PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Notes on Lecture 34: Chap. 11 in F&W

Heat conduction

1. Basic equations
2. Boundary value problems

Physics Colloquium

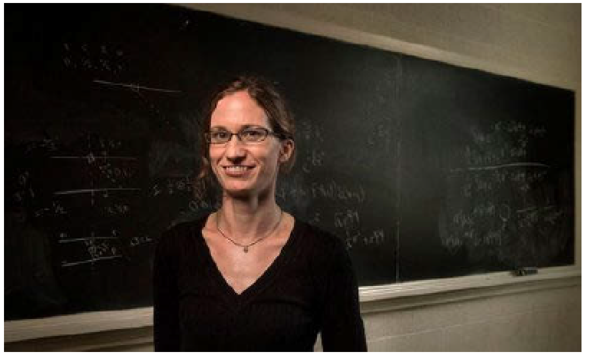
- Thursday -

November 14, 2024

4 PM in Olin 101

The ties that bind: understanding nuclear forces from lattice QCD

There are many open questions in nuclear physics which only lattice QCD may be able to answer. One example is understanding the nature and origin of the fine-tuning of interactions between nucleons and nuclei observed in nature. The first step toward building a bridge between the underlying theory, QCD, and nuclear observables is full control over one- and two-nucleon systems. While enormous strides have been made in recent years in precision calculations of single-nucleon observables, the history of two-nucleon calculations has generated more questions than answers. In particular, there is a controversy in the literature between calculations performed using different theoretical techniques, even for calculations far from the physical point,



Amy Nicholson

Associate Professor, Dept. of Physics and Astronomy

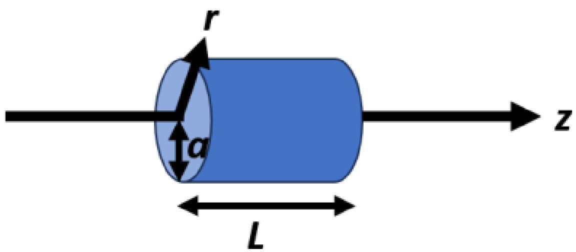
University of North Carolina - Chapel Hill

31	Wed, 11/06/2024	Chap. 9	Non-linear and other wave properties	#24
32	Fri, 11/08/2024	Chap. 10	Surface waves in fluids	#25
33	Mon, 11/11/2024	Chap. 10	Surface waves in fluids; soliton solutions	#26
34	Wed, 11/13/2024	Chap. 11	Heat conduction	#27
35	Fri, 11/15/2024	Chap. 12	Viscous effects in hydrodynamics	
36	Mon, 11/18/2024	Chap. 12	Viscous effects in hydrodynamics	
37	Wed, 11/20/2024	Chap. 13	Elasticity	
38	Fri, 11/22/2024	Chap. 1-13	Review	
39	Mon, 11/25/2024	Chap. 1-13	Review	
	Wed, 11/27/2024	Thanksgiving		
	Fri, 11/29/2024	Thanksgiving		
	Mon, 12/02/2024		Presentations 1	
	Wed, 12/04/2024		Presentations 2	
40	Fri, 12/06/2024	Chap. 1-13	Review	

PHY 711 – Homework # 27

Assigned: 11/13/2024 Due: 11/18/2024

Read Chapter 11 of **Fetter and Walecka**.



1. A cylindrical solid material with cylindrical radius a and length L and thermal diffusivity κ has a time-dependent cylindrically symmetric temperature profile $T(r, z, t)$. In these cylindrical coordinates, the material is contained within $0 \leq r \leq a$ and $0 \leq z \leq L$. In the absense of external heating, the temperature profile is is well-described by the equation of heat conduction

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T.$$

At $t \leq 0$, the material is prepared so that its temperature profile is given by

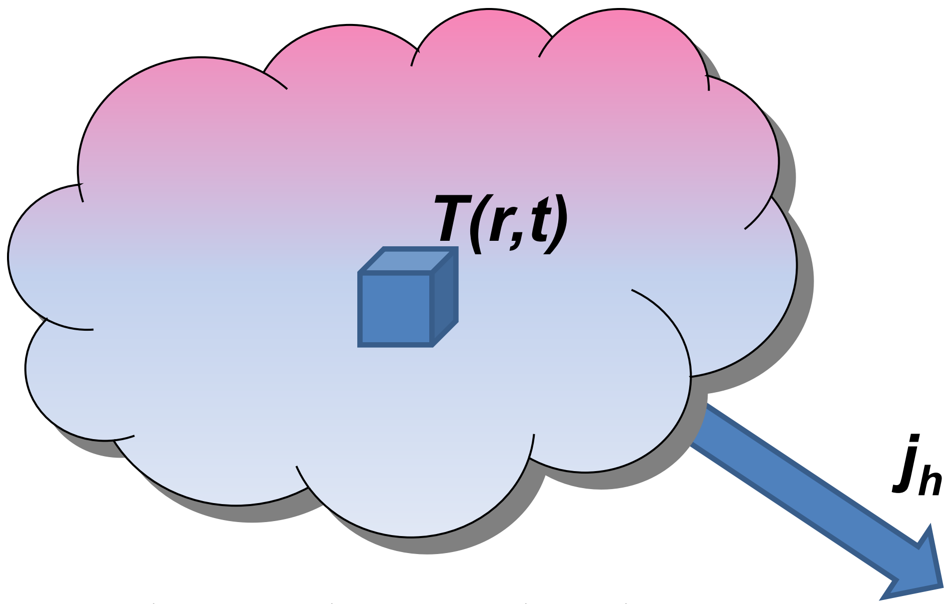
$$T(r, z, t \leq 0) = \begin{cases} 0 & \text{for} \quad r > a \quad \text{and/or} \quad z > L \\ A\cos(\pi z/L) & \text{for} \quad r \leq a \quad \text{and/or} \quad z \leq L, \end{cases}$$

where A is a given constant. The cylindrical solid is placed in a thermally insulated container so that its temperature is well-described by the boundary conditions

$$\hat{\mathbf{n}} \cdot \nabla T(r, z, t) = 0$$

at all of its surfaces. Find an expression for the temperature profile of this system $T(r, z, t)$ for $t > 0$.

Conduction of heat



Enthalpy (H) of a system at constant pressure p

non uniform temperature $T(\mathbf{r},t)$

mass density ρ and heat capacity c_p

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0 \right) d^3r + H_0(T_0, p)$$

Note that in this treatment we are considering a system at constant pressure *p*

Notation:	Heat added to system	-- $dQ = TdS$
	External work done on system	-- $dW = -pdV$
	Internal energy	-- $dE = dQ + dW = TdS - pdV$
	Entropy	-- dS
	Enthalpy	-- $dH = d(E + pV) = TdS + Vdp$

Heat capacity at constant pressure:

$$C_p \equiv \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial H}{\partial T}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p$$

$$C_p = \int \rho c_p d^3 r$$

More generally, note that c_p can depend on *T*; here we are assuming that dependence to be trivial.

Conduction of heat -- continued

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0\right) d^3r + H_0(T_0, p)$$

Time rate of change of enthapy:

$$\frac{dH}{dt} = \int_V \rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} d^3r = -\int_A \mathbf{j}_h \cdot d\mathbf{A} + \int_V \rho \dot{q} d^3r$$

heat flux

heat source

$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

Conduction of heat -- continued

$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

Empirically: $\mathbf{j}_h = -k_{th} \nabla T(\mathbf{r},t)$

$$\Rightarrow \frac{\partial T(\mathbf{r},t)}{\partial t} = \kappa \nabla^2 T(\mathbf{r},t) + \frac{\dot{q}}{c_p}$$

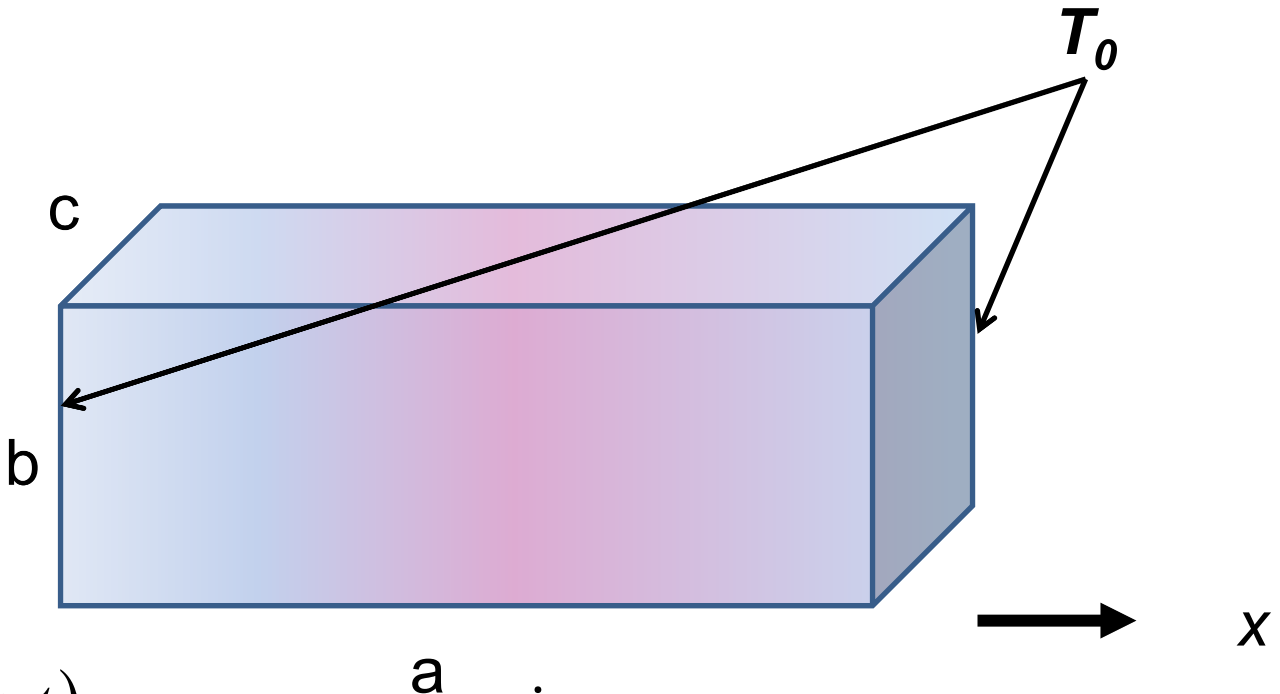
$$\kappa \equiv \frac{k_{th}}{\rho c_p}$$ thermal diffusivity

https://www.engineersedge.com/heat_transfer/thermal_diffusivity_table_13953.htm

Typical values (m^2/s)

Air	$2\text{x}10^{-5}$
Water	$1\text{x}10^{-7}$
Copper	$1\text{x}10^{-4}$

Boundary value problems for heat conduction



$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = \frac{\dot{q}}{c_p}$$

Without source term: $$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = 0$$

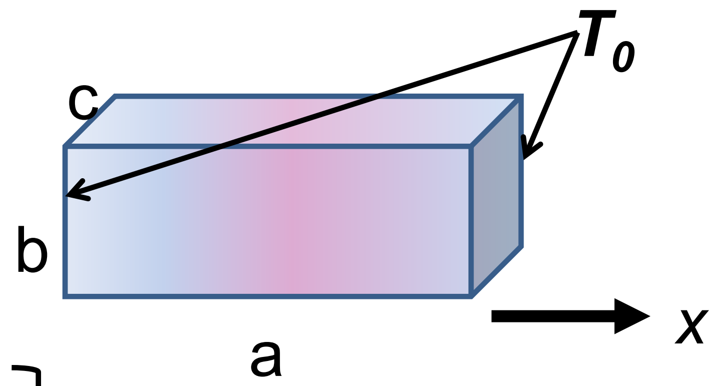
Example with boundary values: $T(0,y,z,t) = T(a,y,z,t) = T_0$

Have you ever encountered the following equation in other contexts and if so where?

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = 0$$

Boundary value problems for heat conduction

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$



$$T(0,y,z,t) = T(a,y,z,t) = T_0$$

$$\frac{\partial T(x,0,z,t)}{\partial y} = \frac{\partial T(x,b,z,t)}{\partial y} = 0$$

$$\frac{\partial T(x,y,0,t)}{\partial z} = \frac{\partial T(x,y,c,t)}{\partial z} = 0$$

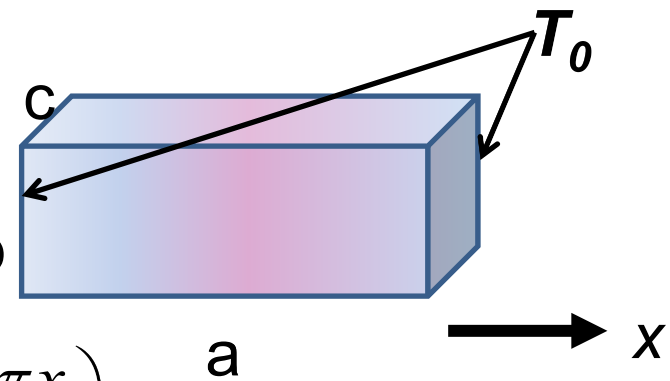
Assuming thermally insulated boundaries

Separation of varables: $T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$

Let $\frac{d^2X}{dx^2} = -\alpha^2 X \quad \frac{d^2Y}{dy^2} = -\beta^2 Y \quad \frac{d^2Z}{dz^2} = -\gamma^2 Z$

$$\Rightarrow -\lambda + \kappa\left(\alpha^2 + \beta^2 + \gamma^2\right) = 0$$

Boundary value problems for heat conduction



$$T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$$

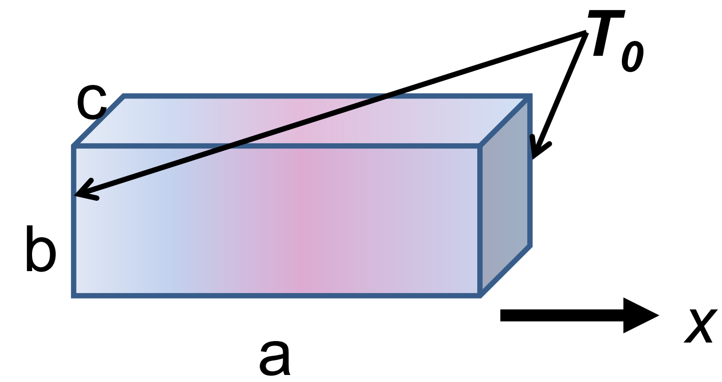
$$X(0) = X(a) = 0 \qquad \Rightarrow X(x) = \sin\left(\frac{m\pi x}{a}\right)$$

$$\frac{dY(0)}{dy} = \frac{dY(b)}{dy} = 0 \qquad \Rightarrow Y(y) = \cos\left(\frac{n\pi y}{b}\right)$$

$$\frac{dZ(0)}{dz} = \frac{dZ(c)}{dz} = 0 \qquad \Rightarrow Z(z) = \cos\left(\frac{p\pi z}{c}\right)$$

$$-\lambda_{nmp} + \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right) = 0$$

Boundary value problems for heat conduction



Full solution:

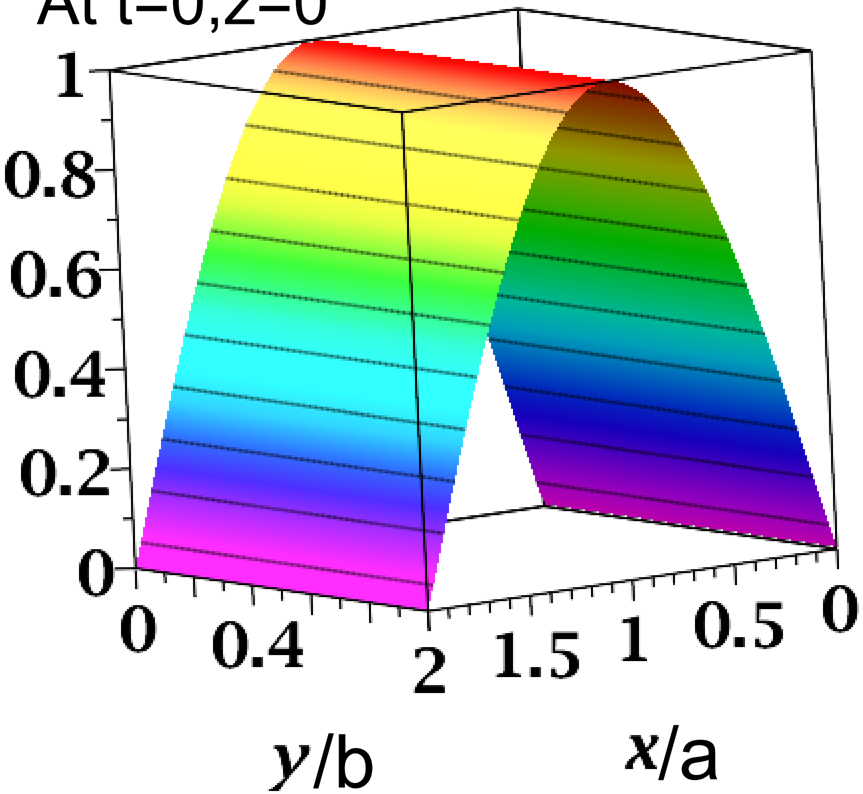
$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp}t}$$

$$\lambda_{nmp} = \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right)$$

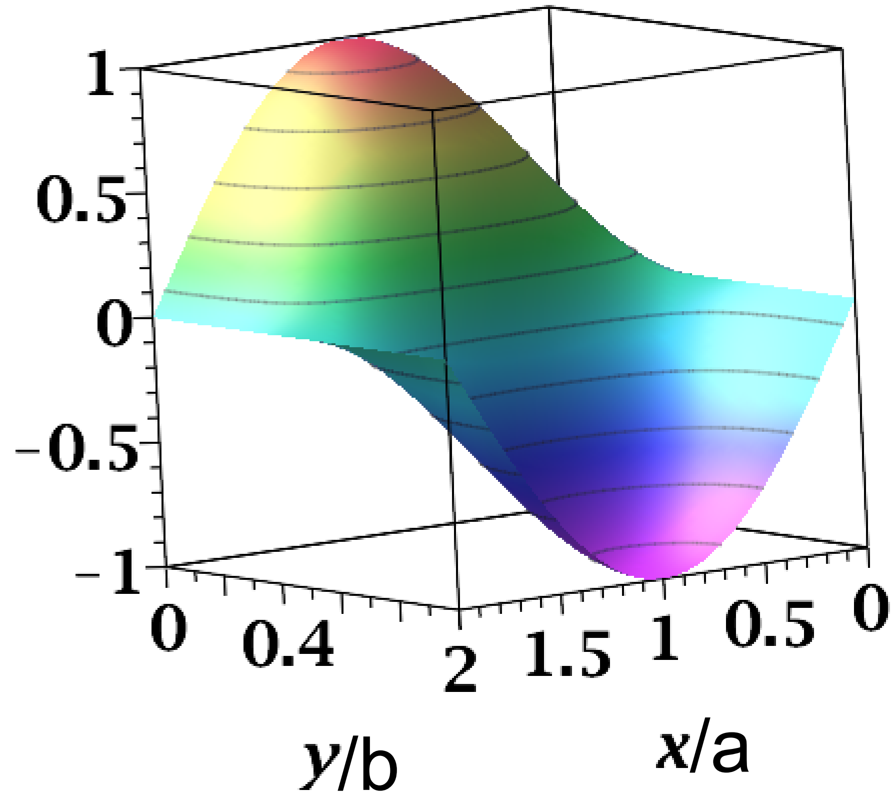
Full solution:

$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp}t}$$

At t=0,z=0



$m=1, n=0, p=0$



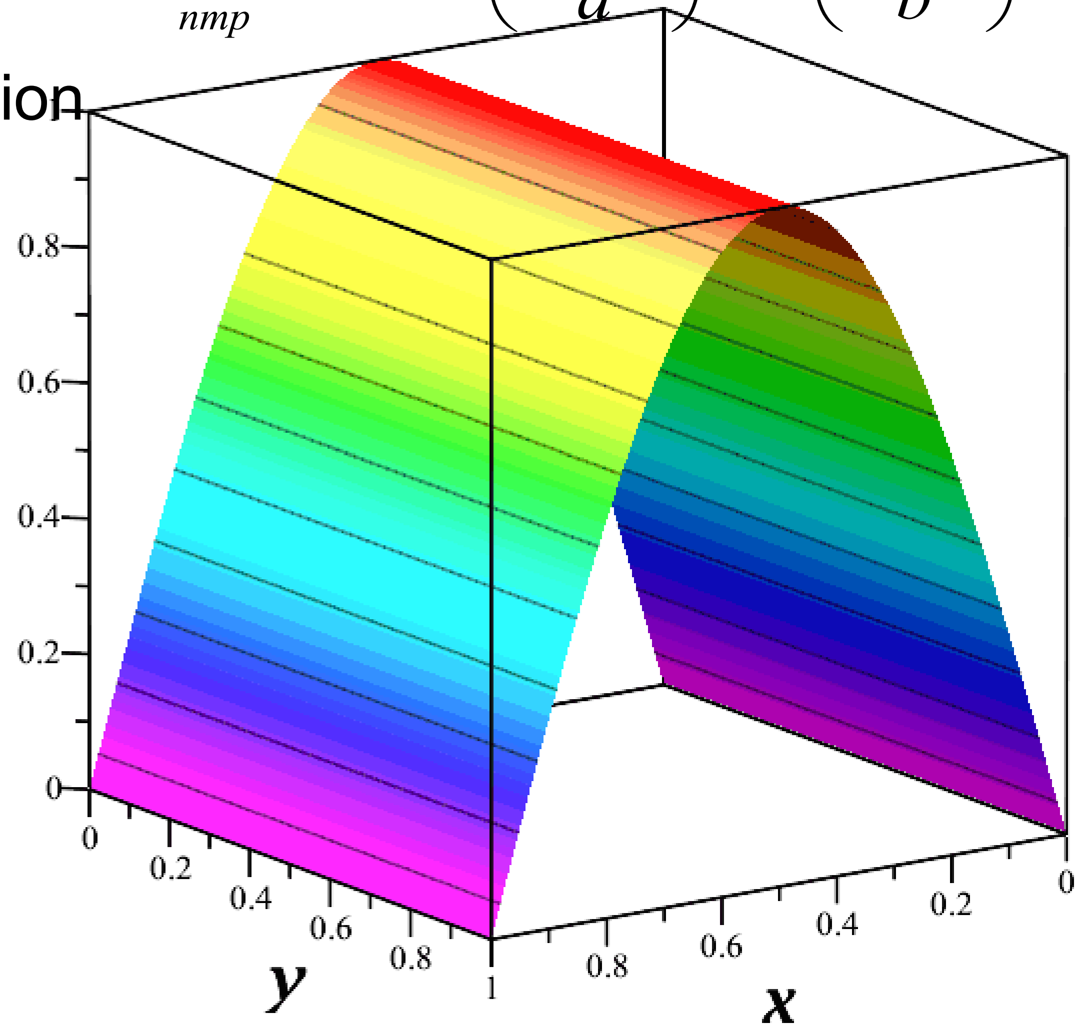
$m=1, n=1, p=0$

Full solution:

$t = 0.$

$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp}t}$$

Time evolution
nmp=100
at *z=0*



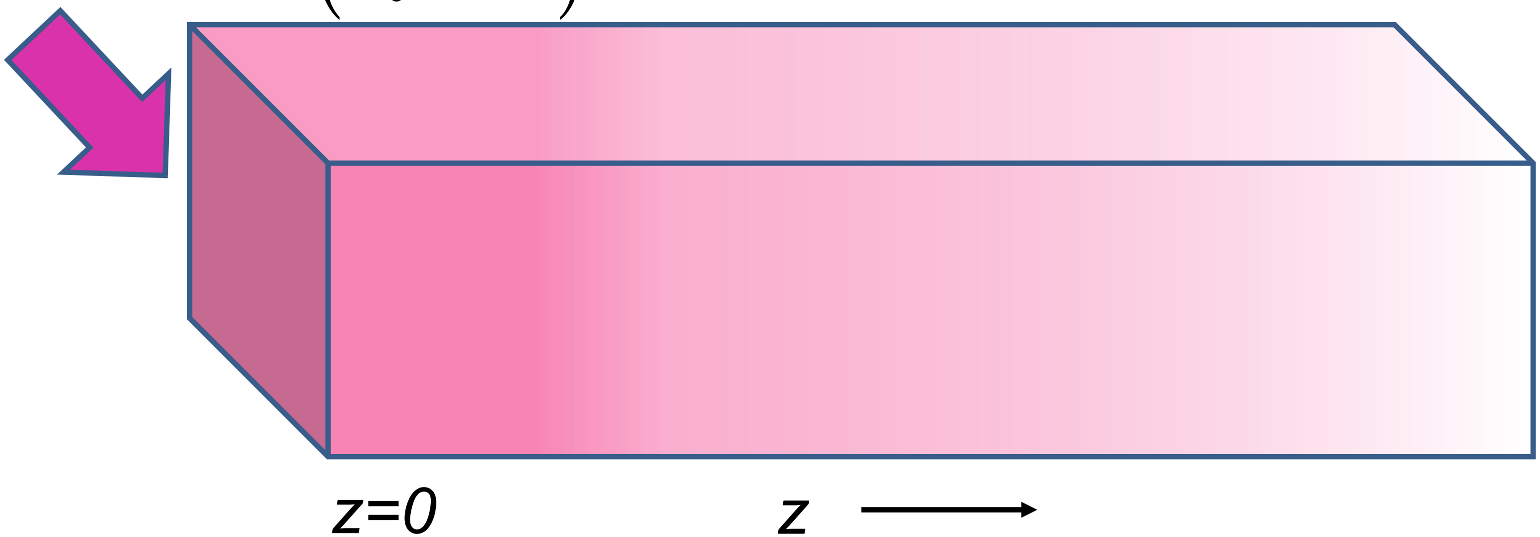
What real system could have such a temperature distribution?

Comment – While one can imagine that the boundary conditions can be readily realized, the single normal mode patterns are much harder.   On the other hand, we see that the smallest values of lambda have the longest time constants.

Oscillatory thermal behavior

Here we assume that the spatial variation is along z

$$T(z=0,t)=\Re\left(T_0 e^{-i\omega t}\right)$$



$$\frac{\partial T}{\partial t}=\kappa\frac{\partial^2 T}{\partial z^2}$$

Assume: $T(z,t)=\Re\left(f(z)e^{-i\omega t}\right)$

$$(-i\omega)f=\kappa\frac{d^2 f}{dz^2}$$

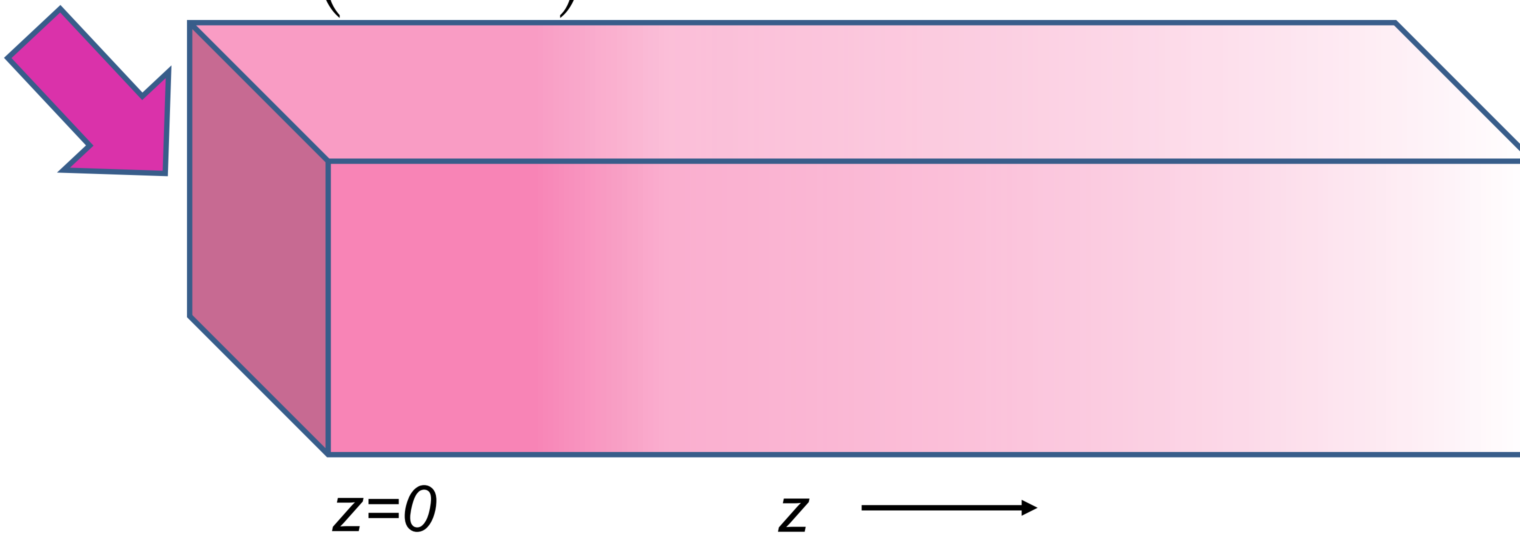
Let $f(z)=Ae^{\alpha z}$

$$\alpha^2=-\frac{i\omega}{\kappa}=e^{3i\pi/2}\ \frac{\omega}{\kappa}$$

$$\alpha=\pm(1-i)\sqrt{\frac{\omega}{2\kappa}}$$

Oscillatory thermal behavior -- continued

$$T(z=0,t) = \Re\left(T_0 e^{-i\omega t}\right)$$

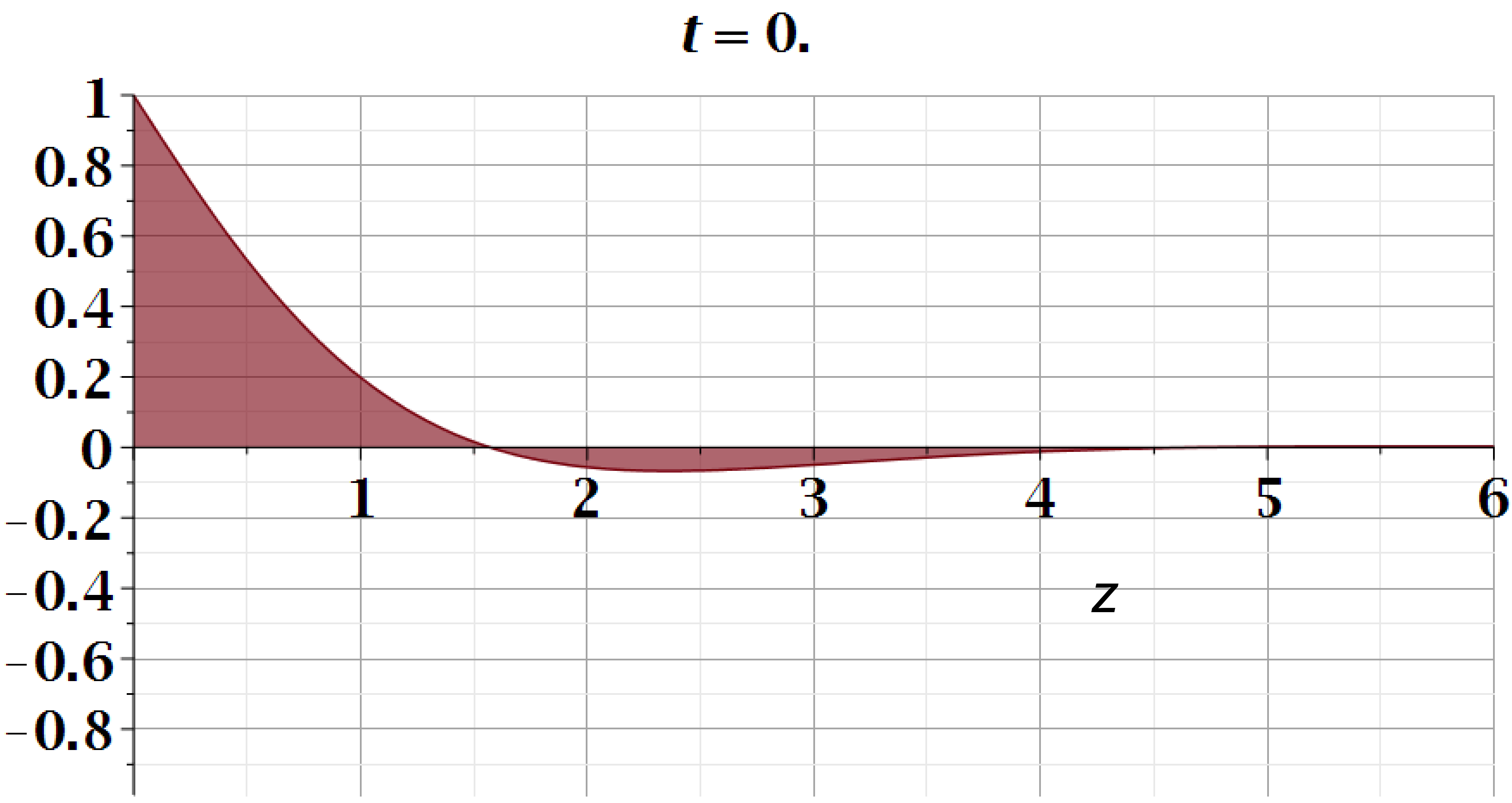


$$T(z,t) = \Re\left(A e^{\pm(1-i)z/\delta} e^{-i\omega t}\right)$$

where $\delta \equiv \sqrt{\dfrac{2\kappa}{\omega}}$

Physical solution: $T(z,t) = T_0 e^{-z/\delta} \cos\left(\dfrac{z}{\delta} - \omega t\right)$

$$T(z,t) = T_0 e^{-z/\delta} \cos\left(\frac{z}{\delta} - \omega t\right)$$



Initial value problem in an infinite domain; Fourier transform

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

$$T(\mathbf{r},0) = f(\mathbf{r})$$

$$\text{Let:} \quad \widetilde{T}(\mathbf{q},t) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} T(\mathbf{r},t)$$

$$\widetilde{f}(\mathbf{q}) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} f(\mathbf{r})$$

$$\Rightarrow \widetilde{T}(\mathbf{q},0) = \widetilde{f}(\mathbf{q})$$

$$\Rightarrow \frac{\partial \widetilde{T}(\mathbf{q},t)}{\partial t} = -\kappa q^2 \widetilde{T}(\mathbf{q},t)$$

$$\widetilde{T}(\mathbf{q},t) = \widetilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

Initial value problem in an infinite domain; Fourier transform

$$\widetilde{T}(\mathbf{q},t)=\int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}}T(\mathbf{r},t) \qquad \Rightarrow T(\mathbf{r},t)=\frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{r}}\widetilde{T}(\mathbf{q},t)$$

$$\widetilde{T}(\mathbf{q},t)=\widetilde{T}(\mathbf{q},0)e^{-\kappa q^2 t}$$

$$T(\mathbf{r},t)=\frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{r}}\widetilde{T}(\mathbf{q},0)e^{-\kappa q^2 t}$$

$$\widetilde{T}(\mathbf{q},0)=\widetilde{f}(\mathbf{q})=\int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}}f(\mathbf{r})$$

$$T(\mathbf{r},t)=\int d^3r' G(\mathbf{r}-\mathbf{r}',t)T(\mathbf{r}',0)$$

$$\text{with } G(\mathbf{r}-\mathbf{r}',t)\equiv\frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}e^{-\kappa q^2 t}$$

Initial value problem in an infinite domain; Fourier transform

$$T(\mathbf{r},t) = \int d^3r' \, G(\mathbf{r}-\mathbf{r}',t) T(\mathbf{r}',0)$$

$$\text{with} \quad G(\mathbf{r}-\mathbf{r}',t) \equiv \frac{1}{(2\pi)^3} \int d^3q \, e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} e^{-\kappa q^2 t}$$

$$G(\mathbf{r}-\mathbf{r}',t) = \frac{1}{(4\pi\kappa t)^{3/2}} e^{-|\mathbf{r}-\mathbf{r}'|^2/(4\kappa t)}$$

Heat equation in half-space

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = 0$$

$$T(\mathbf{r},t) \Rightarrow T(z,t) \quad \text{with initial and boundary values:}$$

$$T(z,t) \equiv 0 \quad \text{for } z < 0$$

$$T(z,0) = 0 \quad \text{for } z > 0$$

$$T(0,t) = T_0 \quad \text{for } t \geq 0$$

$$\text{Solution:} \quad T = T_0 \,\text{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

$$\text{where } \text{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} du$$

More details about the error function --
https://dlmf.nist.gov/7

§7.2(i) Error Functions

7.2.1
$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt,$$

7.2.2
$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt = 1 - \operatorname{erf} z,$$

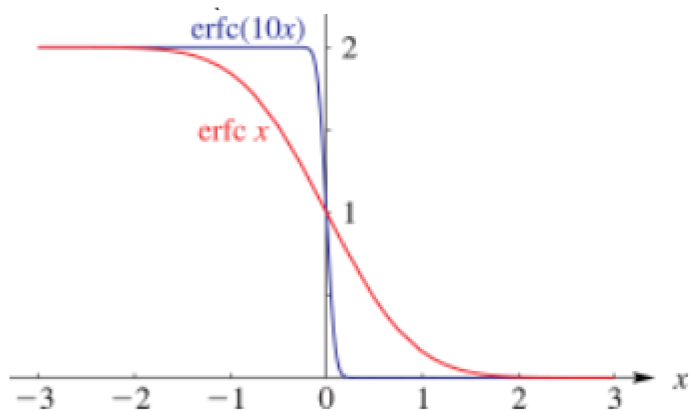


Figure 7.3.1: Complementary error functions $\operatorname{erfc} x$ and $\operatorname{erfc}(10x)$, $-3 \le x \le 3$.

Heat equation in half-space -- continued

$$\frac{\partial T(z,t)}{\partial t} - \kappa \frac{\partial^2 T(z,t)}{\partial z^2} = 0$$

$$\text{Solution:} \quad T = T_0 \,\text{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

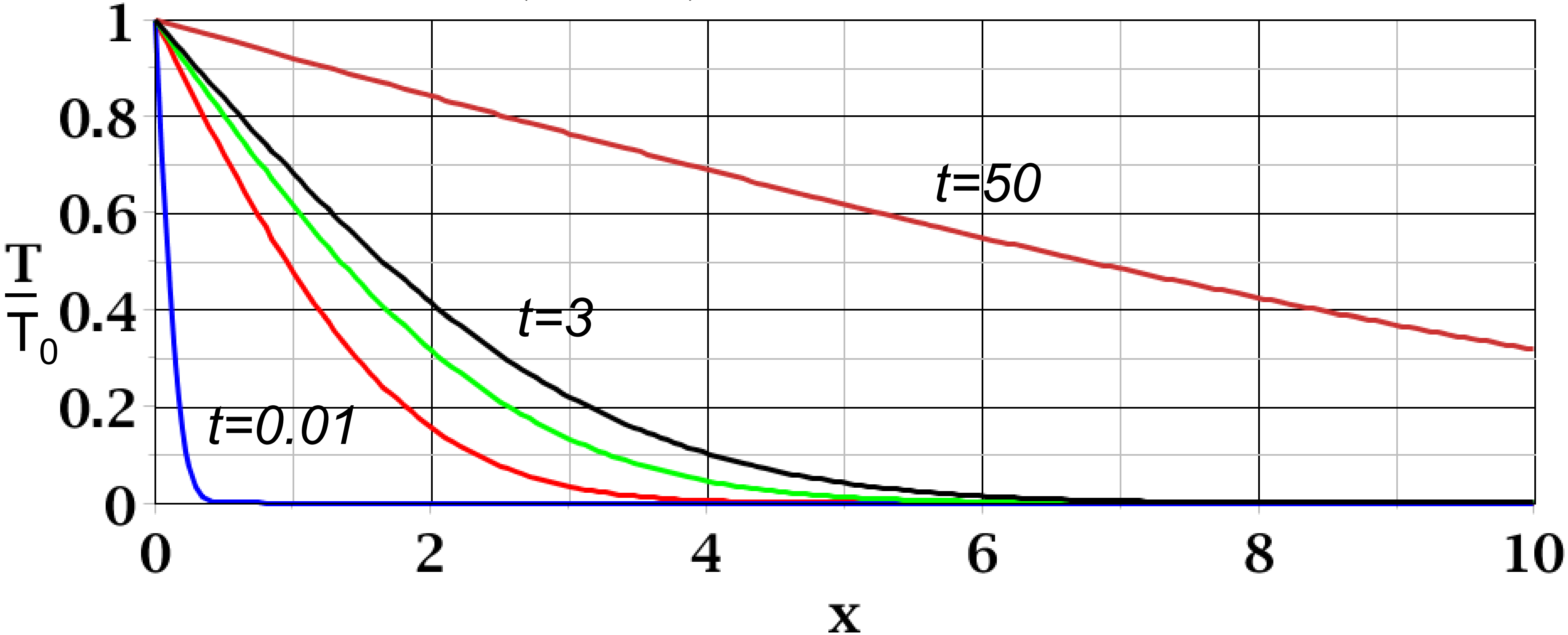
$$\text{where } \text{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$$

$$\text{Note that} \quad \frac{d\,\text{erfc}(x)}{dx} = \frac{d}{dx}\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2} du = -\frac{2}{\sqrt{\pi}} e^{-x^2}$$

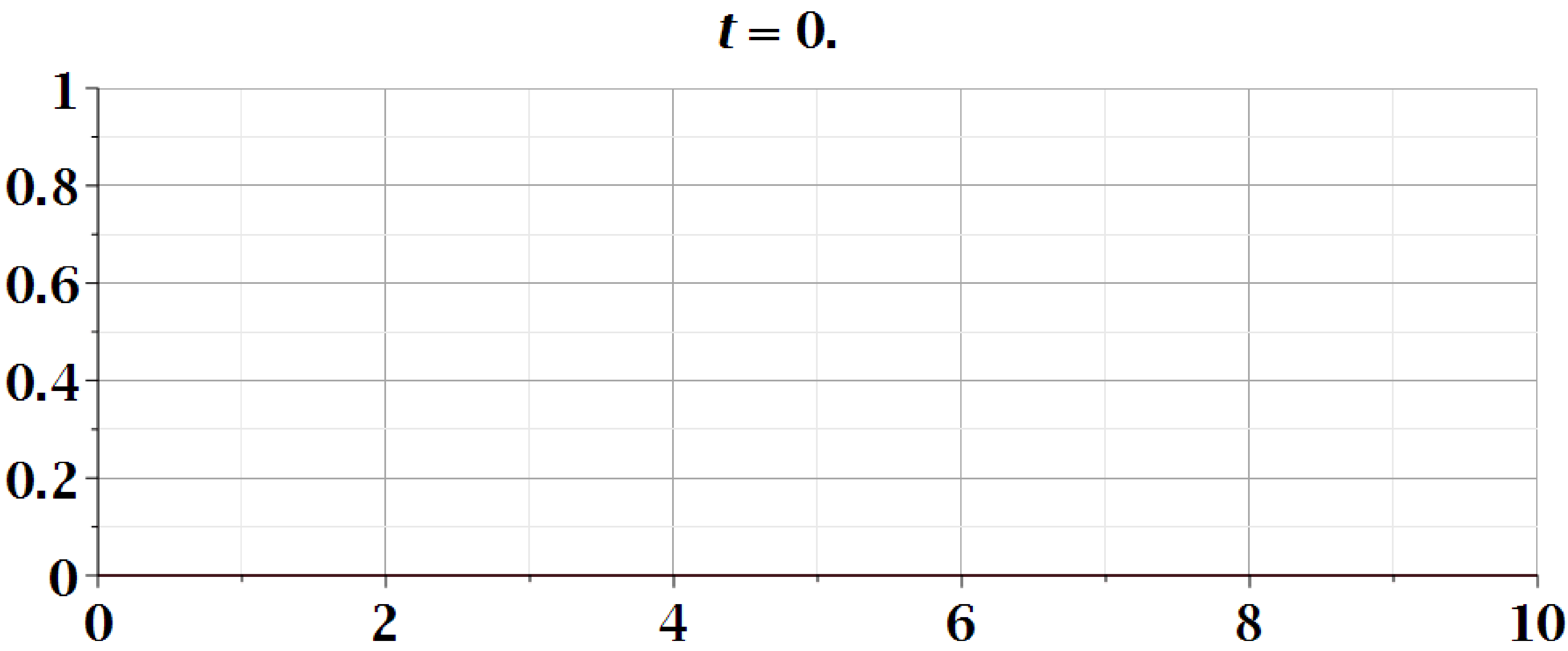
$$\frac{\partial}{\partial t}\text{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)}\left(\frac{z}{4\sqrt{\kappa t^3}}\right)$$

$$\frac{\partial^2}{\partial z^2}\text{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)}\left(\frac{z}{4\kappa\sqrt{\kappa t^3}}\right)$$

$$T = T_0 \ \mathrm{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$



Temperature profile



Another initial value example with one spacial dimension

$$\frac{\partial T(z,t)}{\partial t} - \kappa \frac{\partial^2 T(z,t)}{\partial z^2} = 0$$

With the initial condition $T(z, t=0) = T_0 \delta(z)$

In this case, $T(z,t) = \dfrac{T_0}{\sqrt{4\pi\kappa t}} \exp\left(-\dfrac{z^2}{4\kappa t}\right)$

Here we need to show that $\delta(z) = \lim_{t \to 0}\left(\dfrac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\dfrac{z^2}{4\kappa t}\right)\right)$