PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 36
Continued discussion of viscous fluids: Chap. 12 in F & W

1. Some general comments
2. Navier-Stokes equation
3. Review of results from last time – Stokes "law"
4. Effects on linearized sound waves

33	Mon, 11/11/2024	Chap. 10	Surface waves in fluids; soliton solutions	#26
34	Wed, 11/13/2024	Chap. 11	Heat conduction	#27
35	Fri, 11/15/2024	Chap. 12	Viscous effects in hydrodynamics	#28
36	Mon, 11/18/2024	Chap. 12	Viscous effects in hydrodynamics	#29
37	Wed, 11/20/2024	Chap. 13	Elasticity	#30
38	Fri, 11/22/2024	Chap. 1-13	Review	
39	Mon, 11/25/2024	Chap. 1-13	Review	
	Wed, 11/27/2024	Thanksgiving		
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40	Fri, 12/06/2024	Chap. 1-13	Review	

PHY 711 -- Assignment #29

Assigned: 11/18/2024 Due: 11/25/2024

Continue reading Chapter 12 in **Fetter & Walecka**.

1. Work problem 12.13 at the end of Chapter 12 in **Fetter & Walecka**.

Comment on outstanding homework.

Equations for motion of non-viscous fluid --

Modified Newton-Euler equation in terms of fluid momentum:

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\sum_{j=1}^{3}\frac{\partial(\rho v_j\mathbf{v})}{\partial x_j}=\rho\mathbf{f}_{applied}-\nabla p$$

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\sum_{j=1}^{3}\frac{\partial(\rho v_j\mathbf{v})}{\partial x_j}+\nabla p=\rho\mathbf{f}_{applied}$$

Fluid momentum: $\rho\mathbf{v}$

Stress tensor: $T_{ij}\equiv\rho v_i v_j+p\delta_{ij}$

i^{th} component of Newton-Euler equation:

$$\frac{\partial(\rho v_i)}{\partial t}+\sum_{j=1}^{3}\frac{\partial T_{ij}}{\partial x_j}=\rho f_i$$

Now consider the effects of viscosity

In terms of stress tensor:

$$T_{ij} = T_{ij}^{\text{ideal}} + T_{ij}^{\text{viscous}}$$

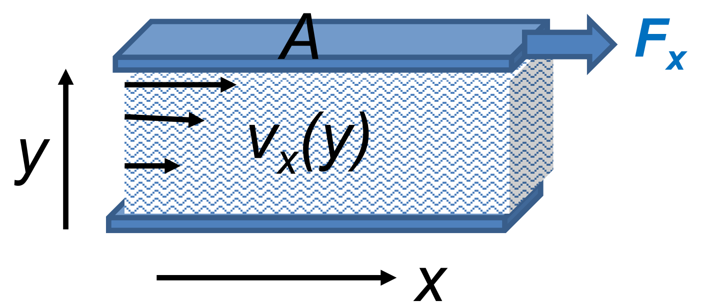
$$T_{ij}^{\text{ideal}} = \rho v_i v_j + p\delta_{ij} = T_{ji}^{\text{ideal}}$$

As an example of a viscous effect, consider --

Newton's "law" of viscosity

$$\frac{F_x}{A} = \eta \frac{\partial v_x}{\partial y}$$

material dependent parameter



Effects of viscosity

Argue that viscosity is due to shear forces in a fluid of the form:

$$\frac{F_{drag}}{A} = \eta \frac{\partial v_x}{\partial y}$$

Formulate viscosity stress tensor with traceless and diagonal terms:

$$T_{kl}^{\text{viscous}} = -\eta\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3}\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)\right) - \zeta\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)$$

viscosity

bulk (or dilational) viscosity

Total stress tensor: $T_{kl} = T_{kl}^{ideal} + T_{kl}^{\text{viscous}}$

$$T_{kl}^{\text{ideal}} = \rho v_k v_l + p\delta_{kl}$$

$$T_{kl}^{\text{viscous}} = -\eta\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3}\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)\right) - \zeta\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)$$

Effects of viscosity -- continued

Incorporating generalized stress tensor into Newton-Euler equations

$$\frac{\partial(\rho v_i)}{\partial t}+\sum_{i=1}^{3}\frac{\partial T_{ij}}{\partial x_j}=\rho f_i$$

$$\frac{\partial(\rho v_i)}{\partial t}+\sum_{j=1}^{3}\frac{\partial(\rho v_i v_j)}{\partial x_j}=\rho f_i-\frac{\partial p}{\partial x_i}+\eta\sum_{j=1}^{3}\frac{\partial^2 v_i}{\partial x_j^2}+\left(\zeta+\frac{1}{3}\eta\right)\sum_{j=1}^{3}\frac{\partial^2 v_j}{\partial x_i\partial x_j}$$

Continuity equation

$$\frac{\partial \rho}{\partial t}+\sum_{j=1}^{3}\frac{\partial(\rho v_j)}{\partial x_j}=0$$

Vector form (Navier-Stokes equation)

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}-\frac{1}{\rho}\nabla p+\frac{\eta}{\rho}\nabla^2\mathbf{v}+\frac{1}{\rho}\left(\zeta+\frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity equation

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0$$

Newton-Euler equations for viscous fluids

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

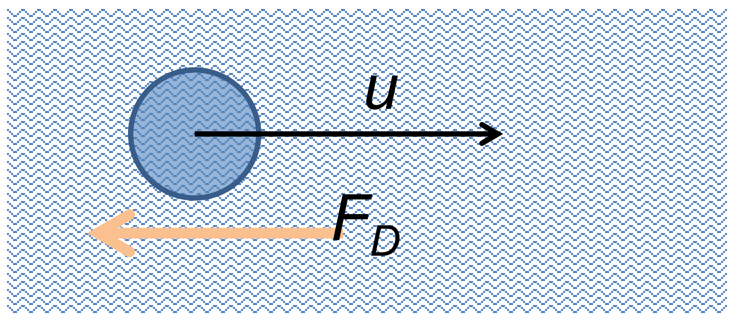
Typical viscosities at 20^o C and 1 atm:

Fluid	η/ρ (m^2/s)	η (Pa s)
Water	1.00×10^{-6}	1×10^{-3}
Air	14.9×10^{-6}	0.018×10^{-3}
Ethyl alcohol	1.52×10^{-6}	1.2×10^{-3}
Glycerine	1183×10^{-6}	1490×10^{-3}

More discussion of viscous effects in incompressible fluids

Stokes' analysis of viscous drag on a sphere of radius R moving at speed u in medium with viscosity η:

$$F_D = -\eta\left(6\pi R u\right)$$



"Derivation"

1. Consider the general effects of viscosity on fluid equations
2. Solve the linearized equations for the case of steady-state flow of a sphere of radius R
3. Infer the drag force needed to maintain the steady-state flow
4. Note that solution is special to the sphere geometry.

Some of the details

For an incompressible fluid: $\nabla \cdot \mathbf{v} = 0$

For irrotational flow: $\nabla \times \mathbf{v} = 0 \quad \Rightarrow \mathbf{v} = -\nabla\Phi(\mathbf{r}) \quad \nabla^2\Phi = 0$

For spherical polar coordinates and asymptotic form:

$$\Phi\left(r \to \infty, \theta\right) = -v_0 r \cos\theta$$

For an ideal fluid (no viscosity): $v_r(a) = 0$

$$\Phi\left(r,\theta\right) = -v_0\left(r + \frac{a^3}{2r^2}\right)\cos\theta \qquad v_r(r) = v_0\left(1 - \frac{a^3}{r^3}\right)\cos\theta$$

For viscous fluid: $v_r(a,\theta) = 0$ and $v_\theta(a,\theta) = 0$

with $\mathbf{v} = v_0\left(\nabla \times \left(\nabla \times f\left(r\right)\hat{\mathbf{z}}\right) + \hat{\mathbf{z}}\right)$ and $\nabla^4 f(r) = 0.$

$$v_r = v_0 \cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right) \qquad v_\theta = -v_0 \sin\theta\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)$$

General equations for hydrodynamics --

Vector form (Navier-Stokes equation)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla \cdot \mathbf{v})$$

Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Additional effects of viscosity – allowing for changes in entropy -- particularly in the case of sound waves in air

$$p(\rho, s) = p_0 + \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \delta s$$

Newton-Euler equations for viscous fluids

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla \cdot \mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Newton-Euler equations for viscous fluids – effects on sound

Without viscosity terms:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p \qquad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Assume: $\mathbf{v} = 0 + \delta\mathbf{v}$ $\quad \mathbf{f} = 0$ $\quad \rho = \rho_0 + \delta\rho$

$$p = p_0 + \delta p = p_0 + \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv p_0 + c^2\delta\rho$$

Linearized equations: $\displaystyle \frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla\delta\rho \qquad \frac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot(\delta\mathbf{v}) = 0$

Let $\delta\mathbf{v} \equiv \delta\mathbf{v}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ $\qquad \delta\rho \equiv \delta\rho_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

Sound waves without viscosity -- continued

Linearized equations: $\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho \qquad \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot (\delta \mathbf{v}) = 0$

Let $\quad \delta\mathbf{v} \equiv \delta\mathbf{v}_0 \; e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \delta\rho \equiv \delta\rho_0 \; e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho \qquad \Rightarrow \omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k}$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot (\delta \mathbf{v}) = 0 \qquad \Rightarrow -\omega\delta\rho_0 + \rho_0\mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\Rightarrow k^2 = \frac{\omega^2}{c^2} \qquad \frac{\delta\rho_0}{\rho_0} = \frac{\hat{\mathbf{k}}\cdot\delta\mathbf{v}_0}{c}$$

➔Pure longitudinal harmonic wave solutions

Newton-Euler equations for viscous fluids – effects on sound

Recall full equations:

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla \cdot \mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Assume: $\mathbf{v} = 0 + \delta \mathbf{v}$ $\quad$ $\mathbf{f} = 0$ $\quad$ $\rho = \rho_0 + \delta\rho$

$$p = p_0 + \delta p = p_0 + c^2 \delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \delta s$$

$$\text{where } c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s$$

viscosity causes heat transfer

Newton-Euler equations for viscous fluids – effects on sound

Note that pressure now depends both on density and entropy so that entropy must be coupled into the equations

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}-\frac{1}{\rho}\nabla p+\frac{\eta}{\rho}\nabla^2\mathbf{v}+\frac{1}{\rho}\left(\zeta+\frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0 \qquad\qquad \rho T\frac{\partial s}{\partial t}=k_{th}\nabla^2 T$$

Assume: $\mathbf{v}=0+\delta\mathbf{v}$ $\quad\mathbf{f}=0\quad$ $\rho=\rho_0+\delta\rho$

$$p=p_0+\delta p=p_0+c^2\delta\rho+\left(\frac{\partial p}{\partial s}\right)_\rho\delta s \quad \text{where } c^2\equiv\left(\frac{\partial p}{\partial \rho}\right)_s$$

$$T=T_0+\delta T=T_0+\left(\frac{\partial T}{\partial \rho}\right)_s\delta\rho+\left(\frac{\partial T}{\partial s}\right)_\rho\delta s$$

$$s=s_0+\delta s$$

Newton-Euler equations for viscous fluids – linearized equations

$$\frac{\partial \mathbf{v}}{\partial t}+\left(\mathbf{v}\cdot\nabla\right)\mathbf{v}=\mathbf{f}-\frac{1}{\rho}\nabla p+\frac{\eta}{\rho}\nabla^2\mathbf{v}+\frac{1}{\rho}\left(\zeta+\frac{1}{3}\eta\right)\nabla\left(\nabla\cdot\mathbf{v}\right)$$

$$\Rightarrow \frac{\partial \delta\mathbf{v}}{\partial t}=\underbrace{-\frac{1}{\rho_0}\nabla\delta p}+\frac{\eta}{\rho}\nabla^2\delta\mathbf{v}+\frac{1}{\rho_0}\left(\zeta+\frac{1}{3}\eta\right)\nabla\left(\nabla\cdot\delta\mathbf{v}\right)$$

$$-\frac{1}{\rho_0}\left\{\left(\frac{\partial p}{\partial\rho}\right)_s\nabla\delta\rho+\left(\frac{\partial p}{\partial s}\right)_\rho\nabla\delta s\right\}=-\frac{c^2}{\rho_0}\nabla\delta\rho-\rho_0\left(\frac{\partial T}{\partial\rho}\right)_s\nabla\delta s$$

Digression -- from the first law of thermodynamics:

$$d\epsilon=Tds+\frac{p}{\rho^2}d\rho$$

$$\left(\frac{\partial}{\partial\rho}\left(\frac{\partial\epsilon}{\partial s}\right)_\rho\right)_s=\left(\frac{\partial T}{\partial\rho}\right)_s \quad\Leftrightarrow\quad \left(\frac{\partial}{\partial s}\left(\frac{\partial\epsilon}{\partial\rho}\right)_s\right)_\rho=\left(\frac{\partial p/\rho^2}{\partial s}\right)_\rho\approx\frac{1}{\rho_0^2}\left(\frac{\partial p}{\partial s}\right)_\rho$$

Newton-Euler equations for viscous fluids – linearized equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \right) = 0$$

$$\Rightarrow \frac{\partial \delta \rho}{\partial t} + \rho_0 \nabla \cdot \left(\delta \mathbf{v} \right) = 0$$

$$\rho T \frac{\partial s}{\partial t} = k_{th} \nabla^2 T$$

$$\Rightarrow \frac{\partial \delta s}{\partial t} = \frac{k_{th}}{\rho_0 T_0} \left(\left(\frac{\partial T}{\partial s} \right)_\rho \nabla^2 \delta s + \left(\frac{\partial T}{\partial \rho} \right)_s \nabla^2 \delta \rho \right)$$

Further relationships:

$$\left(\frac{\partial T}{\partial s} \right)_\rho \approx \frac{T_0}{c_v} \qquad \kappa = \frac{k_{th}}{\rho c_p}$$

heat capacity at constant volume

Newton-Euler equations for viscous fluids – linearized equations

$$\Rightarrow \frac{\partial \delta s}{\partial t} = \left(\gamma\kappa\nabla^2 \delta s + \frac{c_p \kappa}{T_0} \left(\frac{\partial T}{\partial \rho} \right)_s \nabla^2 \delta\rho \right) \quad \text{where } \gamma \equiv \frac{c_p}{c_v}$$

Newton-Euler equations for viscous fluids – effects on sound

Linearized equations (with the help of various thermodynamic relationships):

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho - \rho_0 \left(\frac{\partial T}{\partial \rho}\right)_s \nabla \delta s + \frac{\eta}{\rho_0}\nabla^2 \delta \mathbf{v} + \frac{1}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\nabla\left(\nabla \cdot \delta \mathbf{v}\right)$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \left(\delta \mathbf{v}\right) = 0$$

$$\frac{\partial \delta s}{\partial t} = \gamma\kappa\nabla^2 \delta s + \frac{c_p \kappa}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \delta\rho$$

Here: $\gamma = \frac{c_p}{c_v}$ $\qquad \kappa = \frac{k_{th}}{c_p \rho_0}$

Linearized hydrodynamic equations

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho - \rho_0 \left(\frac{\partial T}{\partial \rho}\right)_s \nabla \delta s + \frac{\eta}{\rho_0}\nabla^2 \delta \mathbf{v} + \frac{1}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\nabla\left(\nabla \cdot \delta \mathbf{v}\right)$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \left(\delta \mathbf{v}\right) = 0$$

$$\frac{\partial \delta s}{\partial t} = \gamma\kappa\nabla^2 \delta s + \frac{c_p \kappa}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \delta\rho$$

It can be shown that

$$\left(\frac{\partial T}{\partial \rho}\right)_s = \frac{Tc^2\beta}{\rho c_p} \qquad \text{where} \qquad \beta \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p \qquad \text{(thermal expansion)}$$

Let $\delta \mathbf{v} \equiv \delta \mathbf{v}_0 \; e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ $\qquad \delta\rho \equiv \delta\rho_0 \; e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ $\qquad \delta s \equiv \delta s_0 \; e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$

Linearized hydrodynamic equations; plane wave solutions:

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} + \frac{T_0\beta c^2}{c_p}\delta s_0\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0 - \frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0$$

In the absense of thermal expansion, $\beta = 0$

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0$$

➔Entropy and mechanical modes are independent

Linearized hydrodynamic equations; full plane wave solutions:

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} + \frac{T_0\beta c^2}{c_p}\delta s_0\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0 - \frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0$$

Longitudinal solutions: $(\delta\mathbf{v}\cdot\mathbf{k} \neq 0)$:

$$\left(\omega^2 - c^2k^2 + i\frac{\omega k^2}{\rho_0}\left(\frac{4}{3}\eta + \zeta\right)\right)\delta\rho_0 - \frac{\rho_0 T_0\beta c^2 k^2}{c_p}\delta s_0 = 0$$

$$\frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0 + \left(\omega + i\gamma\kappa k^2\right)\delta s_0 = 0$$

Linearized hydrodynamic equations; full plane wave solutions:

Longitudinal solutions: $(\delta\mathbf{v}\cdot\mathbf{k}\neq 0)$:

$$\left(\omega^2 - c^2k^2 + i\frac{\omega k^2}{\rho_0}\left(\frac{4}{3}\eta+\zeta\right)\right)\delta\rho_0 - \frac{\rho_0 T_0 \beta c^2 k^2}{c_p}\delta s_0 = 0$$

$$\frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0 + \left(\omega + i\gamma\kappa k^2\right)\delta s_0 = 0$$

Approximate solution: $k = \dfrac{\omega}{c} + i\alpha$

where $\alpha \approx \dfrac{\omega^2}{2c^3\rho_0}\left(\dfrac{4}{3}\eta+\zeta\right) + \dfrac{\kappa T_0 \beta^2\omega^2}{2c_p c}$

$$\delta\rho = \delta\rho_0 e^{-\alpha\hat{\mathbf{k}}\cdot\mathbf{r}} e^{i\frac{\omega}{c}\left(\hat{\mathbf{k}}\cdot\mathbf{r}-ct\right)}$$

Linearized hydrodynamic equations; full plane wave solutions:

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} + \frac{T_0\beta c^2}{c_p}\delta s_0\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0 - \frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0$$

Transverse modes $(\delta\mathbf{v}\cdot\mathbf{k} = 0)$:

$$\delta\rho_0 = 0 \qquad \delta s_0 = 0$$

$$\left(\omega + \frac{i\eta k^2}{\rho_0}\right)\left(\delta\mathbf{v}\times\ \mathbf{k}\right) = 0 \qquad k = \pm\left(\frac{i\omega\rho_0}{\eta}\right)^{1/2}$$

New phenomena in the linear regime

1. Spatial attenuation of waves
2. New transverse modes

Comment on HW 29 -- Analysis of special case of linearized sound wave equations

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} + \frac{T_0\beta c^2}{c_p}\delta s_0\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0 - \frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0$$

For the homework problem, $\eta = 0$ and $\zeta = 0$

Also, note that in these expressions, we have used the expression

$$\left(\frac{\partial T}{\partial\rho}\right)_s = \frac{T_0 c^2\beta}{\rho_0 c_p} \quad \text{alternatively we can use} \quad \left(\frac{\partial T}{\partial\rho}\right)_s = \frac{T_0}{c_p}$$

Eq. 62.24 of your textbook also shows that

$$\left(\frac{\partial T}{\partial\rho}\right)_{s\Rightarrow}^2 = \frac{T_0 c^2}{\rho_0^2 c_p}(\gamma-1) = \left(\frac{T_0 c^2\beta}{\rho_0 c_p}\right)^2 \quad \Rightarrow c^4\beta^2 = (\gamma-1)\frac{c_p c^2}{T_0}$$