferroicity and, in pyrochlore titanates, a spin ice state forms that hosts emergent quasiparticle excitations equivalent to magnetic monopoles. There is an enticing potential of using these functional properties for the development of new types of spin-based electronics. To realize this, thin films are required. I will show that capacitive torque magnetometry is a powerful tool for characterization of structural and magnetic phase transitions in thin film samples. I will present results on bulk and thin-film samples of the spin ice Ho2Ti2O7 [1,2] and on thin films of the spinel vanadate CoV2O4 [3,4]. For the CoV2O4 thin films, I will show that the films display a temperature-induced structural transition to a low temperature noncollinear state with large magnetic anisotropy [4], which is not seen in bulk counterparts. For the pyrochlore titanate (Ho2Ti2O7), the inherent large magneto-crystalline anisotropy leads to a highly degenerate

26	Fri, 10/25/2024	Chap. 8	Wave motion in 2 dimensional membranes	#19
27	Mon, 10/28/2024	Chap. 9	Motion in 3 dimensional ideal fluids	#20
28	Wed, 10/30/2024	Chap. 9	Motion in 3 dimensional ideal fluids	#21
29	Fri, 11/01/2024	Chap. 9	Ideal gas fluids	#22
30	Mon, 11/04/2024	Chap. 9	Traveling and standing waves in the linear approximation	#23
31	Wed, 11/06/2024	Chap. 9	Non-linear and other wave properties	#24
32	Fri, 11/08/2024	Chap. 10	Surface waves in fluids	
33	Mon, 11/11/2024	Chap. 10	Surface waves in fluids	

PHY 711 -- Assignment #24

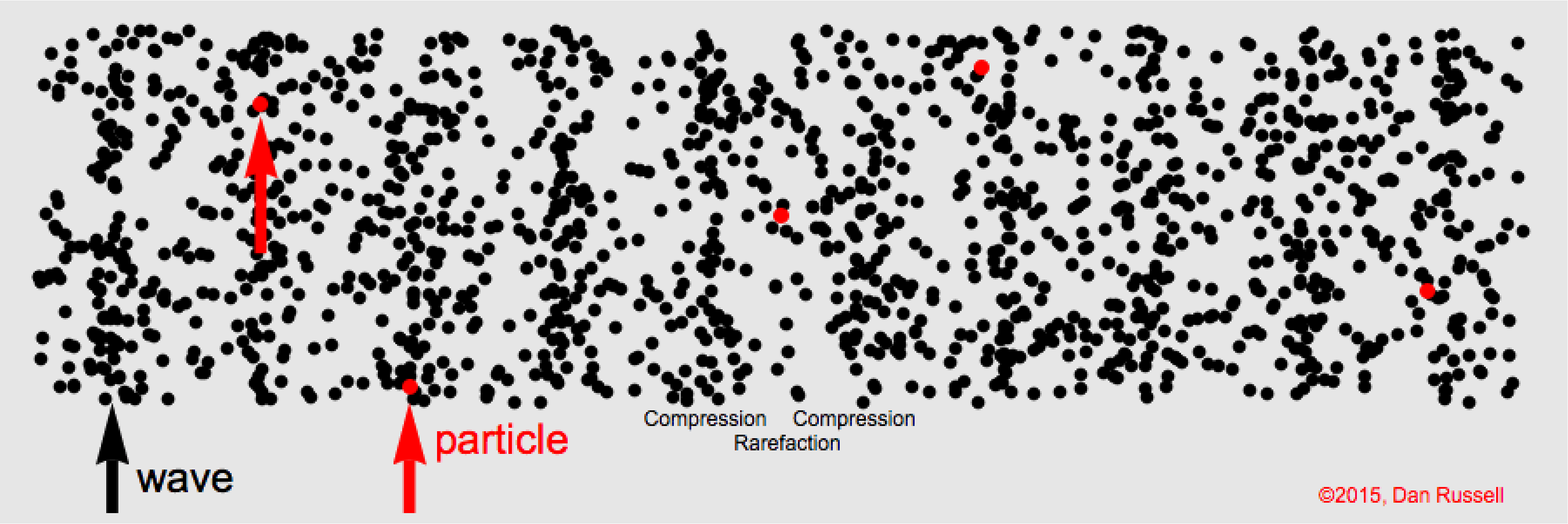
Assigned: 11/06/2024 Due: 11/11/2024

Finish reading Chapter 9 in **Fetter & Walecka**.

1. In class, we discussed how to visualize the non-linear behavior of an adiabatic ideal gas with parameter γ. Using Maple or Mathematica or other software and using a parametric plot formalism, create an animated gif file to show the traveling waveform s(w), where s is a shape of your choice and w=x-u(s(w))t. You will also need to choose the value of γ as well.

Visualization of longitudinal wave motion

From the website:
https://www.acs.psu.edu/drussell/Demos/waves/wavemotion.html



Now consider some non-linear effects in sound

Examples?

We will consider the simple case –

1. One dimension for motion
2. Fluid is assumed to be an ideal gas
3. Adiabatic conditions
4. All variables will be expressed in terms of the density $\rho(x,t)$

Effects of nonlinearities in fluid equations
-- one dimensional case

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

Assume spatial variation confined to x direction ;

assume that $\mathbf{v} = v\hat{\mathbf{x}}$ and $\mathbf{f}_{applied} = 0.$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

Expressing p in terms of ρ: $\quad p = p(\rho)$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} \equiv c^2(\rho)\frac{\partial \rho}{\partial x} \qquad \text{where} \quad \frac{\partial p}{\partial \rho} \equiv c^2(\rho)$$

For adiabatic ideal gas: $\quad \frac{dp}{p} = \gamma\frac{d\rho}{\rho} \qquad p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}$

$$c^2(\rho) = \frac{\gamma p}{\rho} = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \quad \text{where} \quad c_0^2 \equiv \frac{\gamma p_0}{\rho_0} \quad \text{and} \quad \gamma \equiv \frac{C_p}{C_V}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

Expressing variation of v in terms of $v(\rho)$:

$$\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial t} + v\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x} = 0$$

Some more algebra:

From Euler equation: $$\frac{\partial v}{\partial \rho}\left(\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$

From continuity equation: $$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}=-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}$$

Combined equation: $$\frac{\partial v}{\partial \rho}\left(-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$

$$\Rightarrow\left(\frac{\partial v}{\partial \rho}\right)^2=\frac{c^2(\rho)}{\rho^2} \qquad \Rightarrow \qquad \frac{\partial v}{\partial \rho}=\pm\frac{c}{\rho}$$

$$\Rightarrow\frac{\partial \rho}{\partial t}+(v\pm c)\frac{\partial \rho}{\partial x}=0$$

Assuming adiabatic process: $c^2 = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ $\qquad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$\Rightarrow \quad c = c_0 \left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2}$$

$$\frac{\partial v}{\partial \rho} = \frac{dv}{d\rho} = \pm \frac{c}{\rho} \qquad \Rightarrow v = \pm c_0 \int_{\rho_0}^{\rho} \left(\frac{\rho'}{\rho_0}\right)^{(\gamma-1)/2} \frac{d\rho'}{\rho'}$$

$$\Rightarrow v = \pm \frac{2c_0}{\gamma - 1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1 \right)$$

Summary :

$$\frac{dv}{d\rho} = \pm\frac{c}{\rho}$$

$$\frac{\partial\rho}{\partial t} + (v \pm c)\frac{\partial\rho}{\partial x} = 0$$

Assuming adiabatic process : $c^2 = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ $\quad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$c = c_0\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} \qquad v = \pm\frac{2c_0}{\gamma-1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right)$$

Traveling wave solution:

$$\text{Assume:} \quad \rho = \rho_0 + f(x - u(\rho)t)$$

Need to find self-consistent equations for propagation velocity $u(\rho)$ using equations

From previous derivations: $$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Apparently: $$u(\rho) \Leftrightarrow v \pm c$$

For adiabatic ideal gas and + signs:

$$u = v + c = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Traveling wave solution -- continued:

$$\frac{\partial \rho}{\partial t} + \left(v \pm c\right)\frac{\partial \rho}{\partial x} = 0$$

$$\text{Assume:} \quad \rho = \rho_0 + f\left(x - u(\rho)t\right) = \rho_0 + f\left(x - \left(v \pm c\right)t\right)$$

For adiabatic ideal gas and + signs :

$$u = v + c = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Solution in linear approxiation:

$$u = v + c \approx v_0 + c_0 = c_0\left(\frac{\gamma+1}{\gamma-1} - \frac{2}{\gamma-1}\right) = c_0$$

$$\Rightarrow \rho = \rho_0 + f\left(x - c_0 t\right)$$

Some details for the full non-linear case --

Assume: $\rho = \rho_0 + f(x - u(\rho)t)$

Need to find self-consistent equations for propagation velocity $u(\rho)$ using equations

From previous derivations: $\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$

Apparently: $u(\rho) \Leftrightarrow v \pm c$

Note that for $u = v + c$ (choice of + solution)

$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = 0$ is satisfied by a function of the form

$\rho(x,t) = \rho_0 + f(x - u(\rho(x,t))t)$

Let $w \equiv x - u(\rho(x,t))t$

$$\frac{df}{dw}\frac{\partial w}{\partial t} + u\frac{df}{dw}\frac{\partial w}{\partial x} = \frac{df}{dw}(-u + u) = 0$$

Traveling wave solution -- full non-linear case:

Visualization for particular waveform: $\rho = \rho_0 + f(\underbrace{x - u(\rho)t}_{w})$

Assume: $f(w) \equiv \rho_0 s(w)$

$$\frac{\rho}{\rho_0} = 1 + s(x - ut)$$

For adiabatic ideal gas:

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1 + s(x-ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Visualization continued:

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+s(x-ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Plot $s(x-ut)$ for fixed t, as a function of x:

Let $w = x - ut$

$$x = w + ut = w + u(w)t \equiv x(w,t)$$

$$u(w) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+s(w)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Parametric equations:

plot $s(w)$ vs $x(w,t)$ for range of w at each t

Summary

$$\frac{\partial \rho}{\partial t} + u(\rho)\frac{\partial \rho}{\partial x} = 0$$

Solution: $\rho = \rho_0 + f(x - u(\rho)t) = \rho_0\left(1 + s(x - u(\rho)t)\right)$

For linear case: $u(\rho) = c_0$

For non-linear case: $u(\rho) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+s(x-ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$

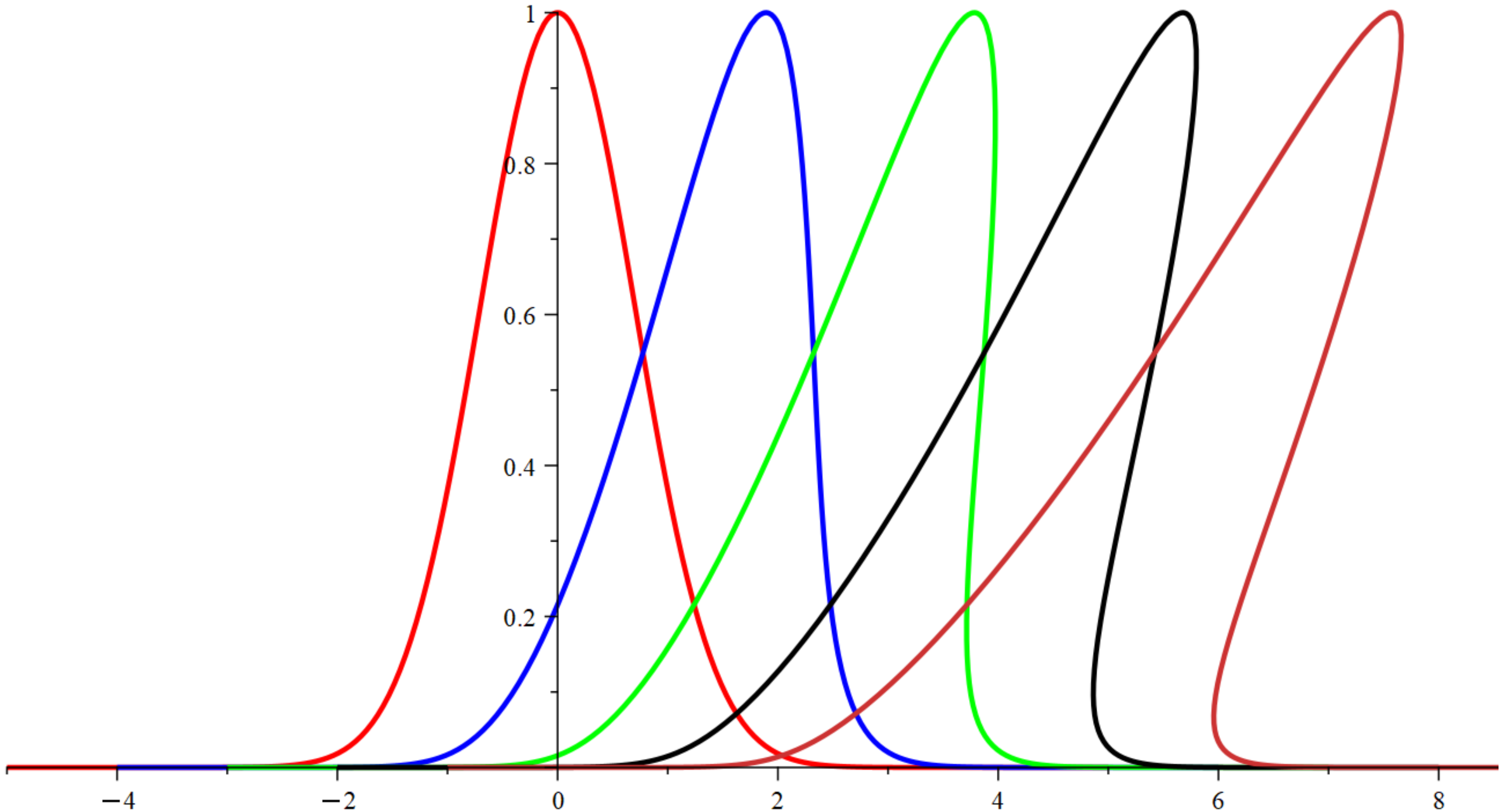
Plot $s(x-ut)$ for fixed t, as a function of x:

Let $w = x - ut \quad \Rightarrow \quad x = w + ut = w + u(w)t \equiv x(w,t)$

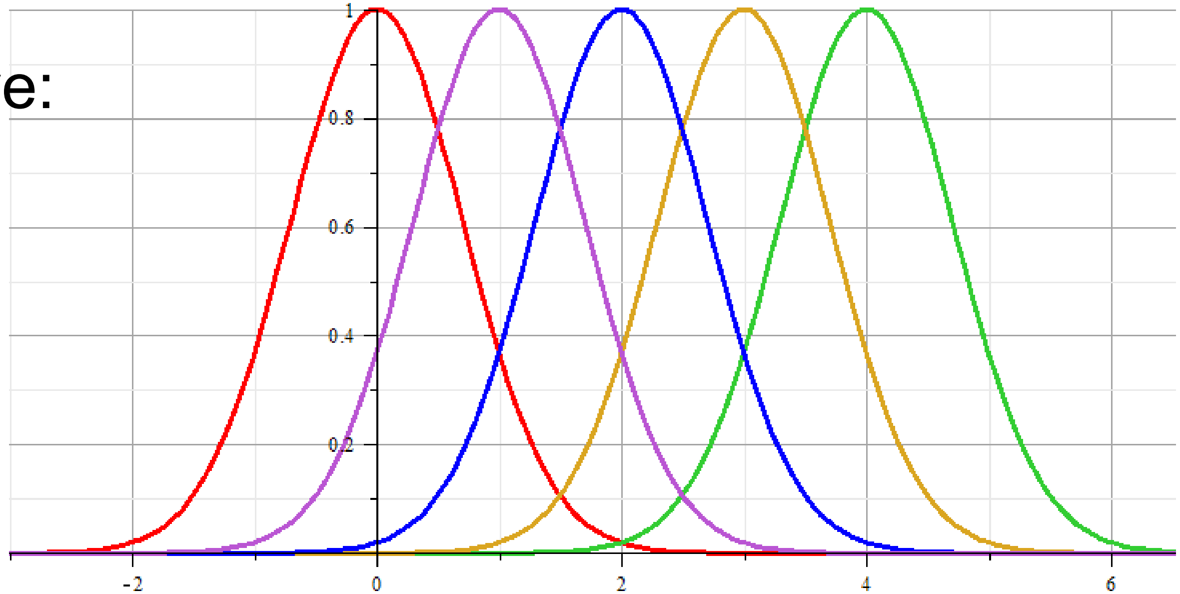
$$u(w) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+s(w)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Parametric equations: plot $s(w)$ vs $x(w,t)$ for range of w

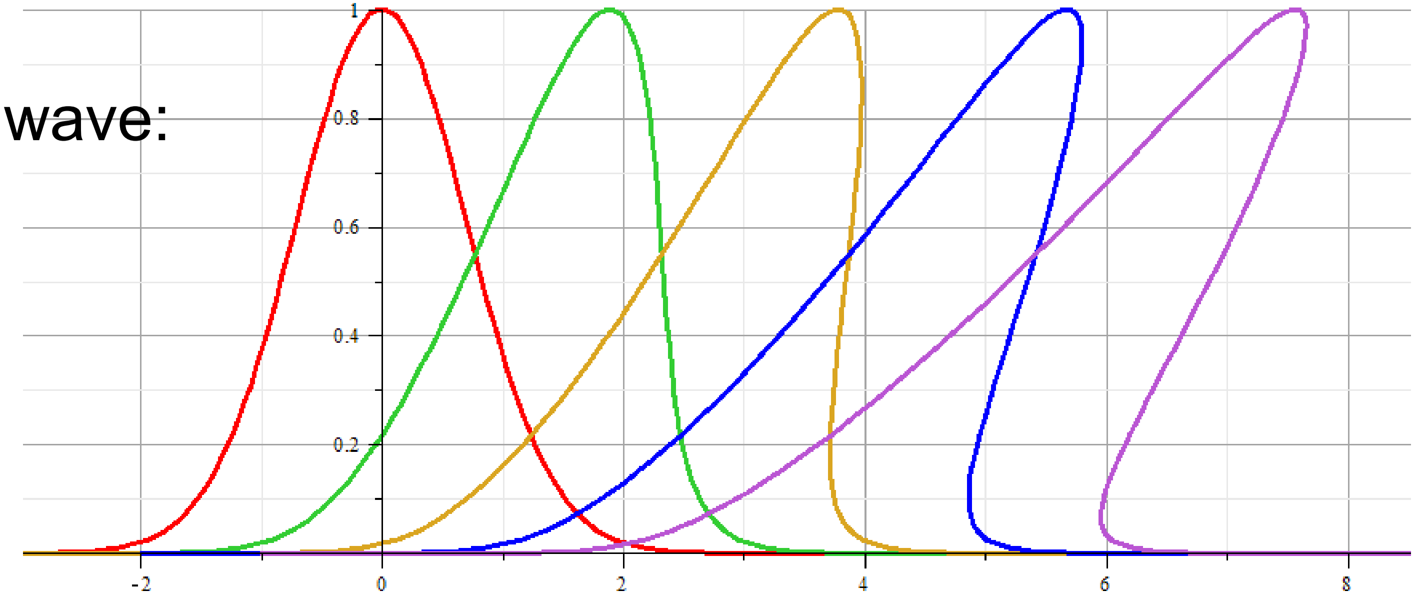
> $plot(\{[x(w, 0, 1.4), f(w), w = -5..5], [x(w, 1, 1.4), f(w), w = -5..5], [x(w, 2, 1.4), f(w), w = -5..5], [x(w, 3, 1.4), f(w), w = -5..5],$
$[x(w, 4, 1.4), f(w), w = -5..5]\}, thickness = 3, color = [red, blue, green, black, orange]);$



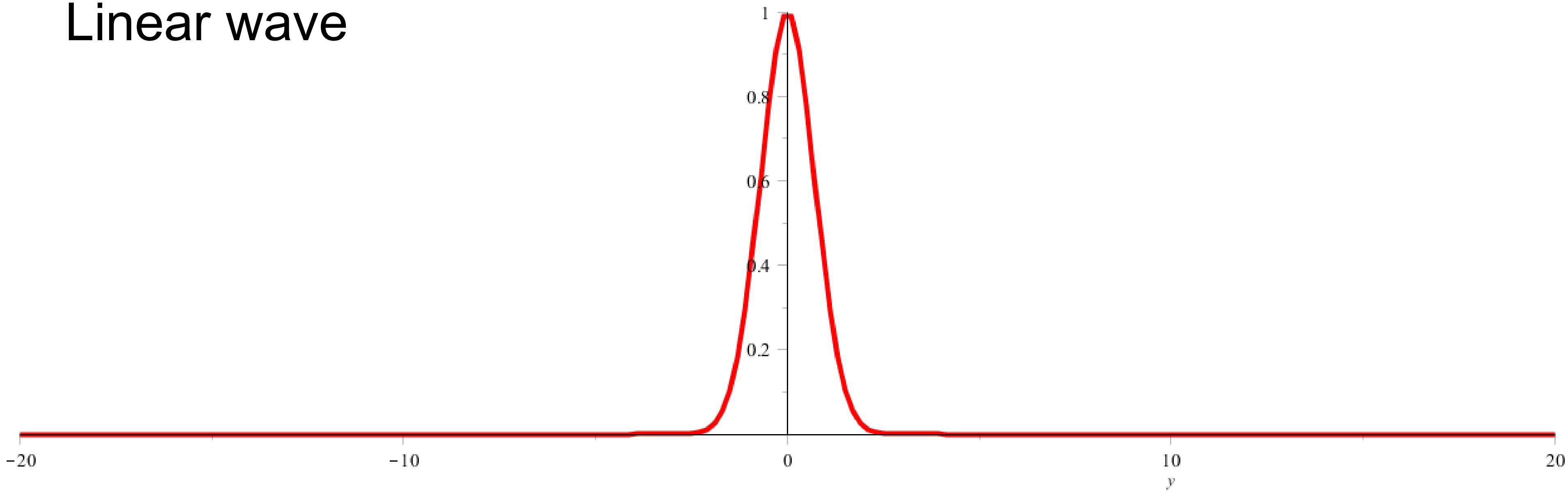
Linear wave:



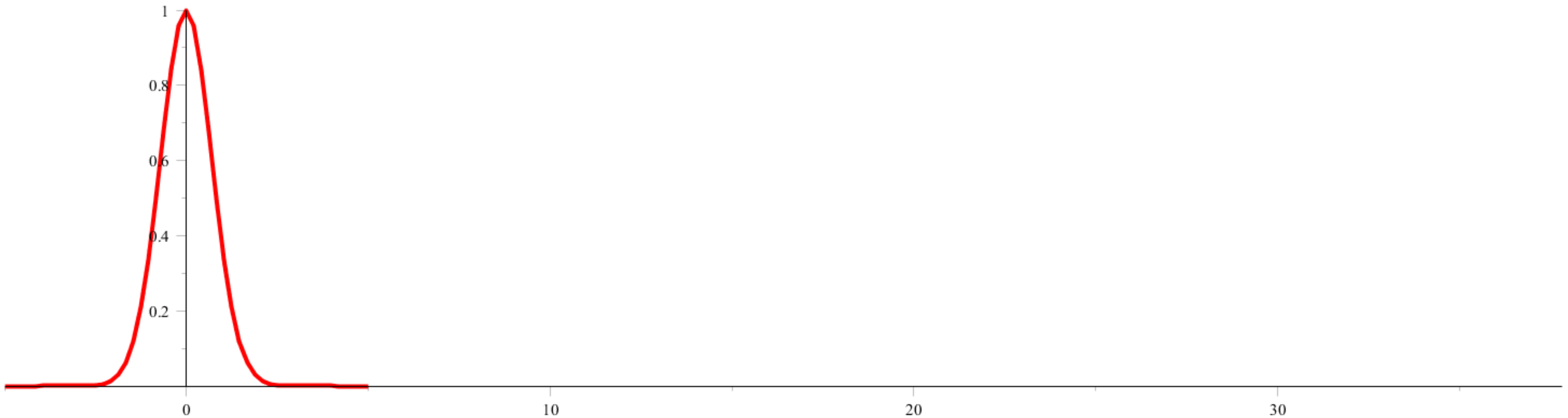
Non-linear wave:



Linear wave

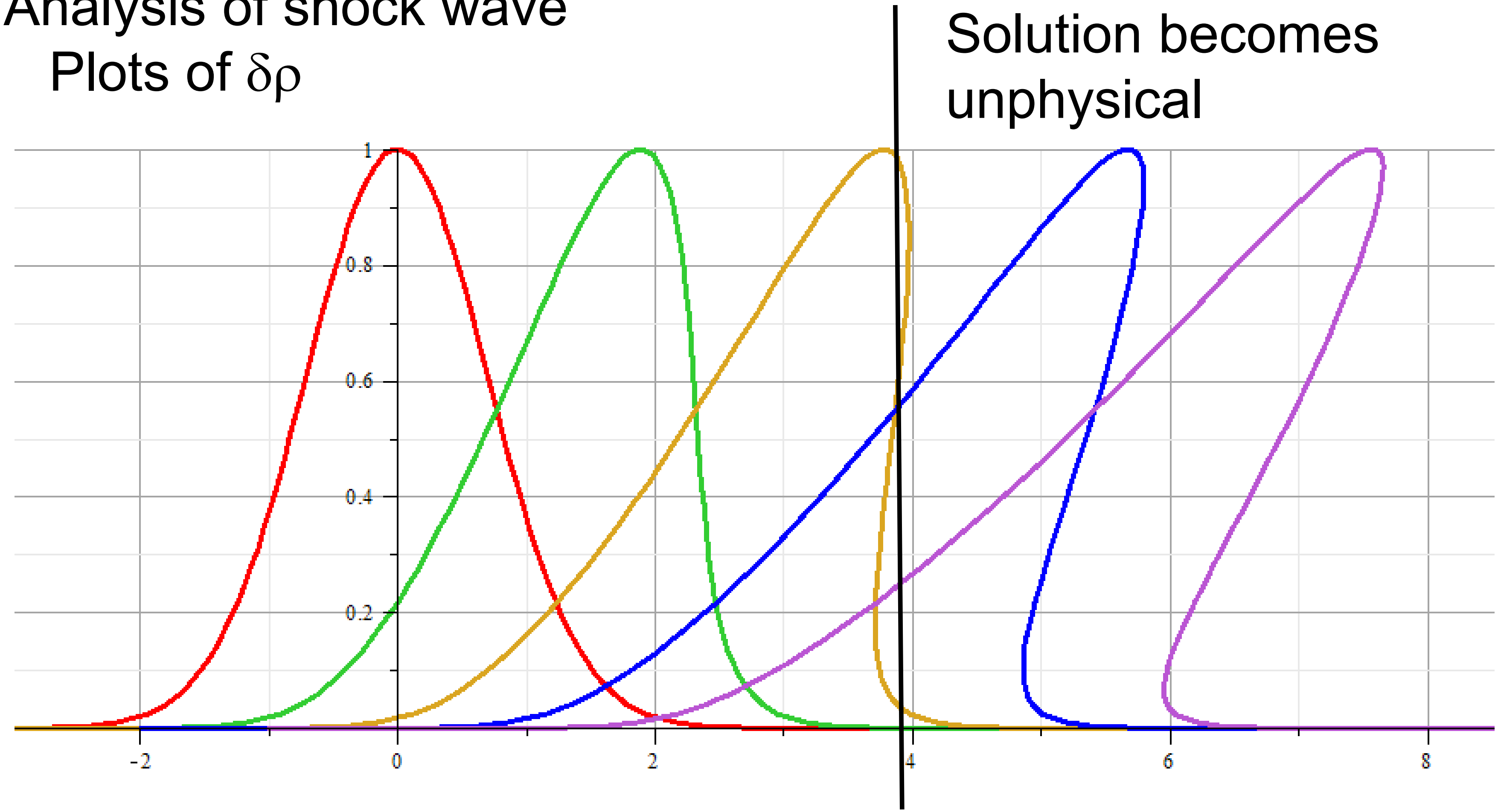


Non-linear wave



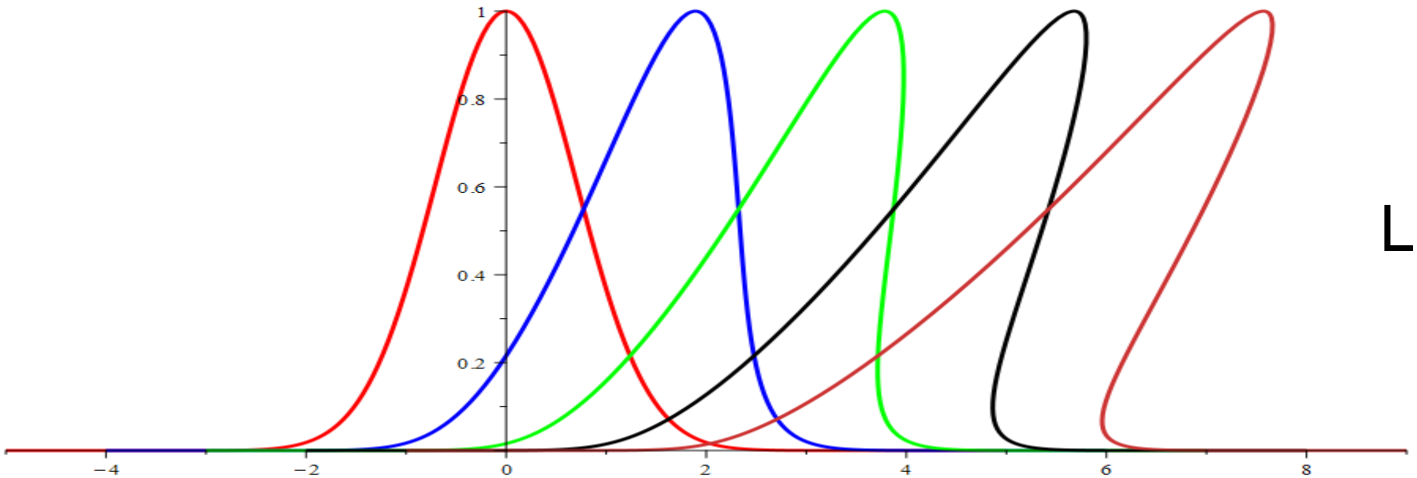
Analysis of shock wave
Plots of $\delta\rho$

Solution becomes unphysical

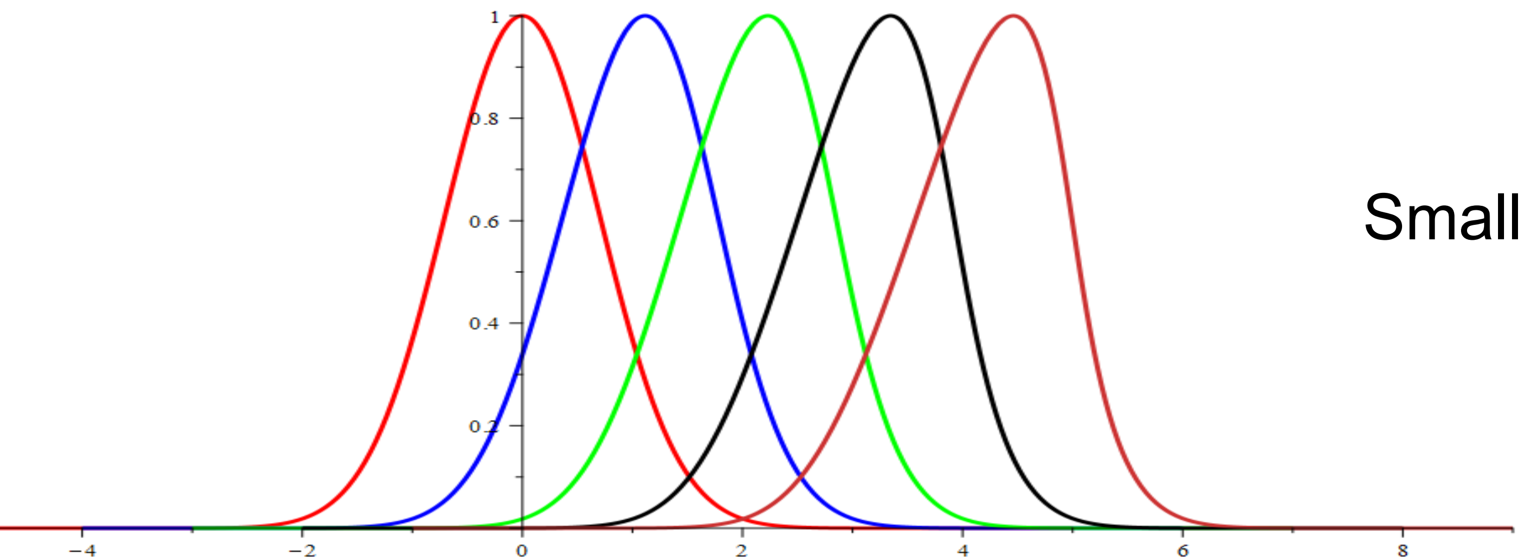


shock

Effects of amplitude of $\delta\rho$



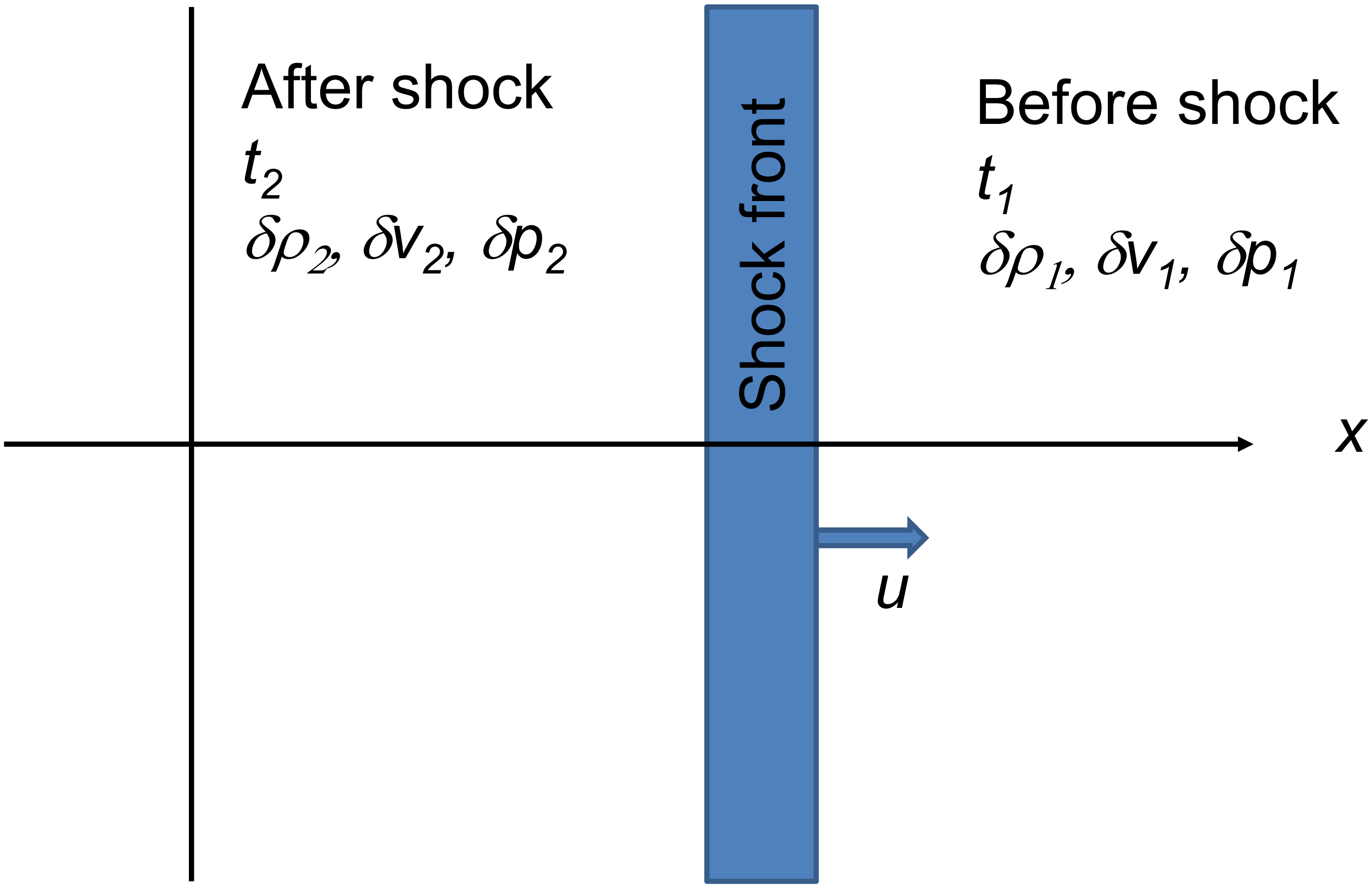
Large amplitude



Smaller amplitude

https://bilderreich.de/1314-16349/image-waves.html

Analysis of shock wave – assumed to moving at velocity u



Note that in this case u is assumed to be a given parameter of the system.

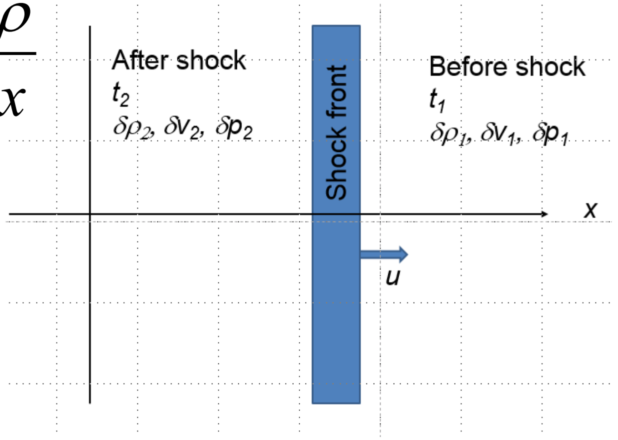
Analysis of shock wave – continued

While analysis in the shock region is complicated, we can use conservation laws to analyze regions 1 and 2

Assume $\rho(x,t) = \rho(x - ut) \quad \Rightarrow \frac{\partial \rho}{\partial t} = -u \frac{\partial \rho}{\partial x}$

$$p(x,t) = p(x - ut)$$

$$v(x,t) = v(x - ut)$$



Continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 = \frac{\partial(\rho v - \rho u)}{\partial x} \qquad \Rightarrow (v_2 - u)\rho_2 = (v_1 - u)\rho_1$$

Conservation of energy and momentum:

$$\Rightarrow p_2 + \rho_2 (v_2 - u)^2 = p_1 + \rho_1 (v_1 - u)^2$$

$$\Rightarrow \epsilon_2 + \frac{1}{2}(v_2 - u)^2 + \frac{p_2}{\rho_2} = \epsilon_1 + \frac{1}{2}(v_1 - u)^2 + \frac{p_1}{\rho_1}$$

Analysis of shock wave – continued

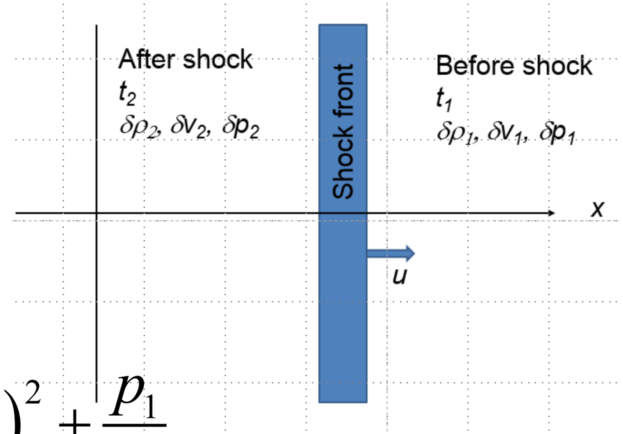
While analysis in the shock region is complicated, we can use conservation laws to analyze regions 1 and 2

Summary of equations

$$\Rightarrow (v_2 - u)\rho_2 = (v_1 - u)\rho_1$$

$$\Rightarrow p_2 + \rho_2 (v_2 - u)^2 = p_1 + \rho_1 (v_1 - u)^2$$

$$\Rightarrow \epsilon_2 + \frac{1}{2}(v_2 - u)^2 + \frac{p_2}{\rho_2} = \epsilon_1 + \frac{1}{2}(v_1 - u)^2 + \frac{p_1}{\rho_1}$$



Assume that within each regions (1 & 2), the ideal gas equations apply

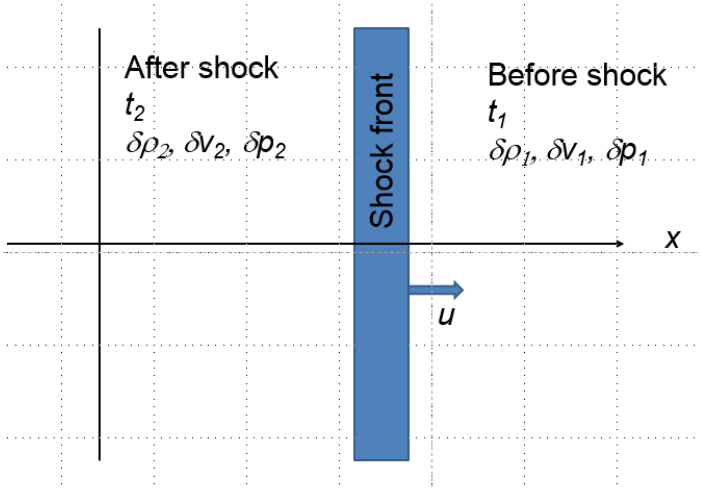
$$\epsilon_1 + \frac{p_1}{\rho_1} = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} \qquad \epsilon_2 + \frac{p_2}{\rho_2} = \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2}$$

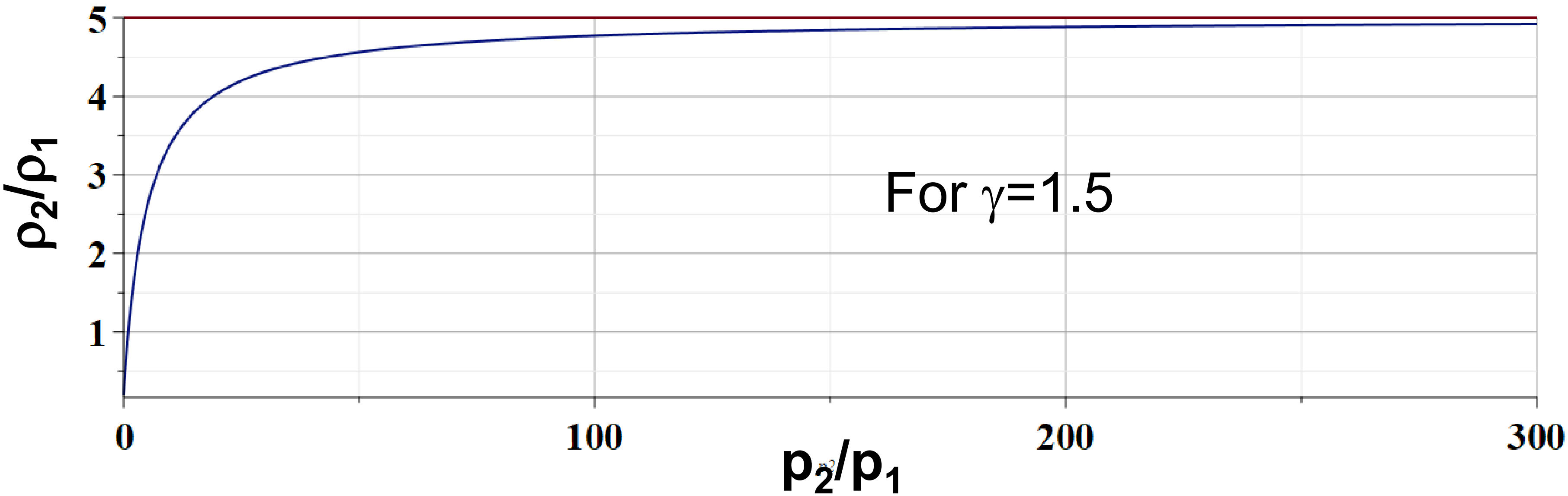
It follows that $$\frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{1}{2}(v_2 - u)^2 = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{1}{2}(v_1 - u)^2$$

Analysis of shock wave – continued

For adiabatic ideal gas, also considering energy and momentum conservation:

$$\frac{\rho_2}{\rho_1} = \frac{\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1}+1}{\frac{\gamma+1}{\gamma-1}+\frac{p_2}{p_1}} \leq \frac{\gamma+1}{\gamma-1}$$





Analysis of shock wave – continued

For adiabatic ideal gas, entropy considerations::

Ideal gas law: $\frac{p}{\rho} = \frac{k_B T}{M_0}$ Adiabatic ideal gas: $\frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma}$

Internal energy density: $\varepsilon \equiv \frac{E_{int}}{M} = \frac{p}{(\gamma - 1)\rho} = \frac{k_B T}{(\gamma - 1) M_0} \equiv c_V T$

First law of thermo: $d\varepsilon = T ds - p d\left(\frac{1}{\rho}\right)$

$$ds = \frac{1}{T}\left(d\left(\frac{p}{(\gamma-1)\rho}\right) + pd\left(\frac{1}{\rho}\right)\right) = \frac{p}{(\gamma-1)\rho T}\left(\frac{dp}{p} - \gamma\frac{d\rho}{\rho}\right) = c_V d\ln\left(\frac{p}{\rho^\gamma}\right)$$

$$s = c_V \ln\left(\frac{p}{\rho^\gamma}\right) + (\text{constant})$$

$$s_2 - s_1 = c_V \ln\left(\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^\gamma\right) \qquad 0 < s_2 - s_1 < c_V\left(\ln\left(\frac{p_2}{p_1}\right) - \gamma\ln\left(\frac{\gamma+1}{\gamma-1}\right)\right)$$