# PHY 711 Classical Mechanics and Mathematical Methods
# 10-10:50 AM MWF in Olin 103

## Notes for Lecture 33: Chapter 10 in F & W

## Surface waves

- Summary of linear surface wave solutions
- Non-linear contributions and soliton solutions

**This material is covered in Chapter 10 of your textbook using similar notation.**

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|---|---|---|---|---|
| **31** | Wed, 11/06/2024 | Chap. 9 | Non-linear and other wave properties | #24 |
| **32** | Fri, 11/08/2024 | Chap. 10 | Surface waves in fluids | #25 |
| **33** | Mon, 11/11/2024 | Chap. 10 | Surface waves in fluids; soliton solutions | #26 |
| **34** | Wed, 11/13/2024 | Chap. 11 | Heat conduction | |
| **35** | Fri, 11/15/2024 | Chap. 12 | Viscous effects in hydrodynamics | |
| **36** | Mon, 11/18/2024 | Chap. 12 | Viscous effects in hydrodynamics | |
| **37** | Wed, 11/20/2024 | Chap. 13 | Elasticity | |
| **38** | Fri, 11/22/2024 | Chap. 1-13 | Review | |
| **39** | Mon, 11/25/2024 | Chap. 1-13 | Review | |
| | Wed, 11/27/2024 | Thanksgiving | | |
| | Fri, 11/29/2024 | Thanksgiving | | |
| **40** | Mon, 12/02/2024 | Chap. 1-13 | Review | |
| | Wed, 12/04/2024 | | Presentations 1 | |
| | Fri, 12/06/2023 | | Presentations 2 | |

# PHY 711 – Homework # 26

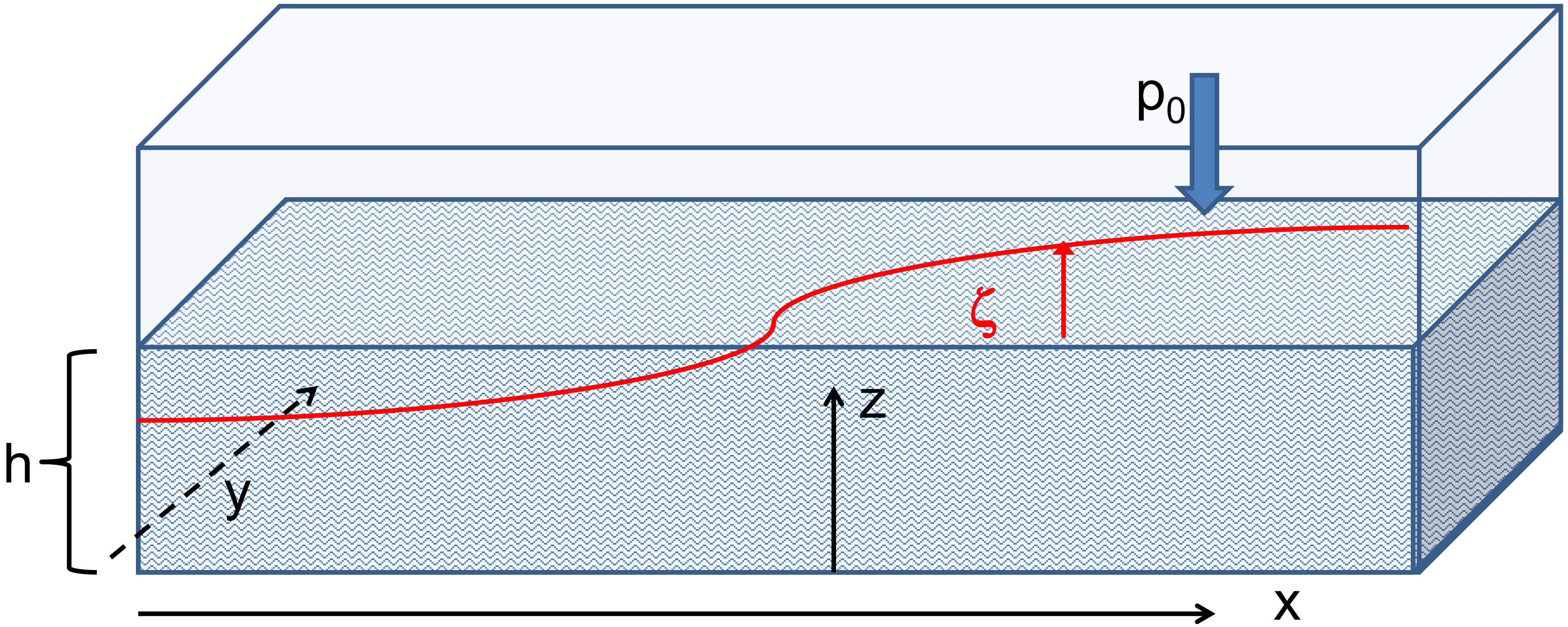
Assigned: 11/11/2024 Due: 11/18/2024

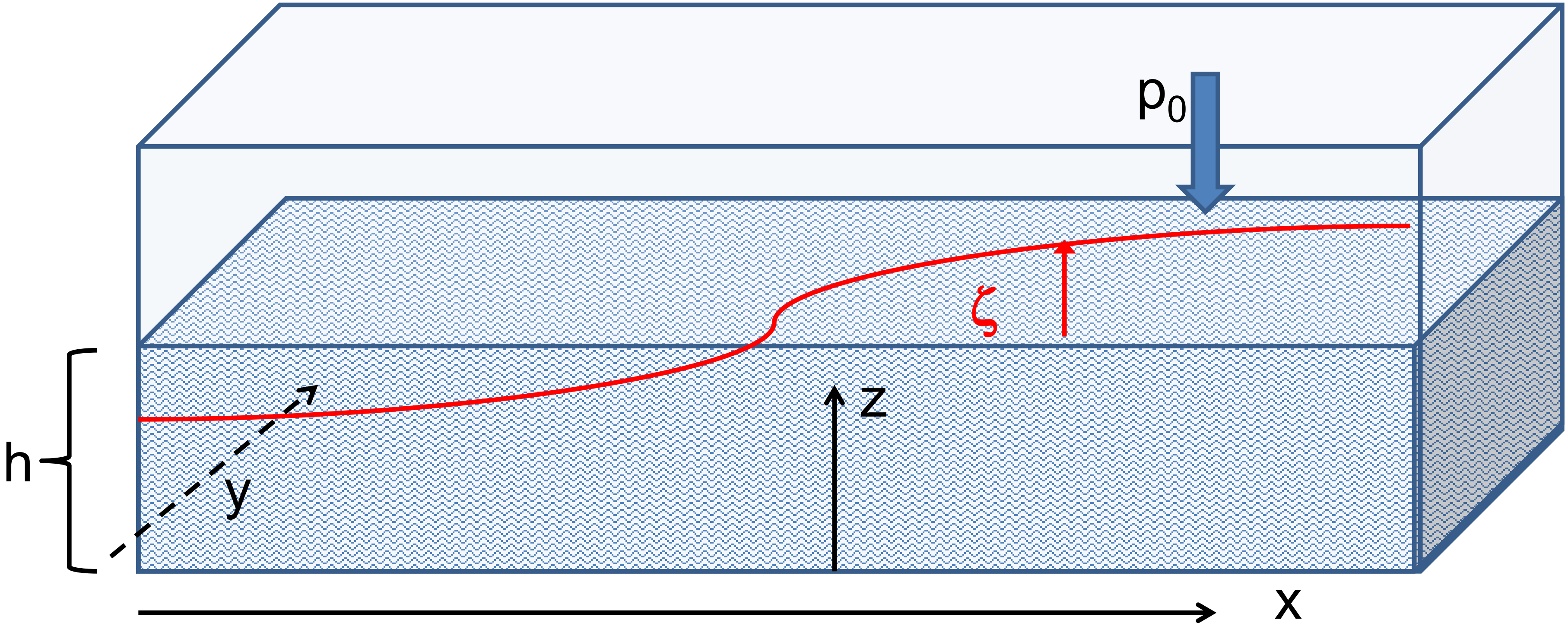
Read Chapter 10 of **Fetter and Walecka**.

1. In your textbook and in class, we discussed the traveling wave soliton solutions to the surface height function $\eta(u)$ as a function of position $x$ and time $t$, $u = x - ct$ takes the form
$$\eta(u) = \frac{\eta_0}{\cosh^2(\sqrt{\frac{3\eta_0}{4h^3}}u)}.$$
Here $\eta_0$ is a scale factor for the height which is related to the acceleration of gravity $g$, the average water height $h$, and the speed parameter of the soliton $c$ according to
$$\eta_0 = h\left(1 - \frac{gh}{c^2}\right).$$
With this information, show that $\eta(u)$ is a solution to the non-linear equation
$$\frac{\eta_0}{h}\eta(u) - \frac{3}{2h}\eta^2(u) - \frac{h^2}{3}\frac{d^2\eta(u)}{du^2} = 0.$$

Consider a container of water with average height h and surface h+$\zeta$(x,y,t)

Atmospheric pressure $p_0$ is in equilibrium at the surface






Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = f_{applied} - \frac{\nabla p}{\rho} = -\nabla U - \frac{\nabla p}{\rho}$$

Continuity equation within the fluid

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{v} = 0$$

For irrotational flow -- $\mathbf{v} = -\nabla \Phi$

Linearized equation: $\nabla\left( -\frac{\partial \Phi}{\partial t} + g(z-h) + \frac{p}{\rho} \right) = 0$

At surface: $z = h + \zeta \qquad -\frac{\partial \Phi}{\partial t} + g\zeta + \frac{p_0}{\rho} = 0$

Keep only linear terms and assume that horizontal variation is only along *x*:

For $\quad 0 \leq z \leq h + \zeta : \qquad \nabla^2 \Phi = \left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} \right) \Phi(x, z, t) = 0$

Consider and periodic waveform: $\quad \Phi(x, z, t) = Z(z) \cos\big( k(x - ct) \big)$

$$\Rightarrow \left( \frac{d^2}{dz^2} - k^2 \right) Z(z) = 0$$

Boundary condition at bottom of tank: $\quad v_z(x, 0, t) = 0$

$$\Rightarrow \frac{dZ}{dz}\big(0\big) = 0 \qquad\qquad Z(z) = A \cosh(kz)$$

At surface: $\quad z = h + \zeta \qquad \frac{\partial \zeta}{\partial t} = v_z\big(x, h + \zeta, t\big) = -\frac{\partial \Phi\big(x, h + \zeta, t\big)}{\partial z}$

Also: $\qquad -\frac{\partial \Phi\big(x, h + \zeta, t\big)}{\partial t} + g\zeta + \frac{p_0}{\rho} = 0$

$$\Rightarrow -\frac{\partial^2 \Phi\big(x, h + \zeta, t\big)}{\partial t^2} + g \frac{\partial \zeta}{\partial t} = -\frac{\partial^2 \Phi\big(x, h + \zeta, t\big)}{\partial t^2} - g \frac{\partial \Phi\big(x, h + \zeta, t\big)}{\partial z} = 0$$

Velocity potential: $\Phi(x,z,t) = A\cosh(kz)\cos\left(k(x-ct)\right)$

At surface: $\Phi(x,\left(h+\zeta\right),t) = A\cosh(k\left(h+\zeta\right))\cos\left(k(x-ct)\right)$

$$\frac{\partial\zeta}{\partial t} = v_z\left(x,h+\zeta,t\right) = -\frac{\partial\Phi\left(x,h+\zeta,t\right)}{\partial z} \qquad -\frac{\partial\Phi\left(x,h+\zeta,t\right)}{\partial t} + g\zeta + \frac{p_0}{\rho} = 0$$

$$\Rightarrow -\frac{\partial^2\Phi\left(x,h+\zeta,t\right)}{\partial t^2} + g\frac{\partial\zeta}{\partial t} = -\frac{\partial^2\Phi\left(x,h+\zeta,t\right)}{\partial t^2} - g\frac{\partial\Phi\left(x,h+\zeta,t\right)}{\partial z} = 0$$

$$A\cosh(k\left(h+\zeta\right))\cos\left(k(x-ct)\right)\left(k^2c^2 - gk\frac{\sinh(k\left(h+\zeta\right))}{\cosh(k\left(h+\zeta\right))}\right) = 0$$

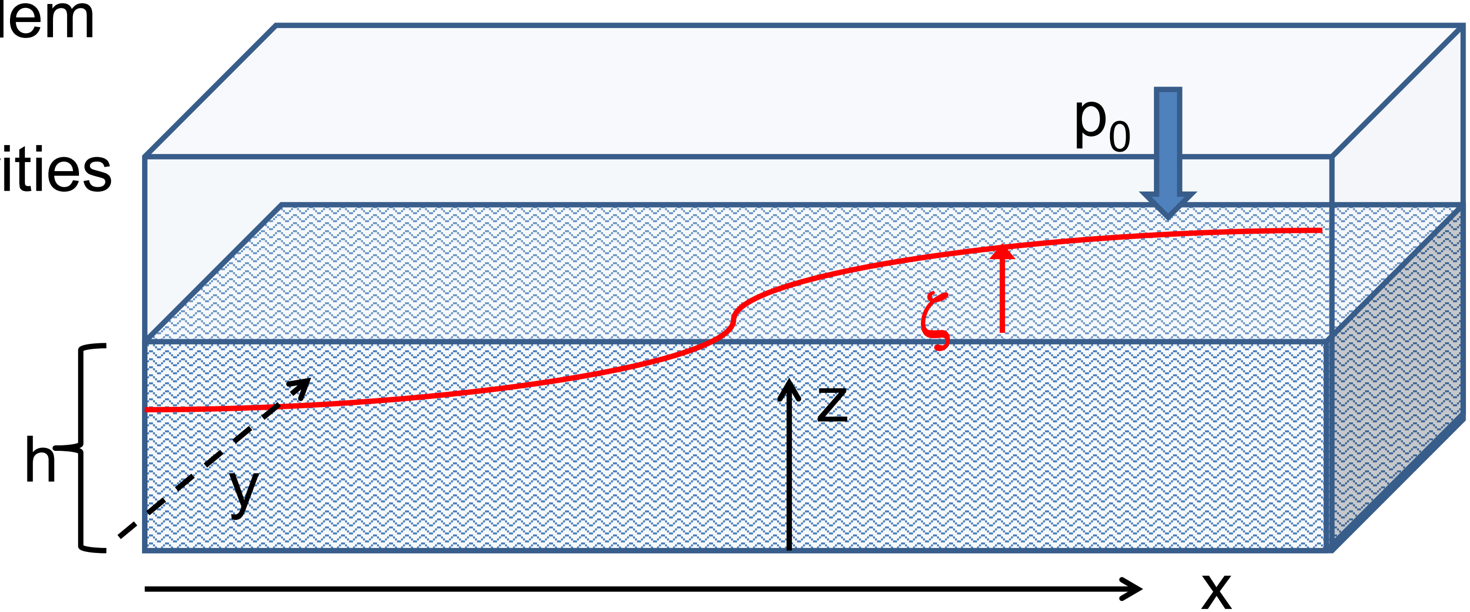
$$\Rightarrow c^2 = \frac{g}{k}\frac{\sinh(k\left(h+\zeta\right))}{\cosh(k\left(h+\zeta\right))} \approx \frac{g}{k}\tanh(kh)$$

Note that this solution represents a pure plane wave. More likely, there would be a linear combination of wavevectors *k*. Additionally, your text considers the effects of surface tension which is ignored here.

**In this lecture, we will focus on the effects of the non-linearities in the Euler, continuity, and surface equations.**

# Surface waves in an incompressible fluid

General problem including non-linearities




Within fluid: $0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} v^2 + g\left(z - h\right) = \text{constant} \qquad \Phi = \Phi(x, y, z, t)$$

$$-\nabla^2 \Phi = 0 \qquad \mathbf{v} = \mathbf{v}(x, y, z, t) = -\nabla \Phi(x, y, z, t)$$

At surface: $z = h + \zeta$ with $\zeta = \zeta\left(x, y, t\right)$

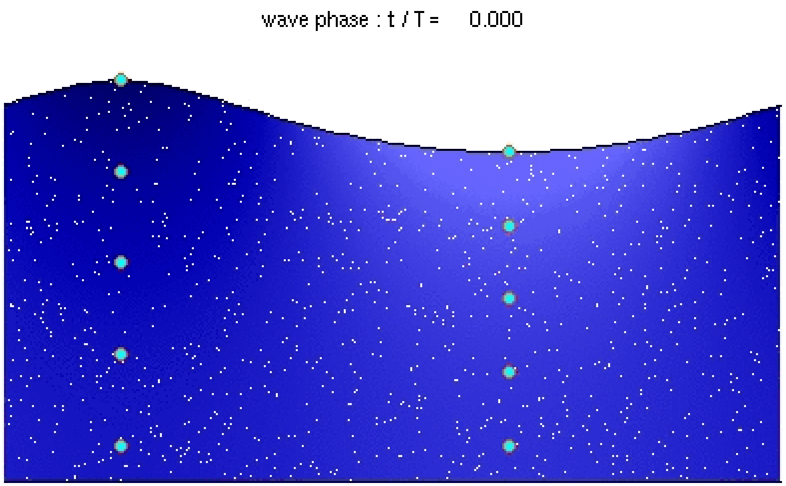
$$v_z(h+\zeta) = \frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} = -\left.\frac{\partial \Phi(x, y, z, t)}{\partial z}\right|_{z=h+\zeta} \quad \text{where } v_{x,y} = v_{x,y}\left(x, y, h+\zeta, t\right)$$

# Some relationships at surface --

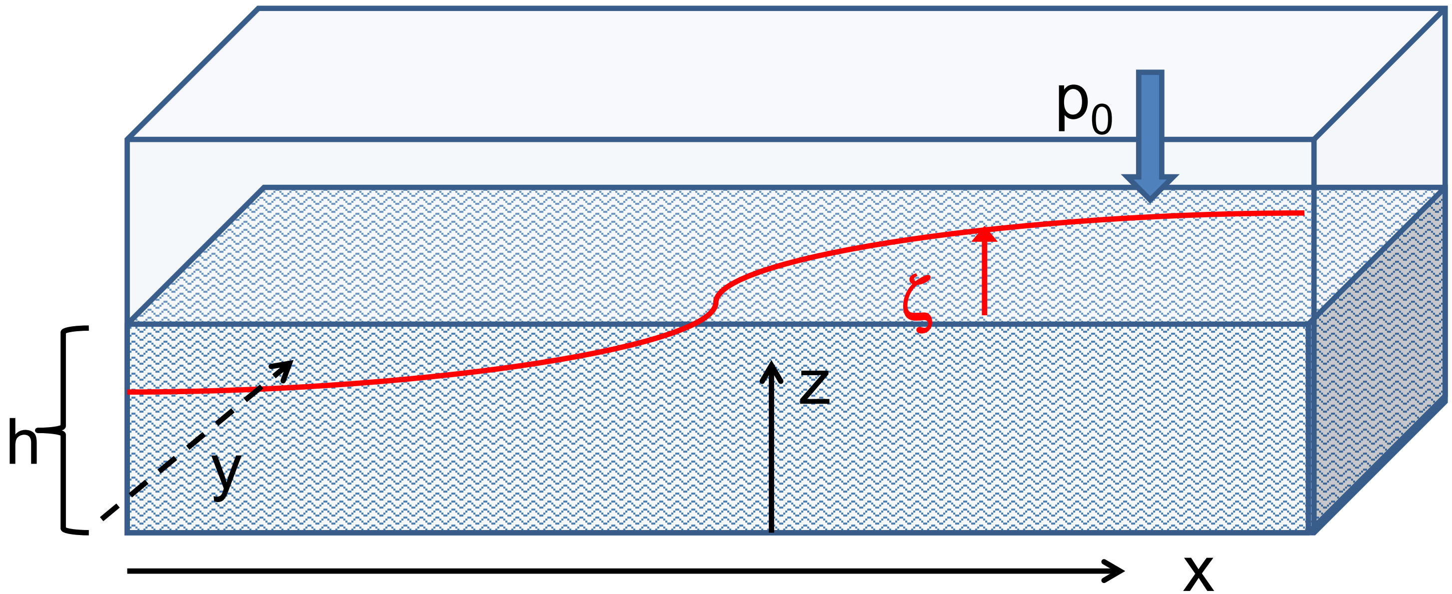
At surface: $z = h + \zeta$ with $\zeta = \zeta\left(x, y, t\right)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} = v_z = -\left.\frac{\partial \Phi(x, y, z, t)}{\partial z}\right|_{z=h+\zeta} \text{; where } v_{x,y} = v_{x,y}\left(x, y, h+\zeta, t\right)$$

Note that $v_z(x, y, h+\zeta, t) = \dfrac{d\zeta}{dt}$




Linear approximation (from Wikipedia)



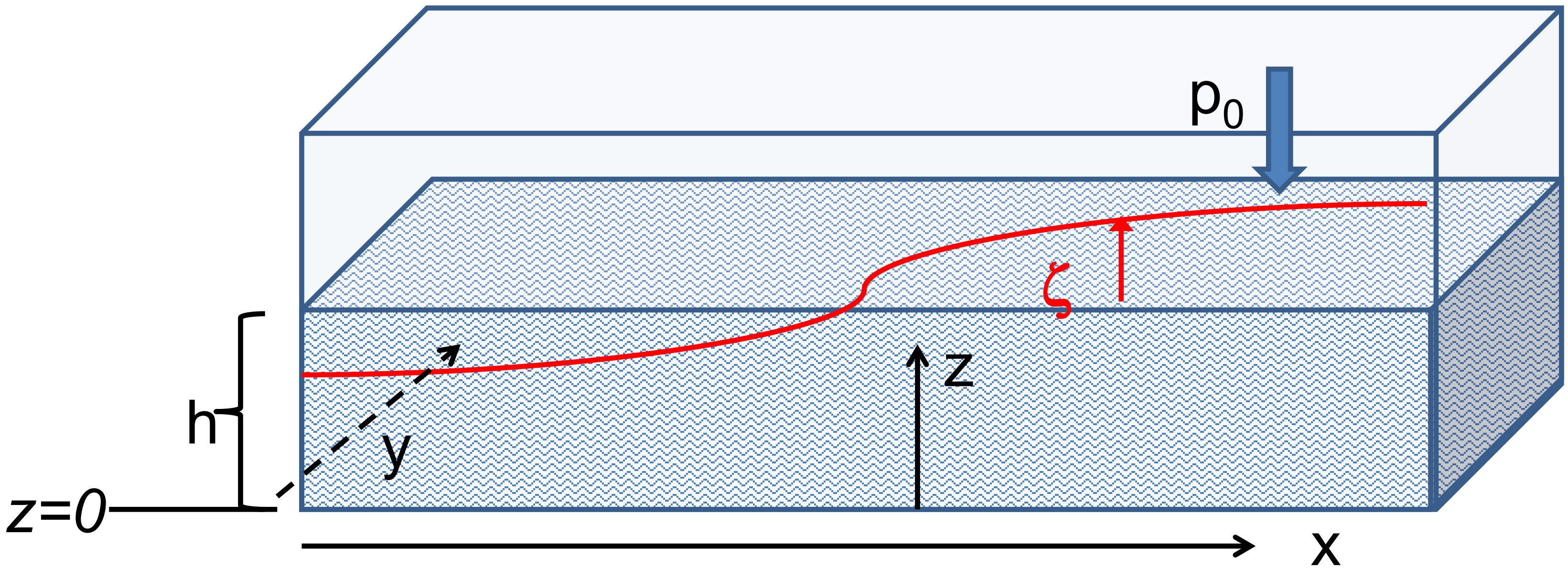

Further simplifications; assume trivial $y$-dependence

$$\Phi = \Phi\left(x, z, t\right) \qquad\qquad \zeta = \zeta\left(x, t\right)$$

Within fluid: $\quad 0 \le z \le h + \zeta$

At surface: $\quad v_z\left(x, z = h + \zeta, t\right) = -\dfrac{\partial \Phi}{\partial z} = \dfrac{d\zeta}{dt}$

# Non-linear effects in surface waves:




Dominant non-linear effects $\Rightarrow$ soliton solutions

$$\zeta(x,t) = \eta_0 \,\mathrm{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right)$$

$\eta_0 =$ constant (representing amplitude)

$$\text{where } c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1+\frac{\eta_0}{2h}\right)$$

# Detailed analysis of non-linear surface waves

[Note that these derivations follow Alexander L. Fetter and John Dirk Walecka, *Theoretical Mechanics of Particles and Continua* (McGraw Hill, 1980), Chapt. 10.]

We assume that we have an incompressible fluid: $\rho = \text{constant}$

Velocity potential: $\Phi(x,z,t)$; $\mathbf{v}(x,z,t) = -\nabla\Phi(x,z,t)$

The surface of the fluid is described by $z=h+\zeta(x,t)$. It is assumed that the fluid is contained in a structure (lake, river, swimming pool, etc.) with a structureless bottom defined by the $z = 0$ plane and filled to an equilibrium height of $z = h$.

Defining equations for $\Phi(x,z,t)$ and $\zeta(x,t)$

where $0 \leq z \leq h + \zeta(x,t)$

Continuity equation:

$$\nabla \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \frac{\partial^2 \Phi(x,z,t)}{\partial x^2} + \frac{\partial^2 \Phi(x,z,t)}{\partial z^2} = 0$$

Bernoulli equation (assuming irrotational flow) and gravitation potential energy

$$-\frac{\partial \Phi(x,z,t)}{\partial t} + \frac{1}{2}\left[\Big(\underbrace{\frac{\partial \Phi(x,z,t)}{\partial x}}_{v_x^2}\Big)^2 + \Big(\underbrace{\frac{\partial \Phi(x,z,t)}{\partial z}}_{v_z^2}\Big)^2\right] + g(z-h) = 0.$$

## Boundary conditions on functions –

Zero velocity at bottom of tank:

$$\frac{\partial \Phi(x,0,t)}{\partial z} = 0.$$

Consistent vertical velocity at water surface

$$v_z(x,z,t)\Big|_{z=h+\zeta} = \frac{d\zeta}{dt} = \mathbf{v}\cdot\nabla\zeta + \frac{\partial \zeta}{\partial t}$$

$$= v_x \frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial t}$$

$$\Rightarrow -\frac{\partial \Phi(x,z,t)}{\partial z} + \frac{\partial \Phi(x,z,t)}{\partial x}\frac{\partial \zeta(x,t)}{\partial x} - \frac{\partial \zeta(x,t)}{\partial t}\Bigg|_{z=h+\zeta} = 0$$

**Analysis assuming water height $z$ is small relative to variations in the direction of wave motion *(x)***

**Taylor's expansion about** $z = 0$:

$$\Phi(x,z,t) \approx \Phi(x,0,t) + z\frac{\partial\Phi}{\partial z}(x,0,t) + \frac{z^2}{2}\frac{\partial^2\Phi}{\partial z^2}(x,0,t) + \frac{z^3}{3!}\frac{\partial^3\Phi}{\partial z^3}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4\Phi}{\partial z^4}(x,0,t)\cdots$$

Note that the zero vertical velocity at the bottom suggest that to a good approximation, that all odd derivatives $\frac{\partial^n\Phi}{\partial z^n}(x,0,t)$ vanish from the Taylor expansion. In addition, the Laplace equation allows us to convert all even derivatives with respect to $z$ to derivatives with respect to $x$.

$$\Phi(x,z,t) \approx \Phi(x,0,t) + \cancel{z\frac{\partial\Phi}{\partial z}(x,0,t)} + \frac{z^2}{2}\frac{\partial^2\Phi}{\partial z^2}(x,0,t) + \cancel{\frac{z^3}{3!}\frac{\partial^3\Phi}{\partial z^3}(x,0,t)} + \frac{z^4}{4!}\frac{\partial^4\Phi}{\partial z^4}(x,0,t)\cdots$$

$$\Rightarrow \frac{\partial^2\Phi(x,z,t)}{\partial x^2} + \frac{\partial^2\Phi(x,z,t)}{\partial z^2} = 0$$

Modified Taylor's expansion: $\Phi(x,z,t) \approx \Phi(x,0,t) - \frac{z^2}{2}\frac{\partial^2\Phi}{\partial x^2}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4\Phi}{\partial x^4}(x,0,t)\cdots$

# Some details --

$$\Phi(x,z,t) \approx \Phi(x,0,t) + z\frac{\partial \Phi}{\partial z}(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^3}{3!}\frac{\partial^3 \Phi}{\partial z^3}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

At bottom: $z = 0$ and $v_z(x,0,t) = 0 \quad \Rightarrow \frac{\partial \Phi}{\partial z}(x,0,t) = 0$

Further, your textbook argues that using Fourier transforms,

$$\Phi(x,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \cosh(kz) e^{ikx} \tilde{f}(k,t) \approx \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \left(1 + \frac{(kz)^2}{2!} + \frac{(kz)^4}{4!} + \ldots\right) e^{ikx} \tilde{f}(k,t)$$

$$\Phi(x,z,t) \approx \Phi(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

Check linearized equations and their solutions:

Bernoulli equations --

Bernoulli equation evaluated at $z = h + \zeta(x,t)$

$$-\frac{\partial \Phi(x,h,t)}{\partial t} + g\zeta(x,t) = 0$$

Consistent vertical velocity at $z = h + \zeta(x,t)$

$$\left. -\frac{\partial \Phi(x,z,t)}{\partial z} - \frac{\partial \zeta(x,t)}{\partial t} \right|_{z=h+\zeta} = 0$$

Using Taylor's expansion results to lowest order

$$-\frac{\partial \Phi(x,h,t)}{\partial z} \approx h \frac{\partial^2 \Phi(x,0,t)}{\partial x^2} = -\frac{\partial \zeta(x,t)}{\partial t} \qquad -\frac{\partial \Phi(x,h,t)}{\partial t} \approx -\frac{\partial \Phi(x,0,t)}{\partial t} = -g\zeta(x,t)$$

Decoupled equations: $\quad \dfrac{\partial^2 \Phi(x,0,t)}{\partial t^2} = gh \dfrac{\partial^2 \Phi(x,0,t)}{\partial x^2}.$

➔linear wave equation with $c^2 = gh$

# Analysis of non-linear equations --

Bernoulli equation evaluated at surface:

$$-\frac{\partial\Phi(x,z,t)}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\Phi(x,z,t)}{\partial x}\right)^2+\left(\frac{\partial\Phi(x,z,t)}{\partial z}\right)^2\right]\Bigg|_{z=h+\zeta}+g\zeta(x,t)=0.$$

Consistency of surface velocity

$$-\frac{\partial\Phi(x,z,t)}{\partial z}+\frac{\partial\Phi(x,z,t)}{\partial x}\frac{\partial\zeta(x,t)}{\partial x}-\frac{\partial\zeta(x,t)}{\partial t}\Bigg|_{z=h+\zeta}=0$$

Representation of velocity potential from Taylor's expansion:

$$\Phi(x,z,t)\approx\Phi(x,0,t)-\frac{z^2}{2}\frac{\partial^2\Phi}{\partial x^2}(x,0,t)+\frac{z^4}{4!}\frac{\partial^4\Phi}{\partial x^4}(x,0,t)\cdots$$

Analysis of non-linear equations -- keeping the lowest order nonlinear terms and include up to 4th order derivatives in the linear terms. Let $\phi(x,t) \equiv \Phi(x,0,t)$

Approximate form of Bernoulli equation evaluated at surface: $z = h + \zeta$

$$-\frac{\partial \phi}{\partial t} + \frac{(h+\zeta)^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left((h+\zeta)\frac{\partial^2 \phi}{\partial x^2}\right)^2\right] + g\zeta = 0$$

$$\Rightarrow -\frac{\partial \phi}{\partial t} + \frac{h^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + g\zeta = 0.$$

Approximate form of surface velocity expression :

$$\Rightarrow \frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial \phi}{\partial x}\right) - \frac{h^3}{3!}\frac{\partial^4 \phi}{\partial x^4} - \frac{\partial \zeta}{\partial t} = 0.$$

These equations represent non-linear coupling of $\phi(x,t)$ and $\zeta(x,t)$.

Coupled equations: $-\frac{\partial \phi}{\partial t}+\frac{h^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2}+\frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2+g\zeta=0.$

$$\frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial \phi}{\partial x}\right)-\frac{h^3}{3!}\frac{\partial^4 \phi}{\partial x^4}-\frac{\partial \zeta}{\partial t}=0.$$

Traveling wave solutions with new notation:

$u \equiv x-ct \qquad \phi(x,t)\equiv \chi(u) \qquad$ and $\quad \zeta(x,t)\equiv \eta(u)$

Note that the wave "speed" $c$ will be determined Consistently --

$$c\frac{d\chi(u)}{du}-\frac{ch^2}{2}\frac{d^3\chi(u)}{du^3}+\frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2+g\eta(u)=0.$$

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right)-\frac{h^3}{6}\frac{d^4\chi(u)}{du^4}+c\frac{d\eta(u)}{du}=0.$$

Integrating and re-arranging coupled equations

$$c\frac{d\chi(u)}{du}-\frac{ch^2}{2}\frac{d^3\chi(u)}{du^3}+\frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2+g\eta(u)=0.$$

$$\chi' = -\frac{g}{c}\eta+\frac{h^2}{2}\chi''' - \frac{1}{2c}(\chi')^2 \approx -\frac{g}{c}\eta-\frac{h^2g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2$$

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right)-\frac{h^3}{6}\frac{d^4\chi(u)}{du^4}+c\frac{d\eta(u)}{du}=0.$$

$$\Rightarrow (h+\eta)\frac{d\chi(u)}{du}-\frac{h^3}{6}\frac{d^3\chi(u)}{du^3}+c\eta(u)=0$$

Now we can express $\frac{d\chi(u)}{du}=\chi'$ in terms of $\eta$:

$$\chi' \approx -\frac{g}{c}\eta-\frac{h^2g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2$$

## Integrating and re-arranging coupled equations – continued -- Expressing modified surface velocity equation in terms of $\eta(u)$:

$$(h+\eta)\left(-\frac{g}{c}\eta-\frac{h^2g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2\right)+\frac{h^3g}{6c}\eta''+c\eta=0$$

$$\Rightarrow\left(1-\frac{gh}{c^2}\right)\eta-\frac{gh^3}{3c^2}\eta''-\frac{g}{c^2}\left(1+\frac{gh}{2c^2}\right)\eta^2=0$$

$$\Rightarrow\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}[\eta(u)]^2=0.$$

Note: $c^2=gh+\ldots$

# Solution of the famous Korteweg-de Vries equation

Modified surface amplitude equation in terms of $\eta$

$$\Rightarrow \left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Soliton solution

$$\zeta(x,t)=\eta(x-ct)=\eta_0\ \mathrm{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right)$$

$$c=\sqrt{\frac{gh}{1-\eta_0/h}}\approx\sqrt{gh}\left(1+\frac{\eta_0}{2h}\right)\quad \text{where } \eta_0 \text{ is a constant}$$

Steps to solution

$$\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}[\eta(u)]^2=0.$$

Let $1-\frac{hg}{c^2}\equiv\frac{\eta_0}{h}$ $\qquad\Rightarrow \frac{\eta_0}{h}\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}[\eta(u)]^2=0.$

Multiply equation by $\eta'(u)$ $\qquad\Rightarrow \frac{d}{du}\left(\frac{\eta_0}{2h}\eta^2(u)-\frac{h^2}{6}\eta'^2(u)-\frac{1}{2h}\eta^3(u)\right)=0$

Integrate wrt $u$ and assume solution vanishes for $u\rightarrow\infty$

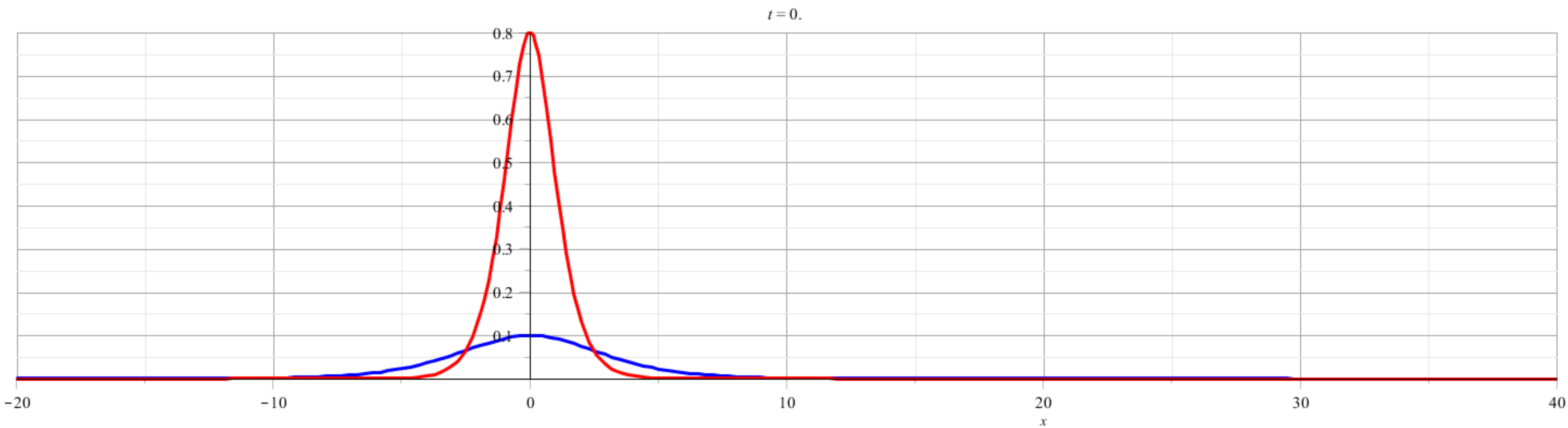
$$\frac{\eta_0}{2h}\eta^2(u)-\frac{h^2}{6}\eta'^2(u)-\frac{1}{2h}\eta^3(u)=0$$

$$\eta'^2(u)=\frac{3}{h^3}\eta^2(u)\left(\eta_0-\eta(u)\right)$$

$$\frac{d\eta}{\eta\left(\eta_0-\eta\right)^{1/2}}=\sqrt{\frac{3}{h^3}}du \qquad \Rightarrow \eta(u)=\frac{\eta_0}{\cosh^2\left(\sqrt{\frac{3\eta_0}{4h^3}}u\right)}=\eta_0\,\mathrm{sech}^2\left(\sqrt{\frac{3\eta_0}{4h^3}}u\right)$$

$$\zeta(x,t) = \eta(x-ct) = \eta_0 \ \mathrm{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right)$$

Two soliton solutions with different amplitudes --



# Relationship to "standard" form of Korteweg-de Vries equation

New variables:

$$\beta = 2\eta_0, \qquad \bar{x} = \sqrt{\frac{3}{2h}}\frac{x}{h}, \qquad \text{and} \qquad \bar{t} = \sqrt{\frac{3}{2h}}\frac{ct}{2\eta_0 h}.$$

Standard Korteweg-de Vries equation

$$\frac{\partial \eta}{\partial \bar{t}} + 6\eta\frac{\partial \eta}{\partial \bar{x}} + \frac{\partial^3 \eta}{\partial \bar{x}^3} = 0.$$

Soliton solution:

$$\eta(\bar{x},\bar{t}) = \frac{\beta}{2}\operatorname{sech}^2\left[\frac{\sqrt{\beta}}{2}(\bar{x} - \beta\bar{t})\right].$$

# More details

Modified surface amplitude equation in terms of $\eta$ :

$$\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Some identities: $\frac{\eta_0}{h}=1-\frac{gh}{c^2}; \quad \frac{\partial\eta}{\partial t}=-c\frac{d\eta}{du}; \quad \frac{\partial\eta}{\partial x}=\frac{d\eta}{du}.$

Derivative of surface amplitude equation:

$$\frac{\eta_0}{h}\eta'-\frac{h^2}{3}\eta'''-\frac{3}{h}\eta\eta'=0.$$

Expression in terms of $x$ and $t$:

$$-\frac{\eta_0}{ch}\frac{\partial\eta}{\partial t}-\frac{h^2}{3}\frac{\partial^3\eta}{\partial x^3}-\frac{3}{h}\eta\frac{\partial\eta}{\partial x}=0.$$

Expression in terms of $\overline{x}$ and $\overline{t}$:

$$\frac{\partial\eta}{\partial\overline{t}}+6\eta\frac{\partial\eta}{\partial\overline{x}}+\frac{\partial^3\eta}{\partial\overline{x}^3}=0.$$

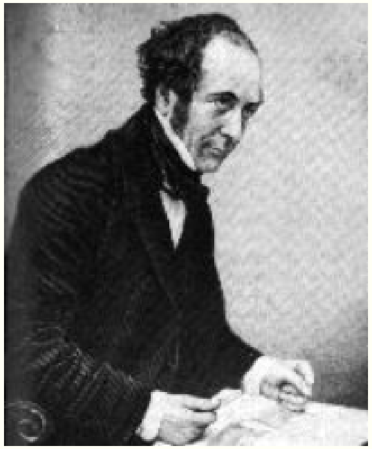
# Summary

Soliton solution

$$\zeta(x,t) = \eta(x-ct) = \eta_0 \ \mathrm{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\ \frac{x-ct}{2h}\right)$$

$$c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1+\frac{\eta_0}{2h}\right) \quad \text{where } \eta_0 \text{ is a constant}$$

# John Scott Russell and the solitary wave



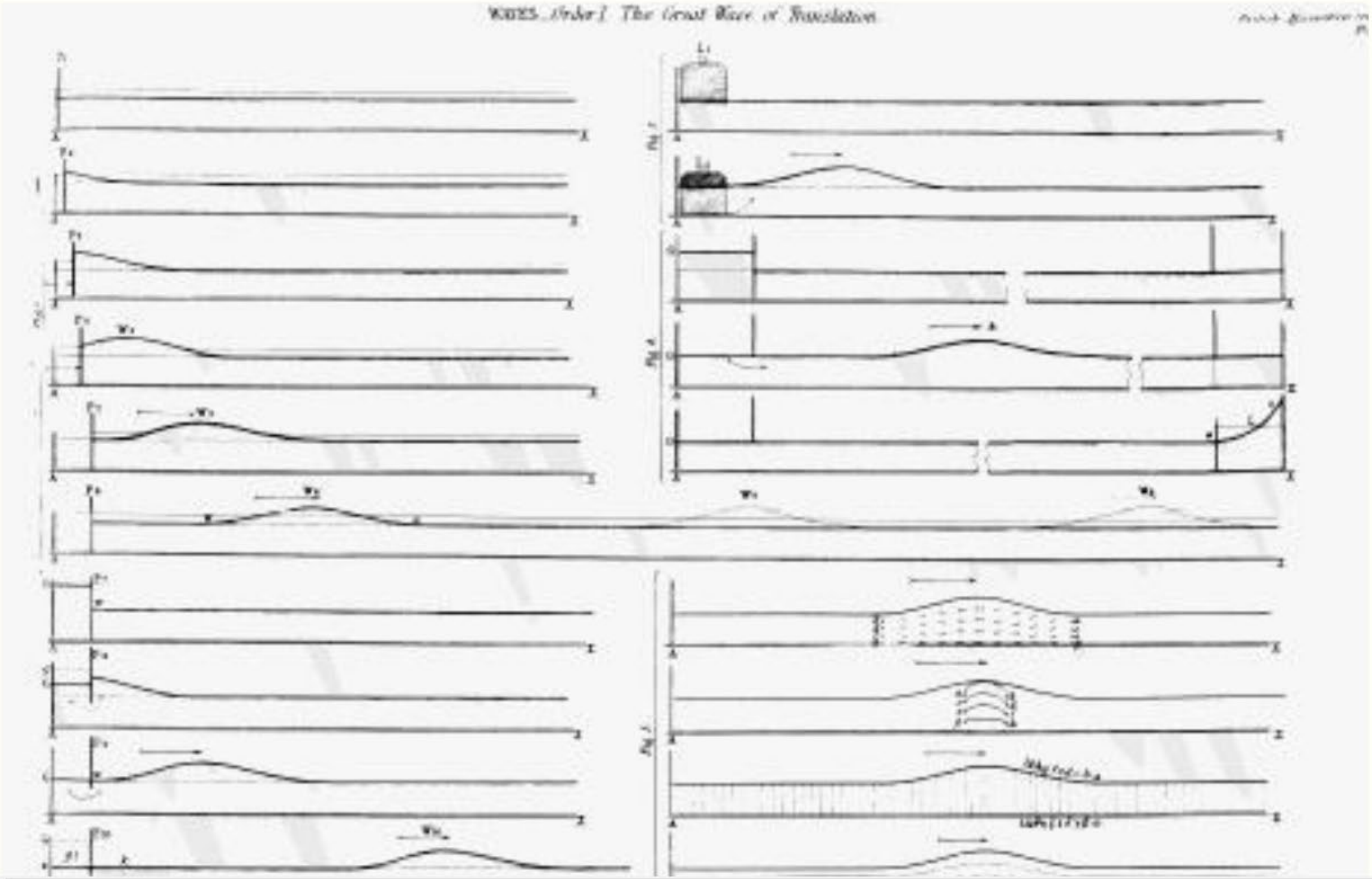
Over one hundred and fifty years ago, while conducting experiments to determine the most efficient design for canal boats, a young Scottish engineer named John Scott Russell (1808-1882) made a remarkable scientific discovery. As he described it in his "Report on Waves": (Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845), pp 311-390, Plates XLVII-LVII).

https://www.macs.hw.ac.uk/~chris/scott_russell.html

``I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".
(Cet passage en francais)

This event took place on the Union Canal at Hermiston, very close to the Riccarton campus of Heriot-Watt University, Edinburgh.

Following this discovery, Scott Russell built a 30' wave tank in his back garden and made further important observations of the properties of the solitary wave.



Throughout his life Russell remained convinced that his solitary wave (the ``Wave of Translation") was of fundamental importance, but nineteenth and early twentieth century scientists thought otherwise. His fame has rested on other achievements. To mention some of his many and varied activities, he developed the "wave line" system of hull construction which revolutionized nineteenth century naval architecture, and was awarded the gold medal of the Royal Society of Edinburgh in 1837. He began steam carriage service between Glasgow and Paisley in 1834, and made one of the first experimental observations of the "Doppler shift" of sound frequency as a train passes. He reorganized the Royal Society of Arts, founded the Institution of Naval Architects and in 1849 was elected Fellow of the Royal Society of London. He designed (with Brunel) the "Great Eastern" and built it; he designed the Vienna Rotunda and helped to design Britain's first armoured warship (the "Warrior"). He developed a curriculum for technical education in Britain, and it has recently become known that he attempted to negotiate peace during the American Civil War.

Reconstruction of the canal soliton --
http://www.ma.hw.ac.uk/solitons/

Soliton home page at Heriot-Watt

# John Scott Russell's Soliton Wave Re-created

On Wednesday 12 July 1995, an international gathering of scientists witnessed a re-creation of the famous 1834 'first' sighting of a soliton or solitary wave on the Union Canal near Edinburgh. They were attending a conference on nonlinear waves in physics and biology at Heriot-Watt University, near the canal.

The occasion was part of a ceremony to name a new aqueduct after John Scott Russell, the Scottish scientist who made the original observation. The aqueduct carries the Union Canal over the Edinburgh City Bypass.


ENLARGE
*Alwyn Scott naming the aqueduct with Chris Eilbeck and Laura Kruskal looking on.*

ENLARGE
*Setting up the experimental apparatus on the Scott Russell Aqueduct.*

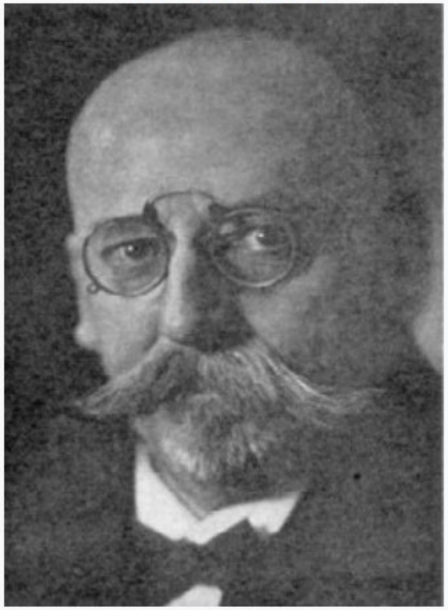
# Photo of canal soliton http://www.ma.hw.ac.uk/solitons/

**Diederik Korteweg**

Diederik Johannes Korteweg

| | |
|---|---|
| **Born** | 31 March 1848<br>Den Bosch |
| **Died** | 10 May 1941 (aged 93)<br>Amsterdam |
| **Nationality** | Dutch |
| **Alma mater** | University of Amsterdam |
| **Known for** | Korteweg–de Vries equation, Moens–Korteweg equation[1] |
| | **Scientific career** |
| **Fields** | Mathematics |
| **Institutions** | University of Amsterdam |

**Gustav de Vries**



| | |
|---|---|
| **Born** | 22 January 1866<br>Amsterdam |
| **Died** | 16 December 1934 (aged 68) |
| **Nationality** | Dutch |
| **Alma mater** | University of Amsterdam |
| **Known for** | Korteweg–De Vries equation |
| | **Scientific career** |
| **Fields** | Mathematics |