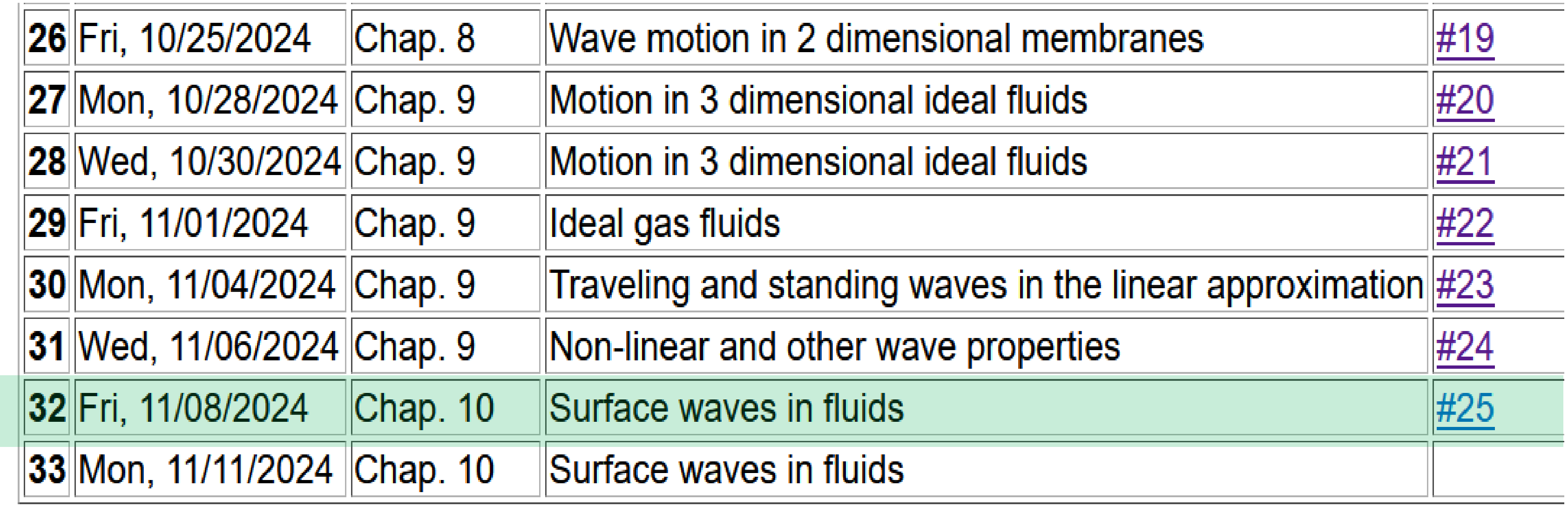
PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 32:

Chapter 10 in F & W: Surface waves

1. **Water waves in a channel**
2. **Wave-like solutions; wave speed**



26	Fri, 10/25/2024	Chap. 8	Wave motion in 2 dimensional membranes	#19
27	Mon, 10/28/2024	Chap. 9	Motion in 3 dimensional ideal fluids	#20
28	Wed, 10/30/2024	Chap. 9	Motion in 3 dimensional ideal fluids	#21
29	Fri, 11/01/2024	Chap. 9	Ideal gas fluids	#22
30	Mon, 11/04/2024	Chap. 9	Traveling and standing waves in the linear approximation	#23
31	Wed, 11/06/2024	Chap. 9	Non-linear and other wave properties	#24
32	Fri, 11/08/2024	Chap. 10	Surface waves in fluids	#25
33	Mon, 11/11/2024	Chap. 10	Surface waves in fluids	

PHY 711 -- Assignment #25

Assigned: 11/08/2024 Due: 11/11/2024

Start reading Chapter 10 in **Fetter & Walecka**.

1. Work Problem 10.3 at the end of Chapter 10 in **Fetter and Walecka**. Note that some of the ideas are discussed in Lecture 32.

Reference: Chapter 10 of Fetter and Walecka

Physics of incompressible fluids and their surfaces

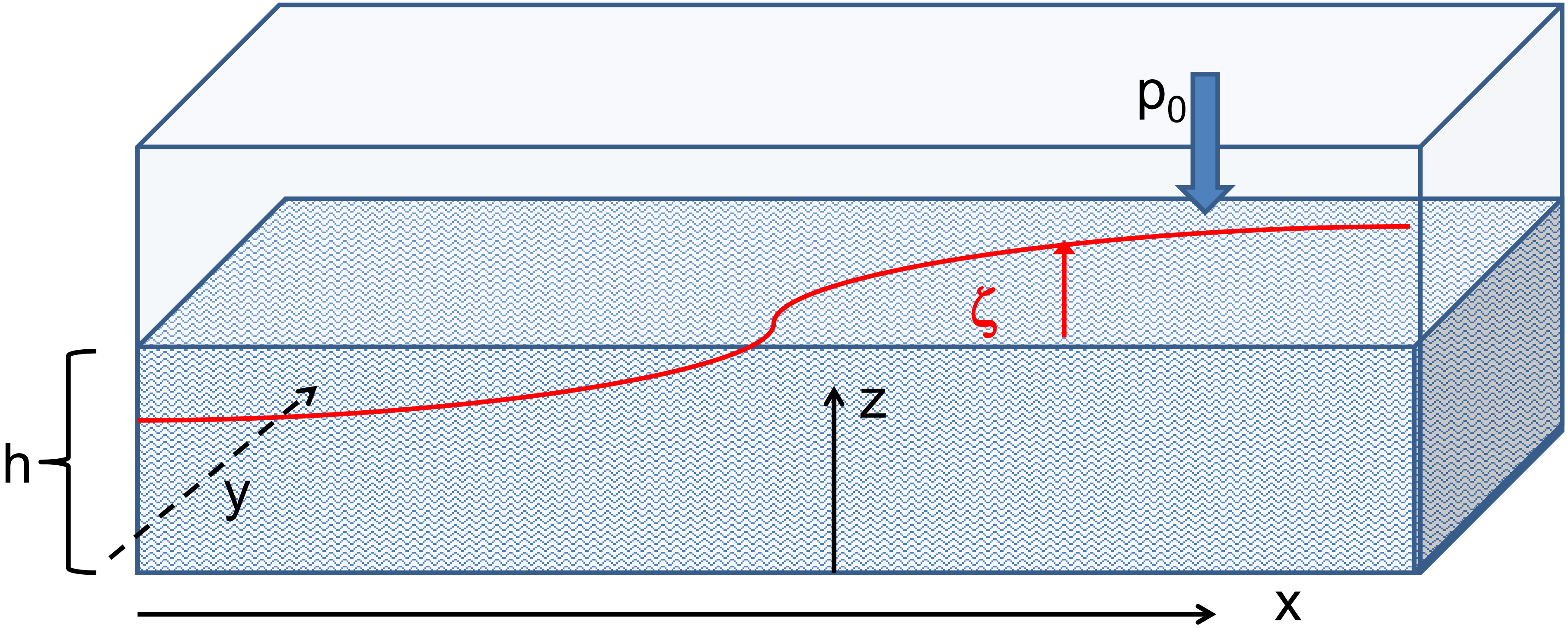
Consider a container of water with average height h and surface h+ζ(x,y,t); (h ←→ z_0 on some of the slides)

Atmospheric pressure is in equilibrium with the surface of water

Pressure at a height z above the bottom where the surface is at a height $h+\zeta$:

$$p(z) = \begin{cases} p_0 + \rho g\left(h+\zeta-z\right) & \text{For } z \leq h+\zeta \\ p_0 & \text{For } z > h+\zeta \end{cases}$$

Here ρ represents density of water



Euler/Newton equation of motion for fluid of density ρ :

$$\frac{d\mathbf{v}}{dt} = f_{applied} - \frac{\nabla p}{\rho} = -g\hat{\mathbf{z}} - \frac{\nabla p}{\rho}$$

This well-describes a fluid like water or like air, but care needs to be exercised when considering the interaction between the two.

In fact, we may not consider ρ_{air} directly because:

a. Because it is a reasonable approximation
b. Because it simplifies the analysis
c. Both of the above

Euler's equation inside a incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = f_{applied} - \frac{\nabla p}{\rho} = -g\hat{\mathbf{z}} - \frac{\nabla p}{\rho}$$

Assume that $v_z << v_x, v_y$ $\Rightarrow -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \approx 0$

$$\Rightarrow p(x,y,z,t) = p_0 + \rho g\left(\zeta(x,y,t) + h - z\right)$$

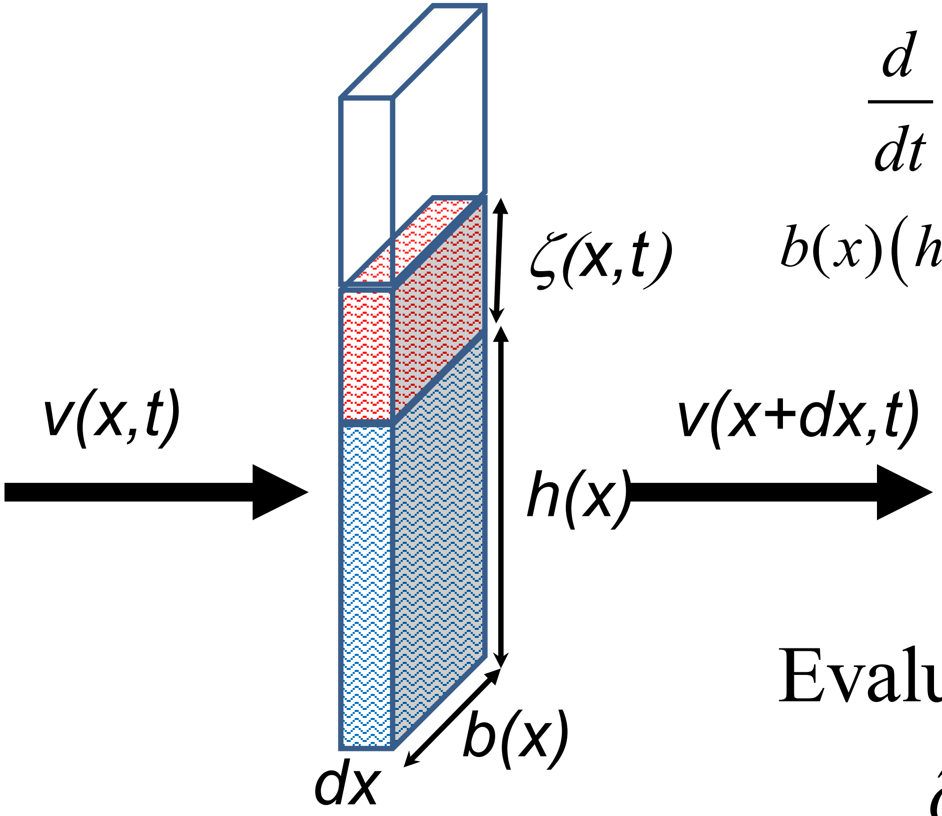
within the water

Horizontal fluid motions (keeping leading terms):

$$\frac{dv_x}{dt} \approx \frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial \zeta}{\partial x}$$

$$\frac{dv_y}{dt} \approx \frac{\partial v_y}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial \zeta}{\partial y}$$

Consider a surface wave moving in the *x*-direction in a channel of width *b(x)* and height *h(x)* +$\zeta(x,t)$:



Continuity condition in integral form:

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \mathbf{v}\cdot d\mathbf{A} = 0$$

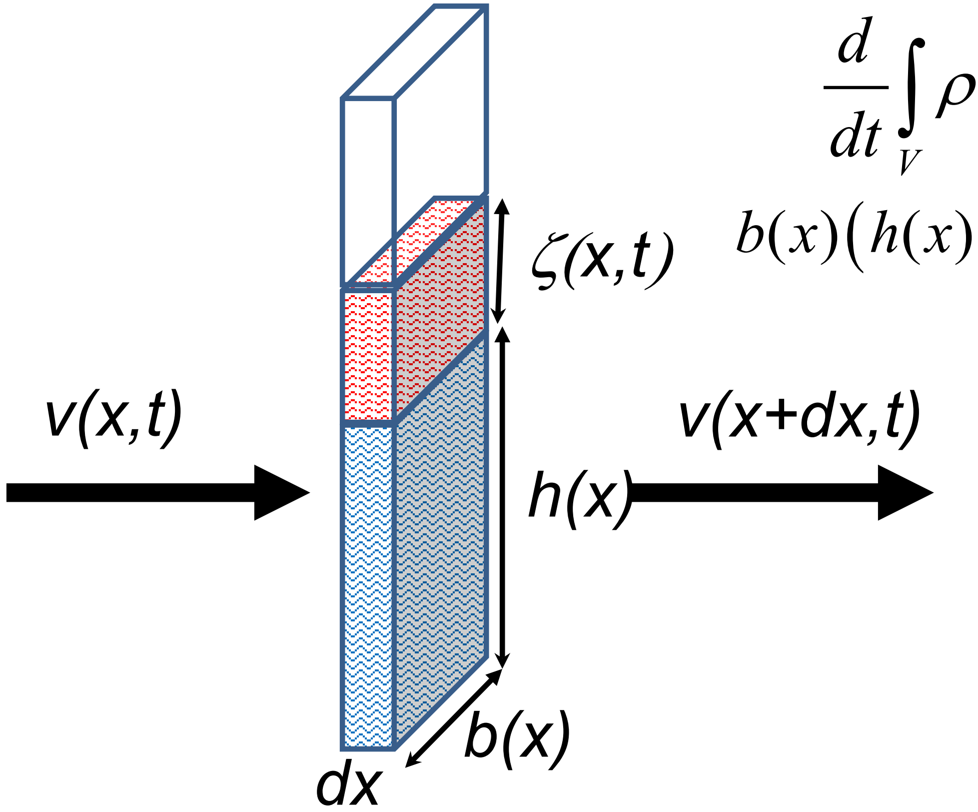
$$b(x)\big(h(x)+\zeta(x,t)\big)\hat{\mathbf{x}}$$

$$b(x)\big(h(x)+\zeta(x,t)\big)dx$$

Evaluating continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big(h(x)b(x)v(x,t)\big)$$

Some details



Continuity condition in integral form:

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \mathbf{v}\cdot d\mathbf{A} = 0$$

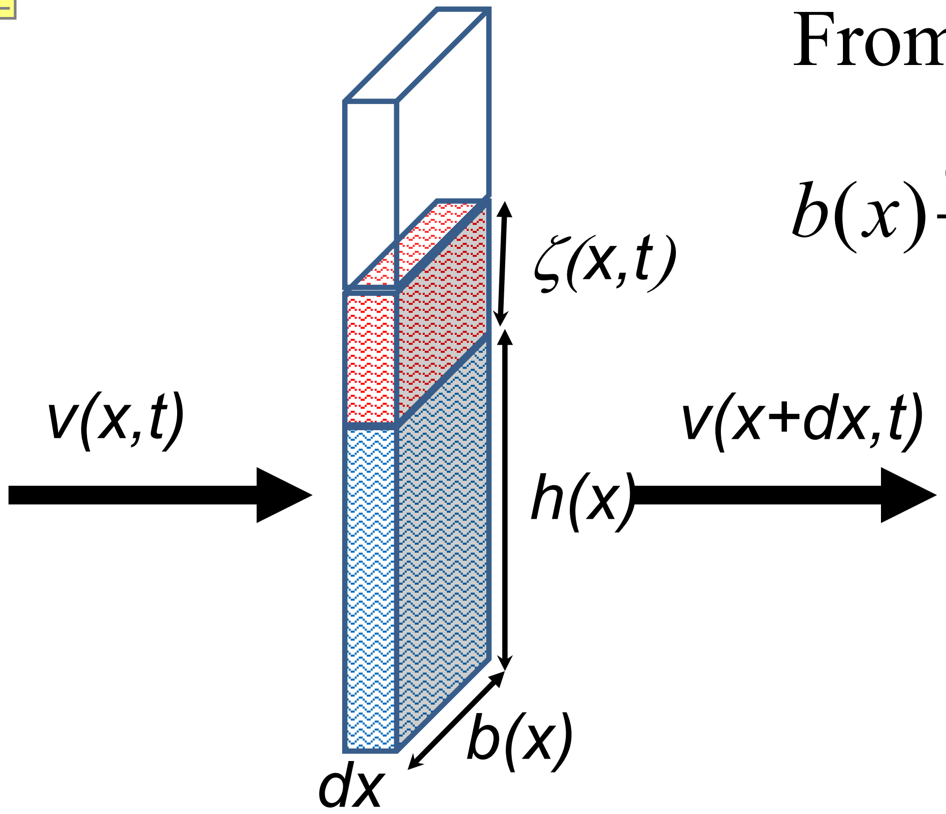
$$b(x)\left(h(x)+\zeta(x,t)\right)dx$$

$$b(x)\left(h(x)+\zeta(x,t)\right)\hat{\mathbf{x}}$$

Here, we are assuming that ρ is constant

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \mathbf{v}\cdot d\mathbf{A} = \rho\int b(x)\frac{\partial\zeta}{\partial t}dx + \rho\int\frac{\partial}{\partial x}\left(b(x)(h(x)+\zeta(x,t))v(x,t)\right)dx = 0$$

$$\Rightarrow b(x)\frac{\partial\zeta}{\partial t} = -\frac{\partial}{\partial x}\left(h(x)b(x)v(x,t)\right)$$



From continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\left(h(x)b(x)v(x,t)\right)$$

Example (Problem 10.3):

$$b(x) = b_0 \qquad h(x) = \kappa x$$

(A special case sometimes found at a beach.)

$$b_0\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\left((\kappa x)b_0 v(x,t)\right)$$

$$\frac{\partial \zeta}{\partial t} = -\kappa\left(v + x\frac{\partial v}{\partial x}\right)$$

From Newton-Euler equation:

$$\frac{dv}{dt} \approx \frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x}$$

Example continued

$$\frac{\partial \zeta}{\partial t} = -\kappa\left(v + x\frac{\partial v}{\partial x}\right) \quad \Rightarrow \quad \frac{\partial^2 \zeta}{\partial t^2} = -\kappa\left(\frac{\partial v}{\partial t} + x\frac{\partial^2 v}{\partial x \partial t}\right)$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x} \quad \Rightarrow \quad \frac{\partial^2 \zeta}{\partial t^2} = \kappa g\left(\frac{\partial \zeta}{\partial x} + x\frac{\partial^2 \zeta}{\partial x^2}\right)$$

It can be shown that a solution can take the form:

$$\zeta(x,t) = CJ_0\left(\frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}\right)\cos(\omega t)$$

Note that $J_0(u)$ satisfies the equation: $\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} + 1\right)J_0(u) = 0$

Therefore, for $u = \frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}$

$$\left(x\frac{d^2}{dx^2} + \frac{d}{dx}\right)J_0(u) = \frac{\omega^2}{\kappa g}\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du}\right)J_0(u) = -\frac{\omega^2}{\kappa g}J_0(u)$$

Therefore, for $u = \frac{2\omega}{\sqrt{\kappa g}}\sqrt{x} \quad \Rightarrow \frac{1}{\sqrt{x}} = \frac{2\omega}{\sqrt{\kappa g}}\frac{1}{u}$

$$\left(x\frac{d^2}{dx^2} + \frac{d}{dx} \right) J_0\left(u\right) = \frac{\omega^2}{\kappa g}\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} \right) J_0(u) = -\frac{\omega^2}{\kappa g} J_0(u)$$

Detail: $\frac{dJ_0(u)}{dx} = \frac{dJ_0(u)}{du}\frac{\omega}{\sqrt{\kappa g}}\frac{1}{\sqrt{x}}$

$$\frac{d^2 J_0(u)}{dx^2} = \frac{d^2 J_0(u)}{du^2}\left(\frac{\omega}{\sqrt{\kappa g}}\frac{1}{\sqrt{x}} \right)^2 - \frac{dJ_0(u)}{du}\frac{\omega}{2\sqrt{\kappa g}}\frac{1}{x\sqrt{x}}$$

Therefore: $\left(x\frac{d^2}{dx^2} + \frac{d}{dx} \right) J_0\left(u\right) = \left(\frac{\omega^2}{\kappa g}\frac{d^2 J_0(u)}{du^2} + \frac{dJ_0(u)}{du}\frac{\omega}{2\sqrt{\kappa g}}\frac{1}{\sqrt{x}} \right)$

$$= \frac{\omega^2}{\kappa g}\left(\frac{d^2 J_0(u)}{du^2} + \frac{dJ_0(u)}{du}\frac{1}{u} \right)$$

Example continued

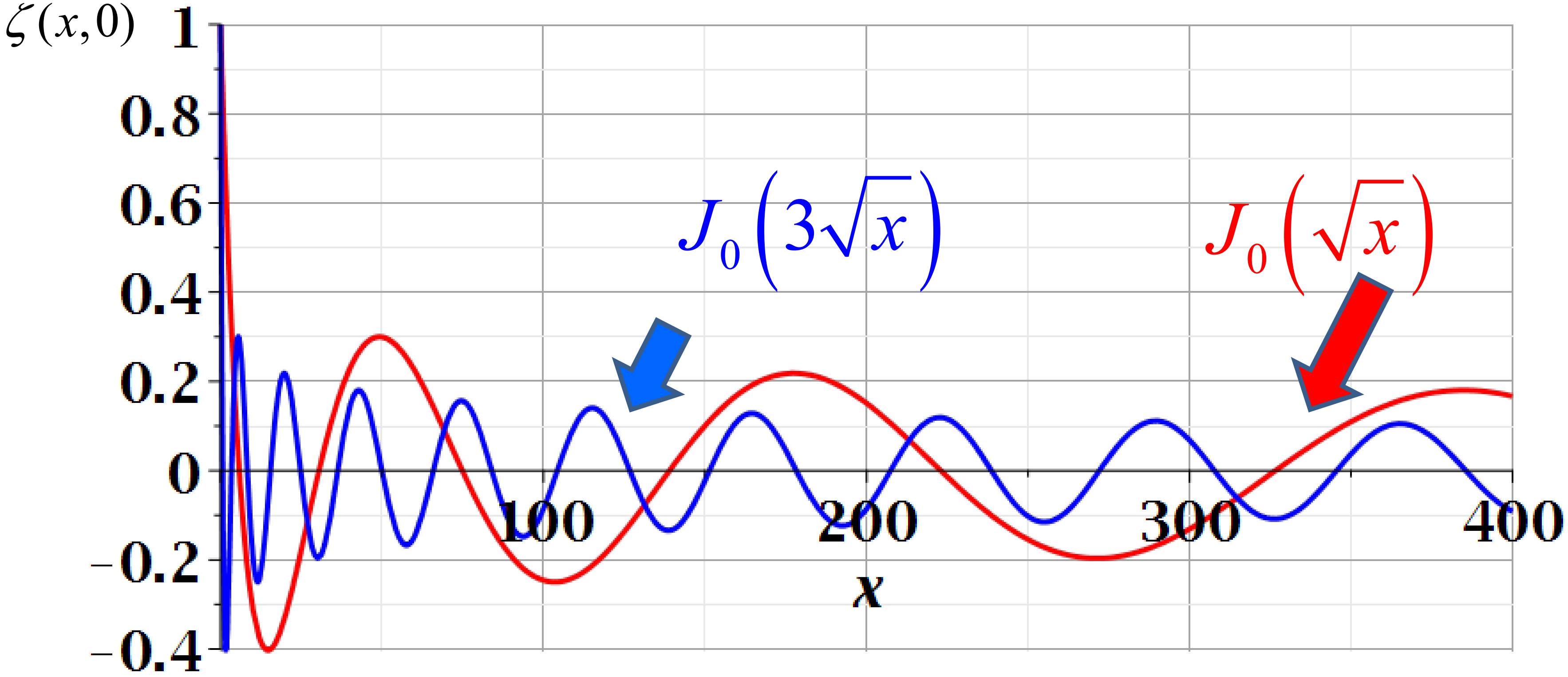
$$\frac{\partial^2 \zeta}{\partial t^2} = \kappa g \left(\frac{\partial \zeta}{\partial x} + x \frac{\partial^2 \zeta}{\partial x^2} \right)$$

$$\Rightarrow \zeta(x,t) = C J_0 \left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t)$$

Check:

$$-\omega^2 C J_0 \left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t) = \kappa g \left(\frac{\partial}{\partial x} + x \frac{\partial^2}{\partial x^2} \right) C J_0 \left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t)$$

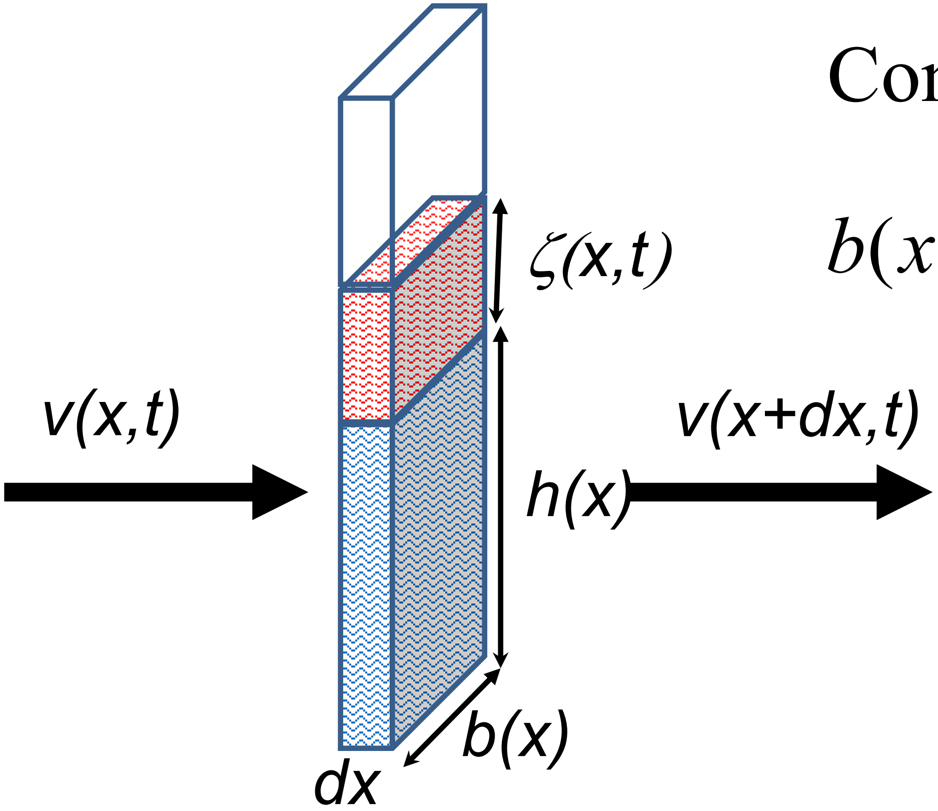
$$\zeta(x,t) = CJ_0\left(\frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}\right)\cos(\omega t)$$



Imagine watching the waves at a beach – can you visualize the configuration for the surface wave pattern to approximation this situation?

a. Long flat beach
b. Beach in which average water level increases
c. Beach in which average water level decreases

A simpler example:



Continuity condition:

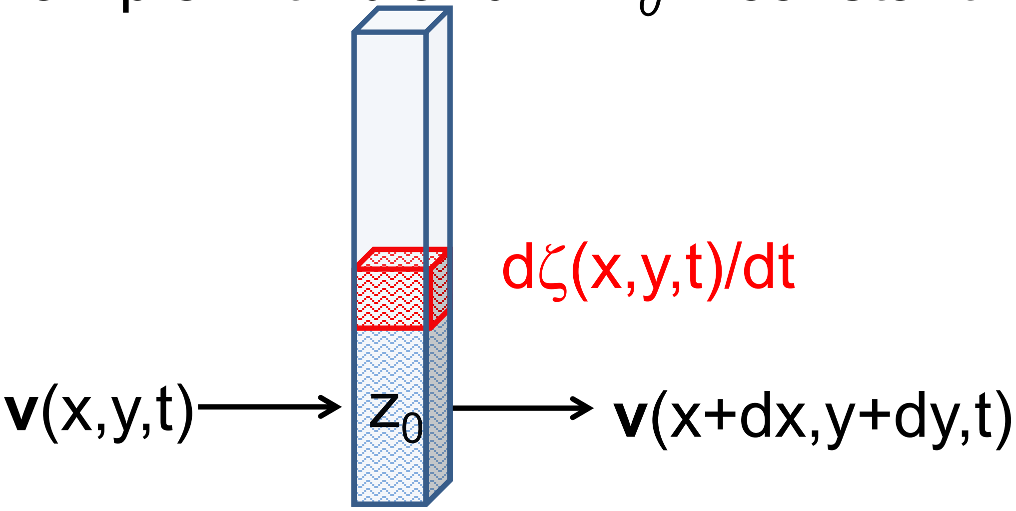
$$b(x)\frac{\partial\zeta}{\partial t} = -\frac{\partial}{\partial x}\big(h(x)b(x)v(x,t)\big)$$

Special case, where *b* and *h* are constant --

For constant b and h:

$$\frac{\partial\zeta}{\partial t} = -h\frac{\partial}{\partial x}\big(v(x,t)\big)$$

Example with b and $h=z_0=$ constant -- continued



Continuity condition for flow of incompressible fluid:

$$\frac{\partial \zeta}{\partial t} + h\nabla \cdot \mathbf{v} = 0$$

From horizontal flow relations: $\frac{\partial \mathbf{v}}{\partial t} = -g\nabla \zeta$

Equation for surface function: $\frac{\partial^2 \zeta}{\partial t^2} - gh\nabla^2 \zeta = 0$

For uniform channel:

Surface wave equation:

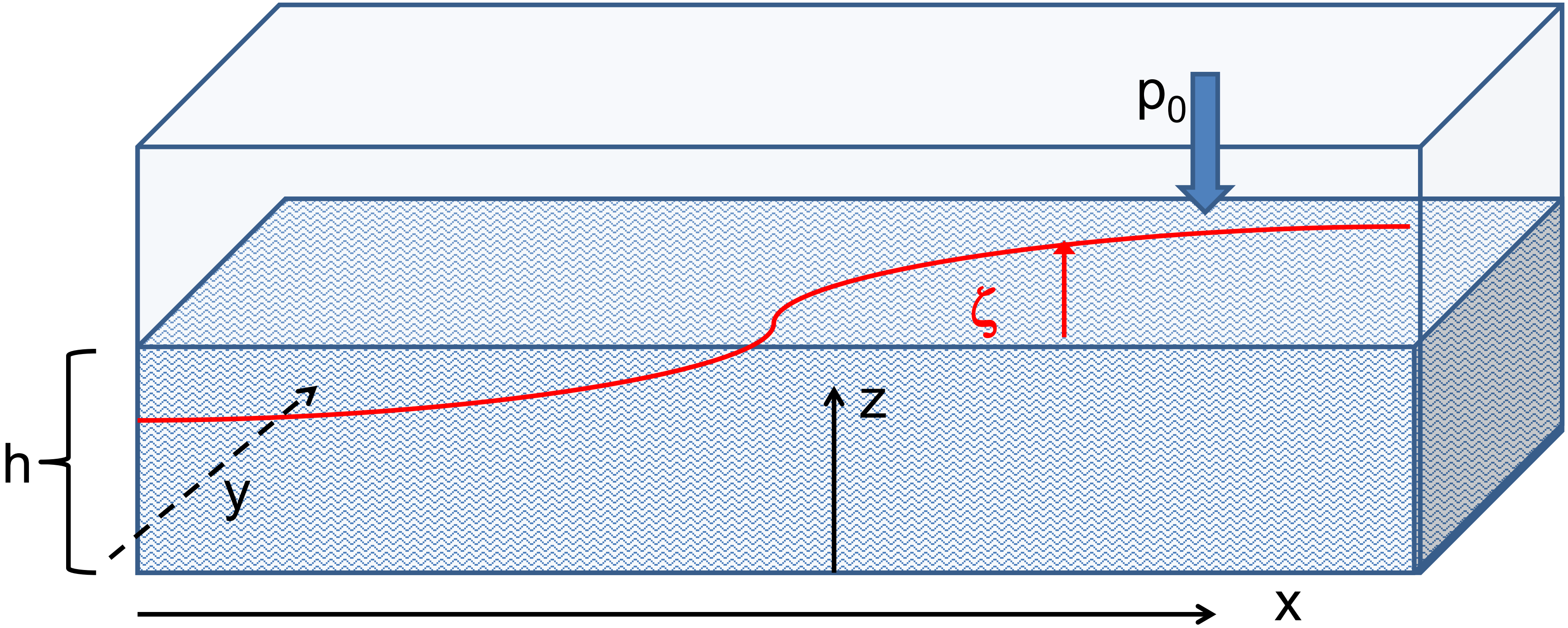
$$\frac{\partial^2 \zeta}{\partial t^2} - c^2 \nabla^2 \zeta = 0 \qquad\qquad c^2 = gh$$

More complete analysis finds:

$$c^2 = \frac{g}{k}\tanh(kh) \qquad \text{where } k = \frac{2\pi}{\lambda}$$

More details: -- recall setup --

Consider a container of water with average height h and surface h+$\zeta(x,y,t)$



Equations describing fluid itself (without boundaries)

Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt}=\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}=\frac{\partial\mathbf{v}}{\partial t}+\nabla\left(\tfrac{1}{2}v^2\right)+\mathbf{v}\times\left(\nabla\times\mathbf{v}\right)=-\nabla U-\frac{\nabla p}{\rho}$$

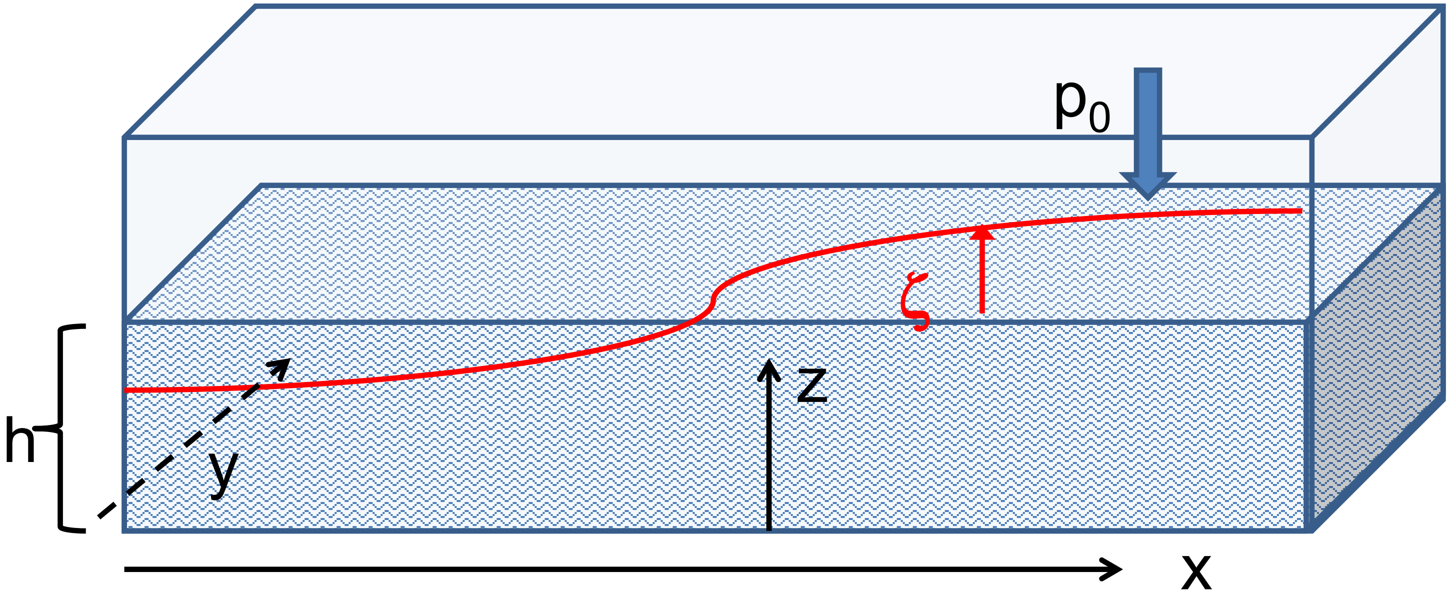
Assume that $\nabla\times\mathbf{v}=0$ (irrotational flow) $\Rightarrow \mathbf{v}=-\nabla\Phi$

$$\Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}v^2+U+\frac{p}{\rho}\right)=0$$

$$\Rightarrow -\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}v^2+U+\frac{p}{\rho}=\text{constant (within the fluid)}$$

For the same system, the continuity condition becomes

$$\nabla\cdot\mathbf{v}=-\nabla^2\Phi=0$$



Within fluid: $0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} v^2 + g\left(z - h\right) = \text{constant}$$

(We have absorbed p$_0$ in “constant”)

$$-\nabla^2 \Phi = 0$$

At surface: $z = h + \zeta$ with $\zeta = \zeta\left(x, y, t\right)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y}$$ where $v_{x,y} = v_{x,y}\left(x, y, h + \zeta, t\right)$

Full equations:

Within fluid: $0 \leq z \leq h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} v^2 + g\left(z - h\right) = \text{constant}$$

(We have absorbed p_0 in “constant”)

$$-\nabla^2 \Phi = 0$$

At surface: $z = h + \zeta$ with $\zeta = \zeta\left(x, y, t\right)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y}$$

where $v_{x,y} = v_{x,y}\left(x, y, h + \zeta, t\right)$

Linearized equations:

For $0 \leq z \leq h + \zeta$: $-\dfrac{\partial \Phi}{\partial t} + g\left(z - h\right) = 0$ $\quad -\nabla^2 \Phi = 0$

At surface : $z = h + \zeta$ $\quad \dfrac{d\zeta}{dt} = \dfrac{\partial \zeta}{\partial t} = v_z\left(x, y, h + \zeta, t\right)$

$$-\frac{\partial \Phi\left(x, y, h + \zeta, t\right)}{\partial t} + g\zeta = 0$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x*:

For $\quad 0 \leq z \leq h + \zeta : \qquad \nabla^2 \Phi = \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} \right) \Phi(x,z,t) = 0$

Consider and periodic waveform: $\quad \Phi(x,z,t) = Z(z)\cos\big(k(x-ct)\big)$

$$\Rightarrow \left(\frac{d^2}{dz^2} - k^2 \right) Z(z) = 0$$

Boundary condition at bottom of tank: $\quad v_z(x,0,t) = 0$

$$\Rightarrow \frac{dZ}{dz}\big(0\big) = 0 \qquad\qquad Z(z) = A\cosh(kz)$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$\text{At surface}: \quad z = h + \zeta \qquad \frac{\partial \zeta}{\partial t} = v_z(x, h+\zeta, t) = -\frac{\partial \Phi(x, h+\zeta, t)}{\partial z}$$

$$-\frac{\partial \Phi(x, h+\zeta, t)}{\partial t} + g\zeta = 0$$

$$-\frac{\partial^2 \Phi(x, h+\zeta, t)}{\partial t^2} + g\frac{\partial \zeta}{\partial t} = -\frac{\partial^2 \Phi(x, h+\zeta, t)}{\partial t^2} - g\frac{\partial \Phi(x, h+\zeta, t)}{\partial z} = 0$$

$$\text{For} \qquad \Phi(x,(h+\zeta),t) = A\cosh(k(h+\zeta))\cos(k(x-ct))$$

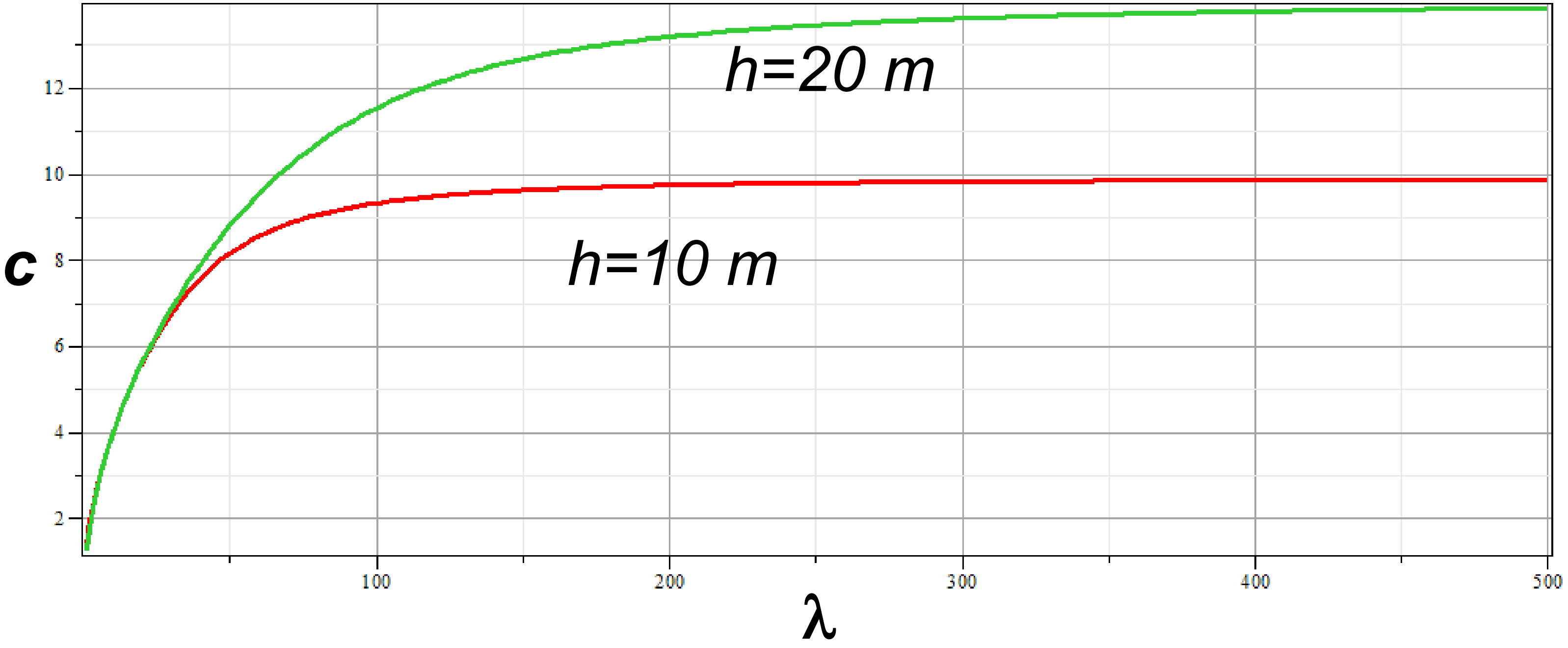
$$A\cosh(k(h+\zeta))\cos(k(x-ct))\left(k^2c^2 - gk\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}\right) = 0$$

$$\Rightarrow \mathrm{c}^2 = \frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 = \frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))} = \frac{g}{k}\tanh\left(k\left(h+\zeta\right)\right)$$

$$\text{Assuming } \zeta << h: \qquad c^2 = \frac{g}{k}\tanh\left(kh\right) \qquad \lambda = \frac{2\pi}{k}$$



For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 \approx \frac{g}{k}\tanh(kh) \qquad \text{For } \lambda >> h,\ c^2 \approx gh$$

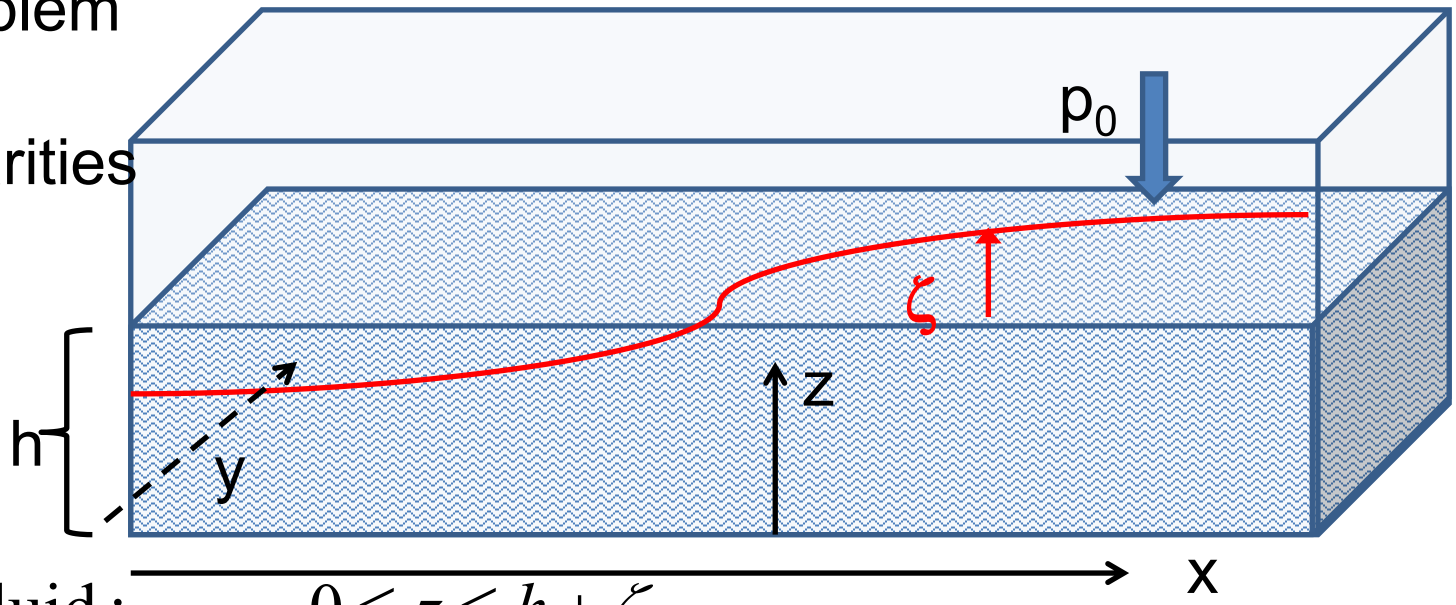
$$\Phi(x,z,t) = A\cosh(kz)\cos(k(x-ct))$$

$$\zeta(x,t) = \frac{1}{g}\frac{\partial\Phi(x,h+\zeta,t)}{\partial t} \approx \frac{kc}{g}A\cosh(kh)\sin(k(x-ct))$$

$$\text{Note that for } \lambda >> h, \quad c^2 \approx gh$$

(solutions are consistent with previous analysis)

General problem
including
non-linearities



$$\text{Within fluid:} \qquad 0 \le z \le h+\zeta$$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}v^2 + g(z-h) = \text{constant} \quad \text{(We have absorbed}$$

$$-\nabla^2 \Phi = 0 \qquad\qquad p_0 \text{ in our constant.)}$$

$$\text{At surface:} \quad z = h+\zeta \qquad \text{with } \zeta = \zeta(x,y,t)$$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} \qquad \text{where } v_{x,y} = v_{x,y}(x,y,h+\zeta,t)$$