PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion for Lecture 37: Chap. 13 in F & W

Physics of elastic continua –

1. Stress and strain
2. Waves in elastic media

Physics Colloquium

4 PM
- Thursday -
Olin 101
November 21, 2024

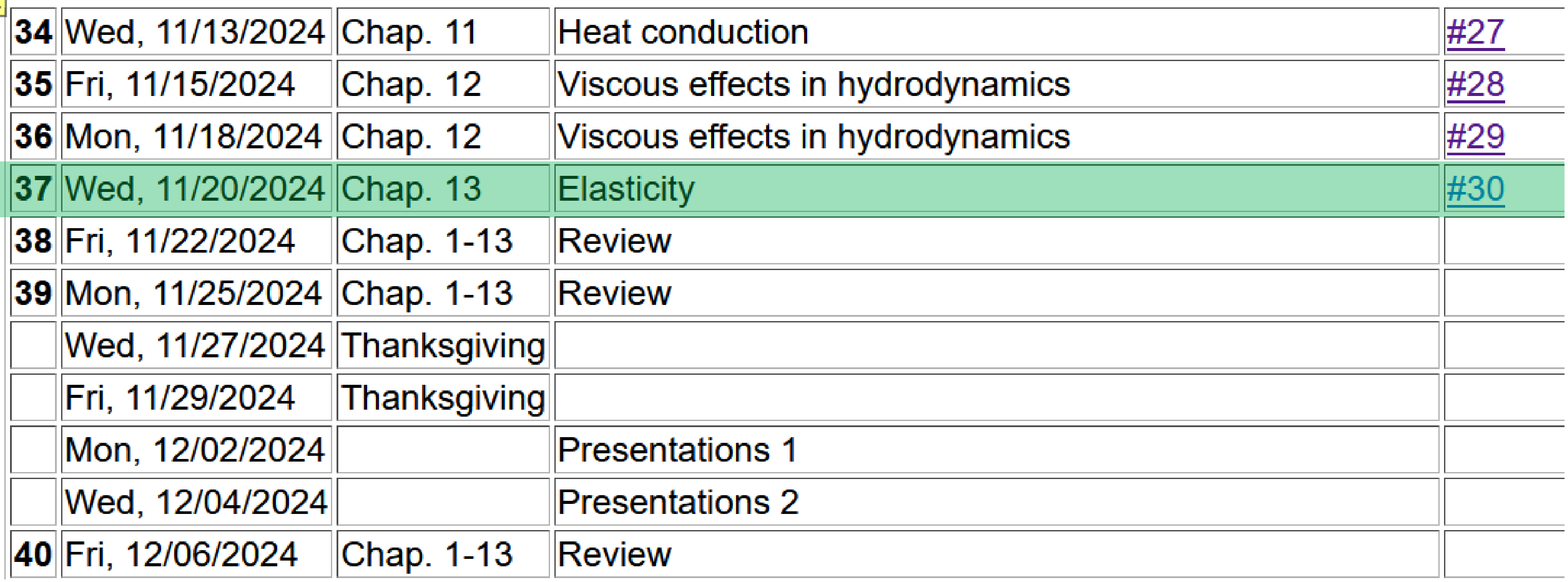
The lives and deaths of star clusters, and the black holes they make along the way

The life cycles of star clusters are an integral part of the formation of galaxies and their black hole populations. In these dense stellar environments, stars and black holes participate in complicated dynamical interactions that can create many unique objects, such as detached black hole binaries, hypervelocity stars, and gravitational-wave sources. In this talk, I will review our current understanding of the evolution of dense star clusters in the Milky Way, and their complicated relationship with their black hole populations. I will then describe a project to self-consistently evolve star clusters formed in a high-resolution MHD simulation of a Milky Way-mass galaxy, from their formation from collapsing giant molecular clouds to their destruction by galactic tidal fields. Finally, I will show how the birth conditions of these star clusters create massive black holes---from the 30 solar mass binaries detected by LIGO and Gaia to the ever illusive intermediate-mass black holes.

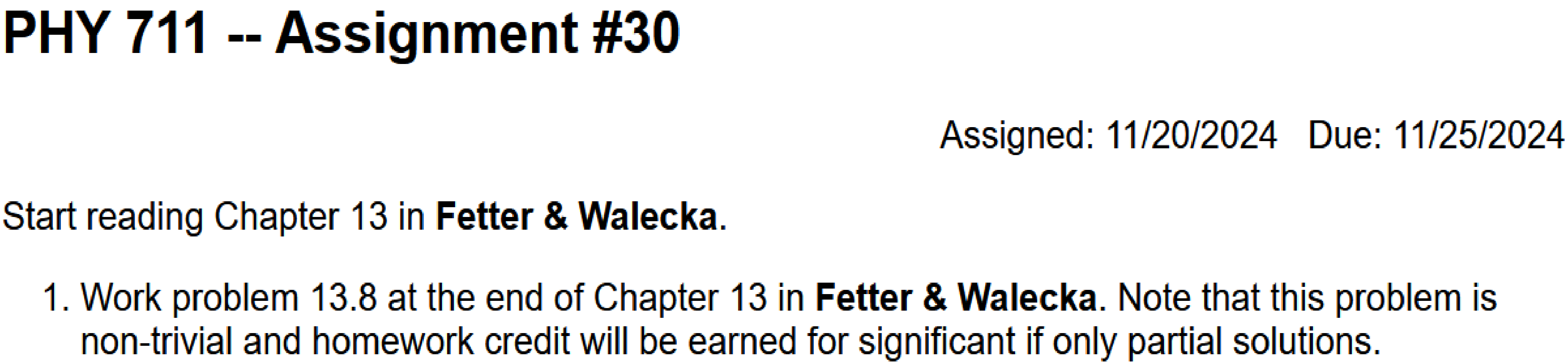
Professor Carl Rodriguez
University of North Carolina
Chapel Hill

Reception 3:30
Olin Lobby

Colloquium 4:00



34	Wed, 11/13/2024	Chap. 11	Heat conduction	#27
35	Fri, 11/15/2024	Chap. 12	Viscous effects in hydrodynamics	#28
36	Mon, 11/18/2024	Chap. 12	Viscous effects in hydrodynamics	#29
37	Wed, 11/20/2024	Chap. 13	Elasticity	#30
38	Fri, 11/22/2024	Chap. 1-13	Review	
39	Mon, 11/25/2024	Chap. 1-13	Review	
	Wed, 11/27/2024	Thanksgiving		
	Fri, 11/29/2024	Thanksgiving		
	Mon, 12/02/2024		Presentations 1	
	Wed, 12/04/2024		Presentations 2	
40	Fri, 12/06/2024	Chap. 1-13	Review	

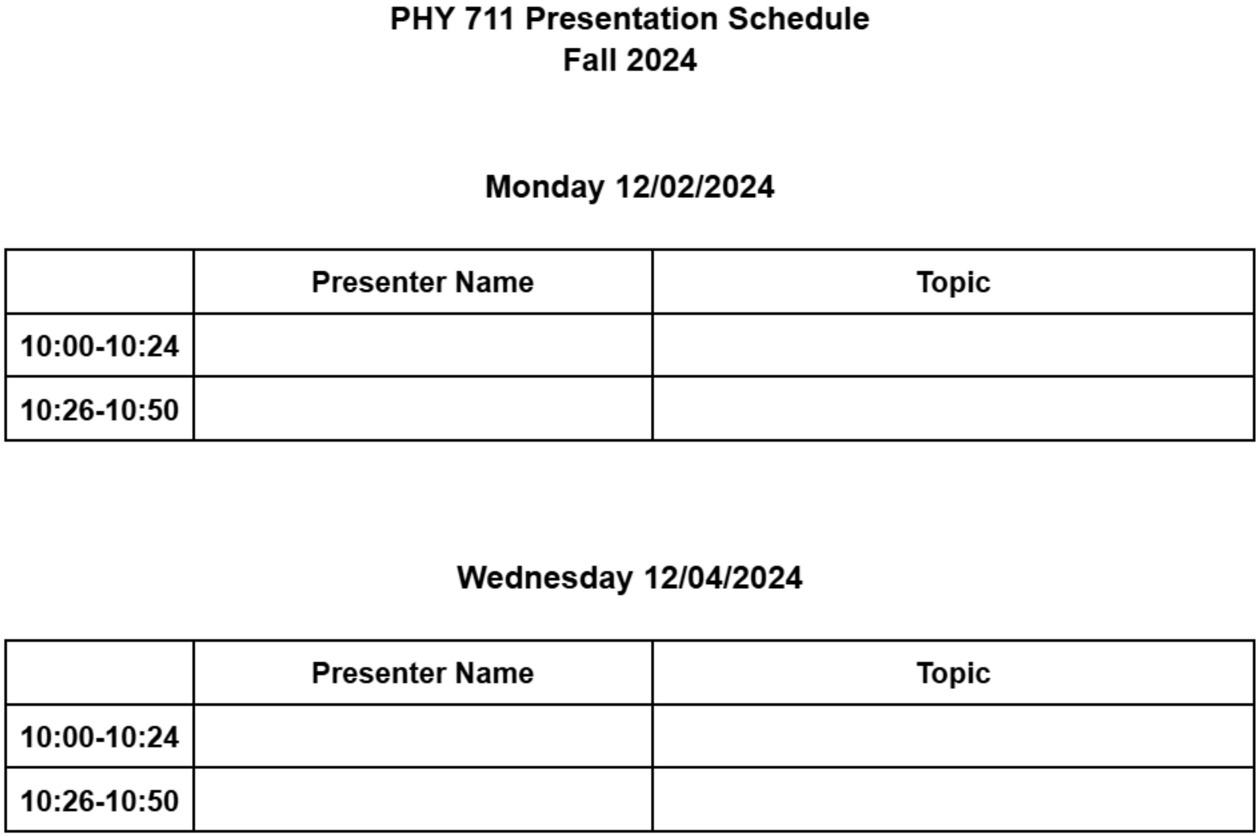


PHY 711 -- Assignment #30

Assigned: 11/20/2024 Due: 11/25/2024

Start reading Chapter 13 in **Fetter & Walecka**.

1. Work problem 13.8 at the end of Chapter 13 in **Fetter & Walecka**. Note that this problem is non-trivial and homework credit will be earned for significant if only partial solutions.



PHY 711 Presentation Schedule
Fall 2024

Monday 12/02/2024

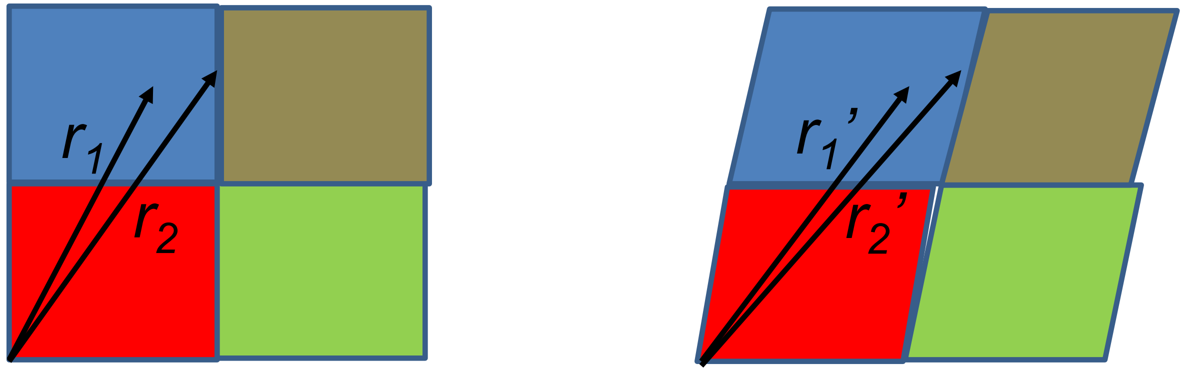
	Presenter Name	Topic
10:00-10:24		
10:26-10:50		

Wednesday 12/04/2024

	Presenter Name	Topic
10:00-10:24		
10:26-10:50		

When would you like to sign up?

Brief introduction to elastic continua



reference

deformation

$$\mathbf{r}_1' = \mathbf{r}_1 + \mathbf{u}(\mathbf{r}_1) \qquad \mathbf{r}_2' = \mathbf{r}_2 + \mathbf{u}(\mathbf{r}_2)$$

$$\mathbf{r}_2' - \mathbf{r}_1' \approx \mathbf{r}_2 - \mathbf{r}_1 + \left((\mathbf{r}_2 - \mathbf{r}_1) \cdot \nabla \right) \mathbf{u}(\mathbf{r}_1) + \ldots$$

Brief introduction to elastic continua -- continued

Deformation components:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\equiv \quad \epsilon_{ij} \quad + \quad O_{ij}$$

elastic strain tensor

rotation of material

Here we have in mind that the materials in our analysis are symmetric so that $\varepsilon_{ij}=\varepsilon_{ji}$.

Brief introduction to elastic continua -- continued

Deformation components:

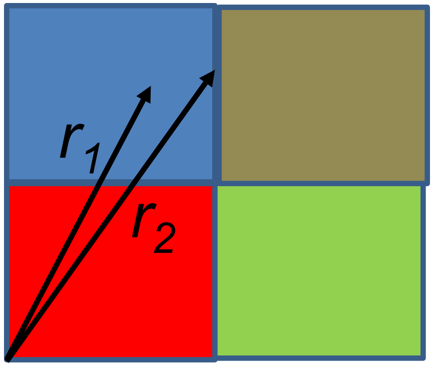
$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\equiv \epsilon_{ij} \qquad + \qquad \xcancel{O_{ij}}$$

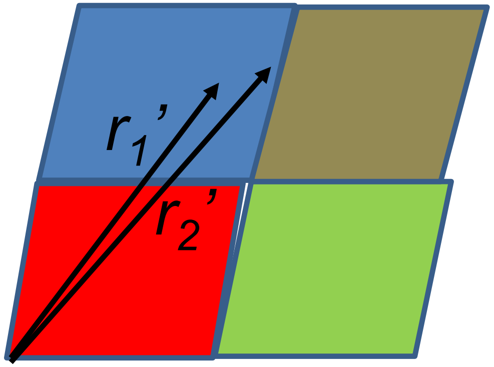
$$V = \mathbf{a}\cdot\left(\mathbf{b}\times\mathbf{c}\right) \qquad V' = \mathbf{a}'\cdot\left(\mathbf{b}'\times\mathbf{c}'\right) \qquad V' = V(1+\nabla\cdot\mathbf{u}) = V\left(1+\mathrm{Tr}\left(\epsilon\right)\right)$$

$$\nabla\cdot\mathbf{u} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \mathrm{Tr}\left(\epsilon\right) = \frac{dV}{V} = -\frac{d\rho}{\rho}$$

Brief introduction to elastic continua -- continued



reference



deformation

$$\mathbf{r}_1{}' = \mathbf{r}_1 + \mathbf{u}(\mathbf{r}_1) \qquad \mathbf{r}_2{}' = \mathbf{r}_2 + \mathbf{u}(\mathbf{r}_2)$$

$$\mathbf{r}_2{}' - \mathbf{r}_1{}' = \mathbf{r}_2 - \mathbf{r}_1 + \Big((\mathbf{r}_2 - \mathbf{r}_1) \cdot \nabla \Big) \mathbf{u}(\mathbf{r}_1) + \ldots$$

$$x_{2i}{}' - x_{1i}{}' \approx x_{2i} - x_{1i} + \sum_{j=1}^{3} \epsilon_{ij} \left(x_{2j} - x_{1j} \right) \qquad \epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Effects of strain on a vector:

$$\mathbf{a}' = \mathbf{a} + a \left(\epsilon_{11} \hat{\mathbf{x}} + \epsilon_{21} \hat{\mathbf{y}} + \epsilon_{31} \hat{\mathbf{z}} \right) \quad \text{where } \mathbf{a} = a \hat{\mathbf{x}}$$

$$a' = \left| \mathbf{a}' \cdot \mathbf{a}' \right|^{1/2} \approx a \left(1 + \epsilon_{11} \right)$$

Deformation

$$\mathbf{a}' = \mathbf{a} + a\left(\epsilon_{11}\hat{\mathbf{x}} + \epsilon_{21}\hat{\mathbf{y}} + \epsilon_{31}\hat{\mathbf{z}}\right)$$

$$\mathbf{b}' = \mathbf{b} + b\left(\epsilon_{12}\hat{\mathbf{x}} + \epsilon_{22}\hat{\mathbf{y}} + \epsilon_{32}\hat{\mathbf{z}}\right)$$

$$\text{for } \ \mathbf{a}\cdot\mathbf{b} = 0 = ab\cos\theta \qquad \Rightarrow \theta = \frac{\pi}{2}$$

$$\mathbf{a}'\cdot\mathbf{b}' \ \approx ab\left(\epsilon_{21} + \epsilon_{12}\right) = 2ab\epsilon_{12} = ab\cos\theta'$$

$$\cos\theta' = \cos\left(\theta + \left(\theta' - \theta\right)\right) = \cos\theta\ \cos\left(\theta' - \theta\right) - \sin\theta\ \sin\left(\theta' - \theta\right)$$

$$\approx -\sin\theta\ \sin\left(\theta' - \theta\right) \approx -\left(\theta' - \theta\right)$$

$$\theta' \approx \theta - 2\epsilon_{12} = \frac{\pi}{2} - 2\epsilon_{12}$$

Elastic stress tensor

$$-\sum_{j=1}^{3} T_{ij} dA_j \Rightarrow i^{th} \text{ component of force acting on surface } \hat{\mathbf{n}} dA \equiv d\mathbf{A}$$

Generalization of Hooke's law, $F_x = -kx$:

Lame' coefficients : $$T_{ij} = -\lambda \delta_{ij} \nabla \cdot \mathbf{u} - \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

or : $$T_{ij} = -\lambda \delta_{ij} \mathrm{Tr}\left(\epsilon\right) - 2\mu\epsilon_{ij}$$

Note: the elastic stress tensor is different from the stress tensor that we encountered in the hydrodynamic analysis!

What is the difference between elastic stress tensor and the stress tensor in hydrodynamics?

Recall Newton's second law for continuum system with density ρ, velocity components v_k and stress tensor T_{ij} :

$$\frac{\partial\left(\rho v_k\right)}{\partial t} = -\sum_{l=1}^{3} \frac{\partial T_{kl}}{\partial x_l} + \rho f_k$$

Force per unit mass

This equation works both for the case of hydrodynamics and for elasticity. The stress tensor T_{kl} has the units of force/area.

Ideal stress tensor for hydrodynamics (no viscosity)

$$T_{ij} \equiv \rho v_i v_j + p\delta_{ij}$$

Stress tensor for elastic medium with elastic coeffients λ and μ

$$T_{ij} = -\lambda\delta_{ij}\nabla\cdot\mathbf{u} - \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Simularities:

Equation of motion: $\frac{\partial\left(\rho v_k\right)}{\partial t} = -\sum_{l=1}^{3}\frac{\partial T_{kl}}{\partial x_l} + \rho f_k$

Differences:

For hydrodynamics, T represents stress due to motions of nearby fluid

For elasticity, T represents restoring forces due to displacements from equilibrium

Elastic stress tensor relationships at equilibrium

$$T_{ij} = -\lambda \delta_{ij} \mathrm{Tr}(\epsilon) - 2\mu\epsilon_{ij}$$

Note that: $\mathrm{Tr}(T) = -3\underbrace{\left(\lambda + \frac{2}{3}\mu\right)}_{K \equiv \text{bulk modulus} = -V\left(\frac{\partial p}{\partial V}\right)} \mathrm{Tr}(\epsilon)$

Inverse Hooke's law: $\epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)} \delta_{ij} Tr(T)\right)$

Stress tensor -- continued

$$\epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)} \delta_{ij} Tr(T) \right)$$

In terms of bulk modulus: $K = \lambda + \frac{2}{3}\mu$

$$\lambda = K - \frac{2}{3}\mu$$

$$\epsilon_{ij} = -\frac{1}{9K}\delta_{ij}Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\delta_{ij}Tr(T) \right)$$

$$\epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)}\ \delta_{ij}Tr(T)\right) = -\frac{1}{9K}\delta_{ij}Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\ \delta_{ij}Tr(T)\right)$$

Example -- hydrostatic pressure: $T_{ij} = \delta_{ij}dp$

$$\mathrm{Tr}(T) = 3dp$$

$$\epsilon_{ij} = -\frac{dp}{3\left(\lambda + \frac{2}{3}\mu\right)}\delta_{ij} \equiv \ -\frac{dp}{3K}\delta_{ij}$$

Note that: $\mathrm{Tr}(\epsilon) = \frac{dV}{V} = -\frac{dp}{K}$

$$\Rightarrow K = -V\frac{\partial p}{\partial V}$$

$$\epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)} \delta_{ij} Tr\left(T\right) \right) = -\frac{1}{9K}\delta_{ij} Tr\left(T\right) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\delta_{ij} Tr\left(T\right) \right)$$

Example -- uniaxial pressure: $T_{ij} = \begin{cases} dp & ij = zz \\ 0 & \text{otherwise} \end{cases}$

$$\epsilon_{zz} = -\frac{1}{E} T_{zz}$$ in terms of Young's modulus

$$E = \frac{9K\mu}{3K + \mu}$$

$$\epsilon_{xx} = \epsilon_{yy} = -\left(\frac{1}{9K} - \frac{1}{6\mu} \right) dp$$

Poisson ratio: $$\sigma = -\frac{\epsilon_{xx}}{\epsilon_{zz}} = -\frac{\epsilon_{yy}}{\epsilon_{zz}} = \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}$$

$$\epsilon_{ij} = -\frac{1}{9K}\delta_{ij}Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\ \delta_{ij}Tr(T)\right)$$

Shear modulus

$$T_{ij} = \begin{cases} -f & \text{for } T_{xy} \text{ or } T_{yx} \\ 0 & \text{otherwise} \end{cases}$$

$$\epsilon_{xy} = \epsilon_{yx} = \frac{f}{2\mu}$$

Relationships between the elastic moduli:

Poisson's ratio:

$$\sigma = -\frac{\epsilon_{xx}}{\epsilon_{zz}} = \frac{1}{2}\frac{3K-2\mu}{3K+\mu}$$

Young's modulus:

$$E = \frac{9K\mu}{3K+\mu}$$

Shear modulus: μ

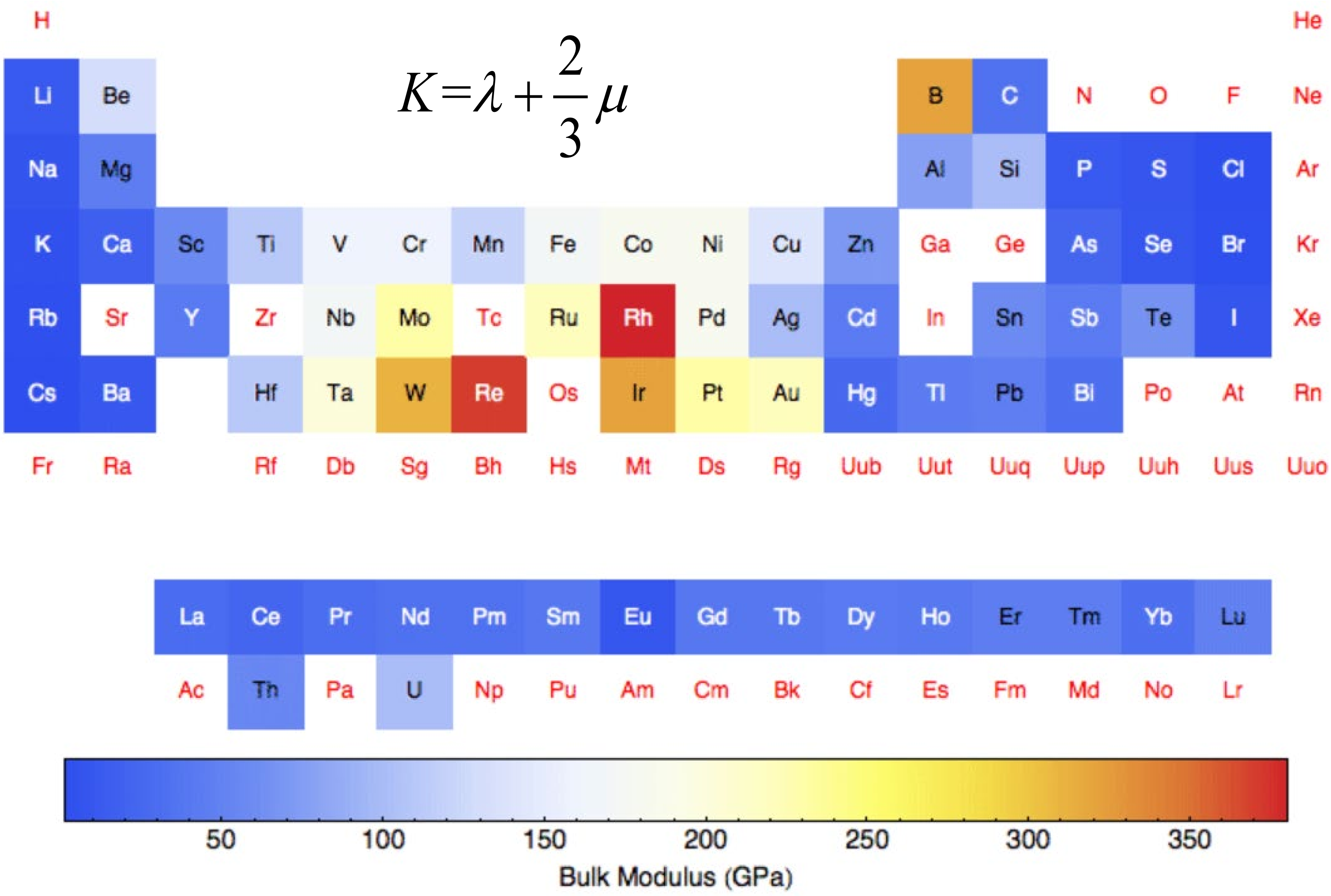
Relationships between elastic constants:

$$K = \frac{1}{3}\frac{E}{1-2\sigma}$$

$$\mu = \frac{1}{2}\frac{E}{1+\sigma}$$

Values of bulk modulus K for elemental materials --

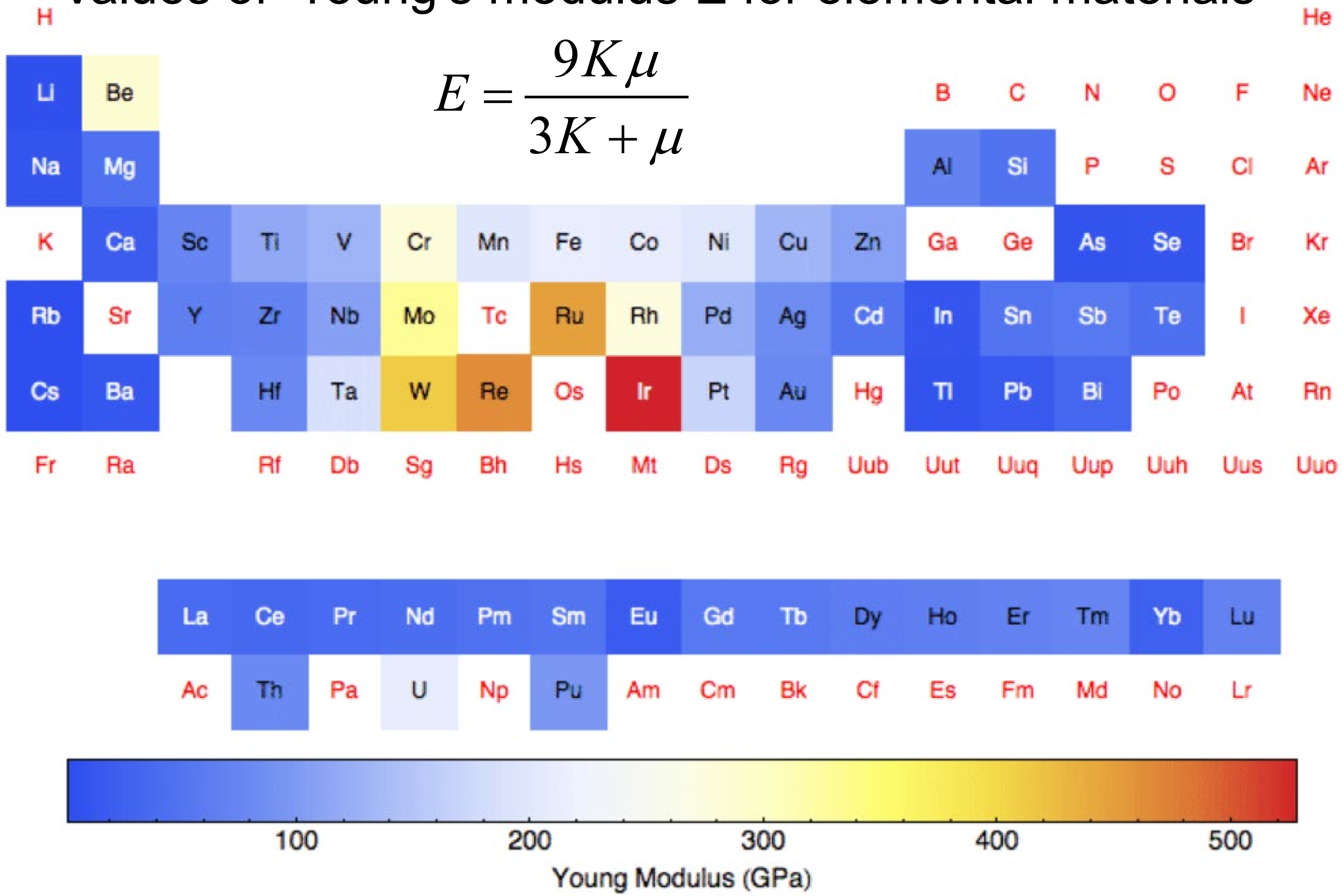
$$K = \lambda + \frac{2}{3}\mu$$



Up to date, curated data provided by *Mathematica*'s ElementData function from Wolfram Research, Inc.

Values of Young's modulus E for elemental materials --

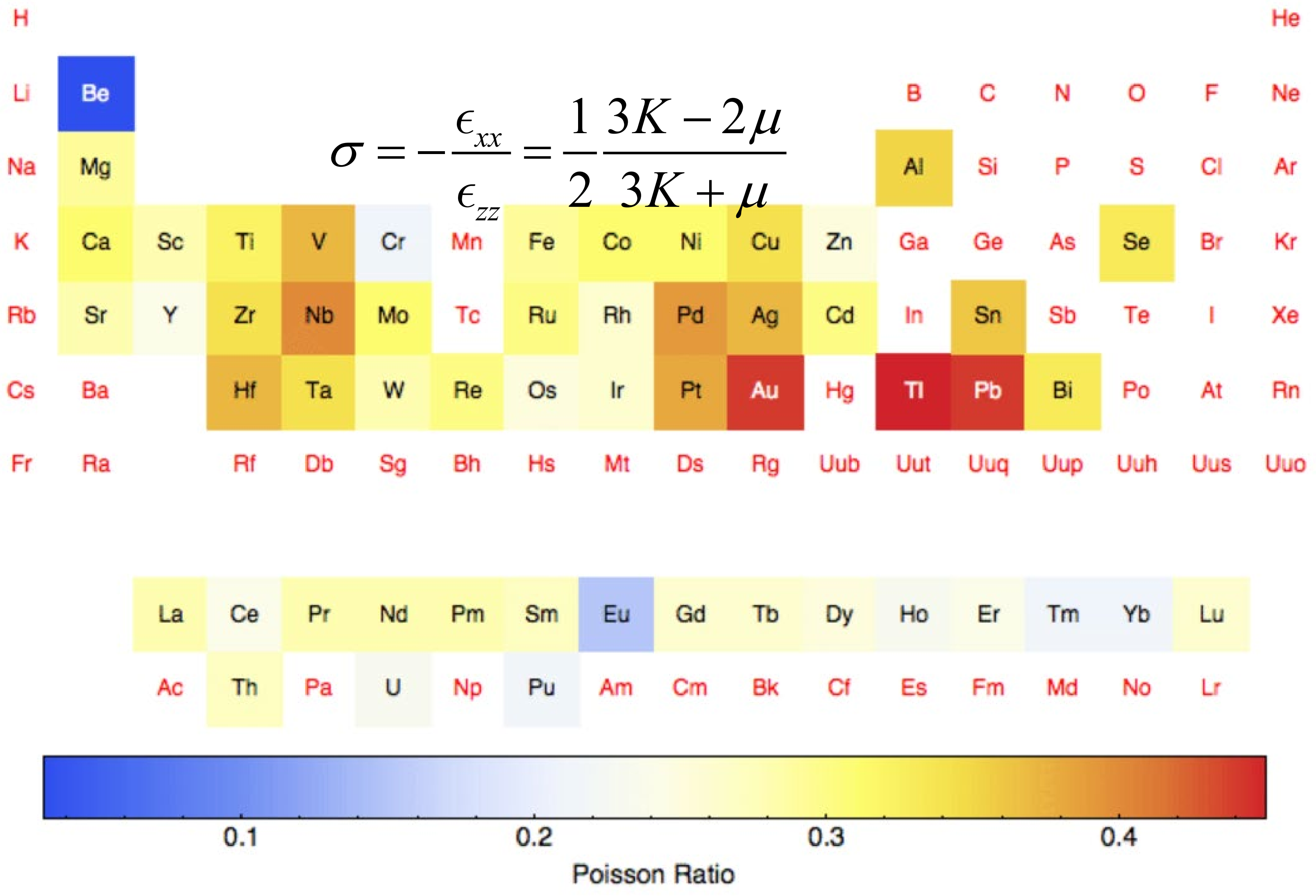
$$E = \frac{9K\mu}{3K+\mu}$$



Up to date, curated data provided by *Mathematica*'s ElementData function from Wolfram Research, Inc.

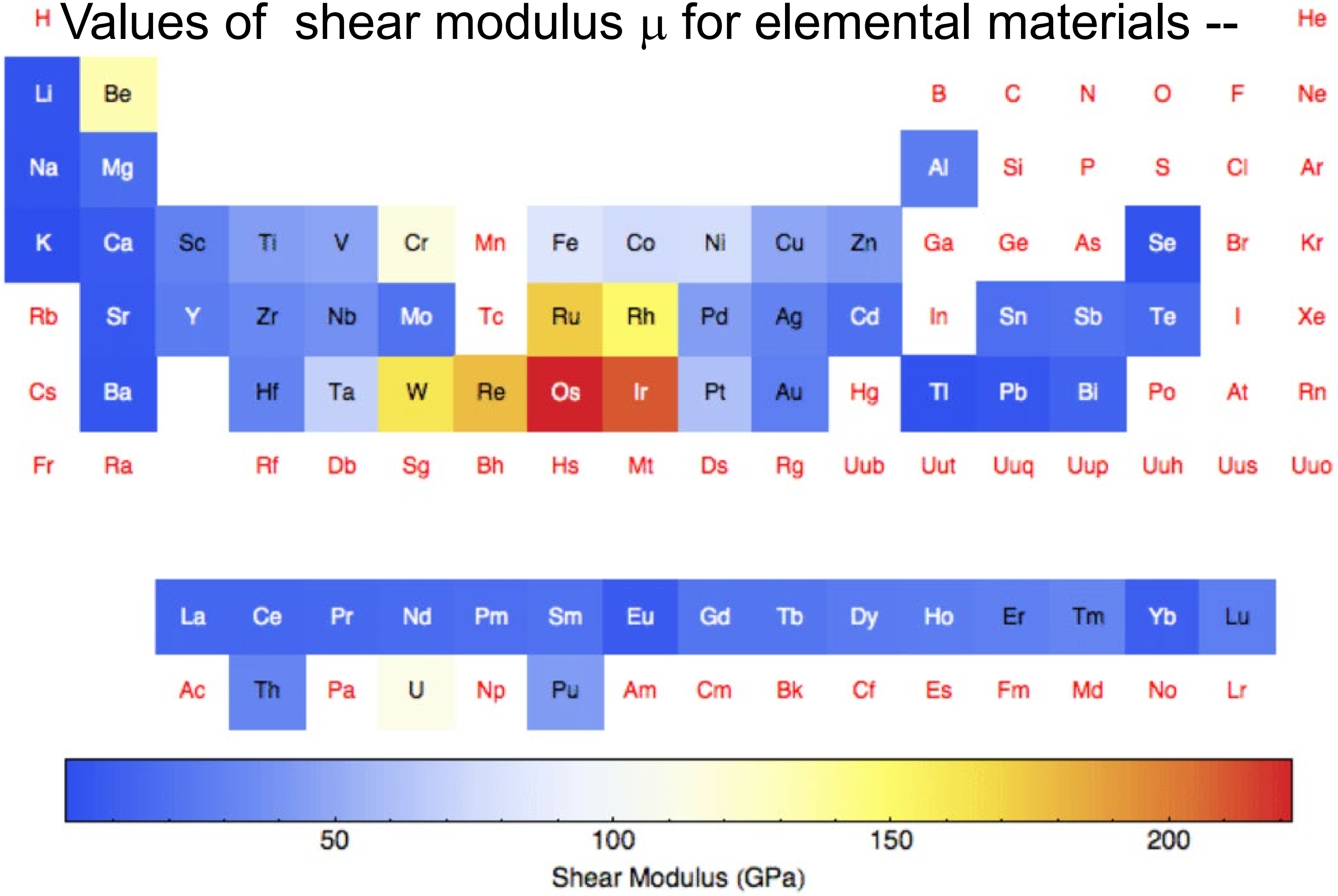
Values of Poisson ratio σ for elemental materials --

$$\sigma = -\frac{\epsilon_{xx}}{\epsilon_{zz}} = \frac{1}{2}\frac{3K-2\mu}{3K+\mu}$$



Up to date, curated data provided by *Mathematica*'s ElementData function from Wolfram Research, Inc.

Values of shear modulus μ for elemental materials --



Up to date, curated data provided by *Mathematica*'s ElementData function from Wolfram Research, Inc.

Summary -- Elastic stress tensor

$$-\sum_{j=1}^{3} T_{ij} dA_j \Rightarrow i^{th} \text{ component of force acting on surface } \hat{\mathbf{n}} dA \equiv d\mathbf{A}$$

Generalization of Hooke's law, $\mathrm{F}_x = -kx$:

Lame' coefficients : $$T_{ij} = -\lambda \delta_{ij} \nabla \cdot \mathbf{u} - \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = -\lambda \delta_{ij} \mathrm{Tr}\left(\epsilon\right) - 2\mu\epsilon_{ij}$$

material-dependent empirical parameters

Bulk modulus: $K = \lambda + \frac{2}{3}\mu$

Young's modulus: $E = \frac{9K\mu}{3K+\mu}$

Poisson ratio: $\sigma = \frac{1}{2}\frac{3K-2\mu}{3K+\mu}$

Shear modulus: μ

Stress

Strain

Hooke's Law: $$T_{ij} = -K\delta_{ij}\mathrm{Tr}(\epsilon) - 2\mu\left(\epsilon_{ij} - \frac{1}{3}\delta_{ij}\mathrm{Tr}(\epsilon)\right)$$

Inverse Hooke's Law: $$\epsilon_{ij} = -\frac{1}{9K}\delta_{ij}Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\,\delta_{ij}Tr(T)\right)$$

Elastic force: $$F_i^{\text{elastic}} = \sum_{j=1}^{3}\frac{\partial T_{ij}}{\partial x_j}$$

Elastic work due to distortion $\delta\mathbf{u}$

$$\delta W^{\text{elastic}} = \int_V d^3x\ \mathbf{F}^{\text{elastic}} \cdot \delta\mathbf{u}$$

$$= \int_V d^3x\ \sum_{ij=1}^{3}\frac{\partial T_{ij}}{\partial x_j}\delta u_i$$

$$= \sum_{ij=1}^{3}\int_A dA_j\ T_{ij}\delta u_i - \sum_{ij=1}^{3}\int_V d^3x T_{ij}\frac{\partial \delta u_i}{\partial x_j}$$

Elastic work due to distortion $\delta\mathbf{u}$

$$\delta W^{\text{elastic}} = \int_V d^3x\ \mathbf{F}^{\text{elastic}} \cdot \delta\mathbf{u}$$

$$= \sum_{ij=1}^{3} \int_A dA_j\ T_{ij}\delta u_i - \sum_{ij=1}^{3} \int_V d^3x T_{ij} \frac{\partial \delta u_i}{\partial x_j}$$

$$= \sum_{ij=1}^{3} \int_A dA_j\ T_{ij}\delta u_i - \sum_{ij=1}^{3} \int_V d^3x T_{ij}\epsilon_{ij}$$

surface contribution

bulk contribution

For large samples: $\delta W^{\text{elastic}} = -\sum_{ij=1}^{3} \int_V d^3x T_{ij}\epsilon_{ij}$

Bulk elastic energy: $\delta W^{\text{elastic}} = -\sum_{ij=1}^{3}\int_V d^3x\, T_{ij}\epsilon_{ij}$

Integrating from 0 to final strain ϵ_{ij} :

$$\delta W^{\text{elastic}} = -\frac{1}{2}\sum_{ij=1}^{3} T_{ij}\epsilon_{ij}$$

Hooke's Law: $T_{ij} = -K\delta_{ij}\mathrm{Tr}(\epsilon) - 2\mu\left(\epsilon_{ij} - \frac{1}{3}\delta_{ij}\mathrm{Tr}(\epsilon)\right)$

$$\delta W^{\text{elastic}} = \frac{1}{2}K\left(\mathrm{Tr}\,\epsilon\right)^2 + \mu\sum_{ij=1}^{3}\left(\epsilon_{ij} - \frac{\delta_{ij}}{3}\mathrm{Tr}\,\epsilon\right)^2$$

$$= \frac{1}{2}\lambda\left(\mathrm{Tr}\,\epsilon\right)^2 + \mu\sum_{ij=1}^{3}\epsilon_{ij}^2$$

Note that the two relations:

$$\delta W^{\text{elastic}} = \frac{1}{2}K\left(\mathrm{Tr}\,\epsilon\right)^2 + \mu\sum_{ij=1}^{3}\left(\epsilon_{ij} - \frac{\delta_{ij}}{3}\mathrm{Tr}\,\epsilon\right)^2$$

$$T_{ij} = -K\delta_{ij}\mathrm{Tr}\left(\epsilon\right) - 2\mu\left(\epsilon_{ij} - \frac{1}{3}\delta_{ij}\mathrm{Tr}\left(\epsilon\right)\right)$$

Ensure that: $\dfrac{\partial \delta W^{\text{elastic}}}{\partial \epsilon_{ij}} = -T_{ij}$

Dynamical equations of motion

Recall Newton's second law for continuum system with density ρ, velocity components v_k and stress tensor T_{ij} :

$$\frac{\partial\left(\rho v_k\right)}{\partial t} = -\sum_{l=1}^{3}\frac{\partial T_{kl}}{\partial x_l} + \rho f_k$$

external force density

For our elastic medium: ρ does not vary in time

Velocity related to displacement: $v_k = \frac{\partial u_k}{\partial t}$

Hooke's law: $T_{kl} = -K\delta_{kl}\mathrm{Tr}\left(\epsilon\right) - 2\mu\left(\epsilon_{kl} - \frac{1}{3}\delta_{kl}\mathrm{Tr}\left(\epsilon\right)\right)$

$$= -K\delta_{kl}\left(\nabla\cdot\mathbf{u}\right) - 2\mu\left(\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) - \frac{1}{3}\delta_{kl}\left(\nabla\cdot\mathbf{u}\right)\right)$$

For: $T_{kl} = -K\delta_{kl}\left(\nabla\cdot\mathbf{u}\right) - 2\mu\left(\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\right)-\frac{1}{3}\delta_{kl}\left(\nabla\cdot\mathbf{u}\right)\right)$

$$\sum_{l=1}^{3}\frac{\partial T_{kl}}{\partial x_l} = -\left(K-\frac{2}{3}\mu\right)\sum_{l=1}^{3}\left(\delta_{kl}\frac{\partial\left(\nabla\cdot\mathbf{u}\right)}{\partial x_l}\right)-\mu\sum_{l=1}^{3}\frac{\partial^2 u_l}{\partial x_k\partial x_l}-\mu\sum_{l=1}^{3}\frac{\partial^2 u_k}{\partial^2 x_l}$$

$$= -\left(K+\frac{1}{3}\mu\right)\sum_{l=1}^{3}\frac{\partial^2 u_l}{\partial x_k\partial x_l}-\mu\sum_{l=1}^{3}\frac{\partial^2 u_k}{\partial^2 x_l}$$

$$\frac{\partial\left(\rho v_k\right)}{\partial t} = -\sum_{l=1}^{3}\frac{\partial T_{kl}}{\partial x_l}+\rho f_k \qquad\qquad v_k = \frac{\partial u_k}{\partial t}$$

$$\rho\frac{\partial^2 u_k}{\partial t^2} = \left(K+\frac{1}{3}\mu\right)\sum_{l=1}^{3}\frac{\partial^2 u_l}{\partial x_k\partial x_l}+\mu\sum_{l=1}^{3}\frac{\partial^2 u_k}{\partial^2 x_l}+\rho f_k$$

Vector form: $\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \left(K+\frac{1}{3}\mu\right)\nabla\left(\nabla\cdot\mathbf{u}\right)+\mu\nabla^2\mathbf{u}+\rho\mathbf{f}$

Dynamical equations of elastic continuum

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=\mu\nabla^2\mathbf{u}+\left(K+\frac{1}{3}\mu\right)\nabla\left(\nabla\cdot\mathbf{u}\right)+\rho\mathbf{f}$$

In the absence of external forces, this reduces to two decoupled wave equations representing longitudinal and transverse modes:

$\mathbf{u}=\mathbf{u}_l+\mathbf{u}_t$

where $\nabla\times\mathbf{u}_l=0$ and $\nabla\cdot\mathbf{u}_t=0$

$$c_l=\left(\frac{K+\frac{4}{3}\mu}{\rho}\right)^{1/2} \quad \text{and} \quad c_t=\left(\frac{\mu}{\rho}\right)^{1/2}$$

Some details -- Dynamical equations of elastic continuum for **f**=0

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=\mu\nabla^2\mathbf{u}+\left(K+\frac{1}{3}\mu\right)\nabla\left(\nabla\cdot\mathbf{u}\right)$$

$\mathbf{u}=\mathbf{u}_l+\mathbf{u}_t$ where $\nabla\times\mathbf{u}_l=0$ and $\nabla\cdot\mathbf{u}_t=0$

$$\rho\frac{\partial^2\mathbf{u}_t}{\partial t^2}=\mu\nabla^2\mathbf{u}_t \quad\Rightarrow\frac{\partial^2\mathbf{u}_t}{\partial t^2}=\frac{\mu}{\rho}\nabla^2\mathbf{u}_t=c_t^2\nabla^2\mathbf{u}_t \quad c_t=\left(\frac{\mu}{\rho}\right)^{1/2}$$

Note that $\nabla\left(\nabla\cdot\mathbf{u}_l\right)=\nabla^2\mathbf{u}_l+\nabla\times\left(\nabla\times\mathbf{u}_l\right)=\nabla^2\mathbf{u}_l$

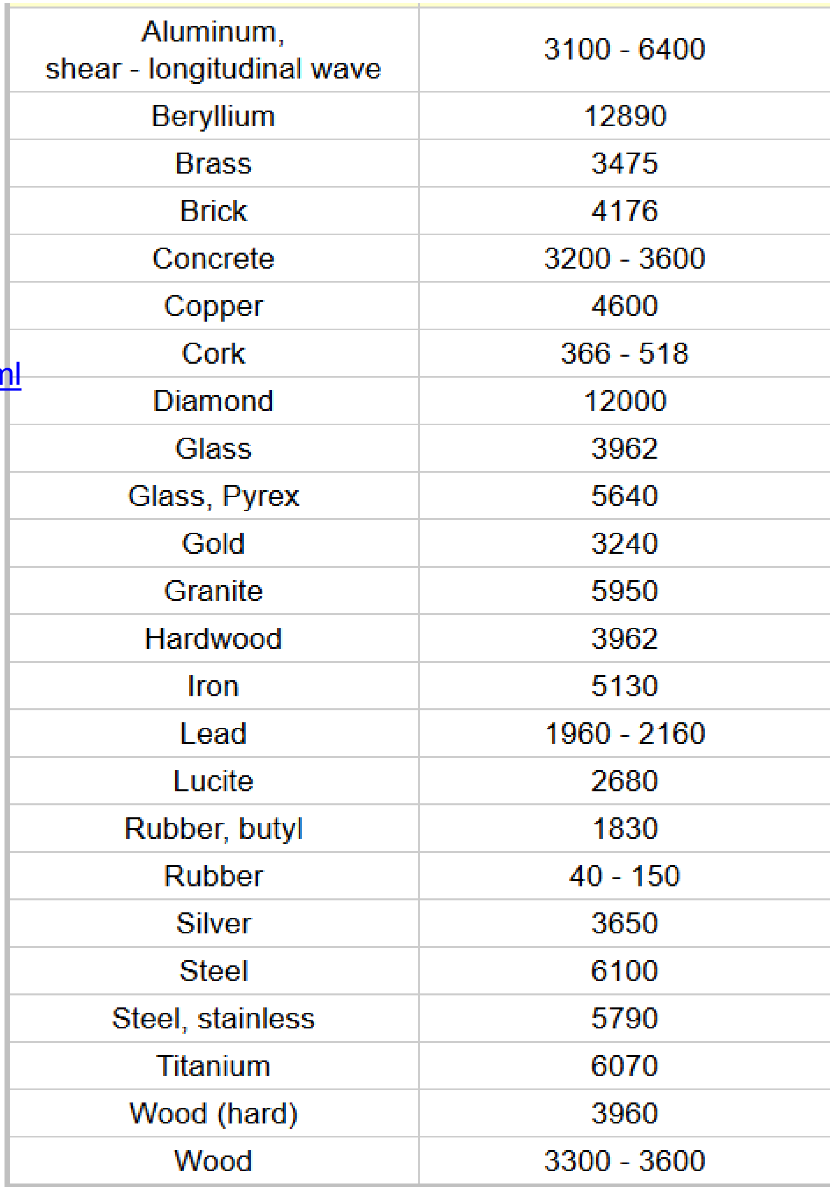
$$\rho\frac{\partial^2\mathbf{u}_l}{\partial t^2}=\mu\nabla^2\mathbf{u}_l+\left(K+\frac{1}{3}\mu\right)\nabla^2\mathbf{u}_l \quad\Rightarrow\frac{\partial^2\mathbf{u}_l}{\partial t^2}=c_l^2\nabla^2\mathbf{u}_l \quad c_l=\left(\frac{K+\frac{4}{3}\mu}{\rho}\right)^{1/2}$$

Typical velocities of longitudinal sound waves

http://www.engineeringtoolbox.com/sound-speed-solids-d_713.html

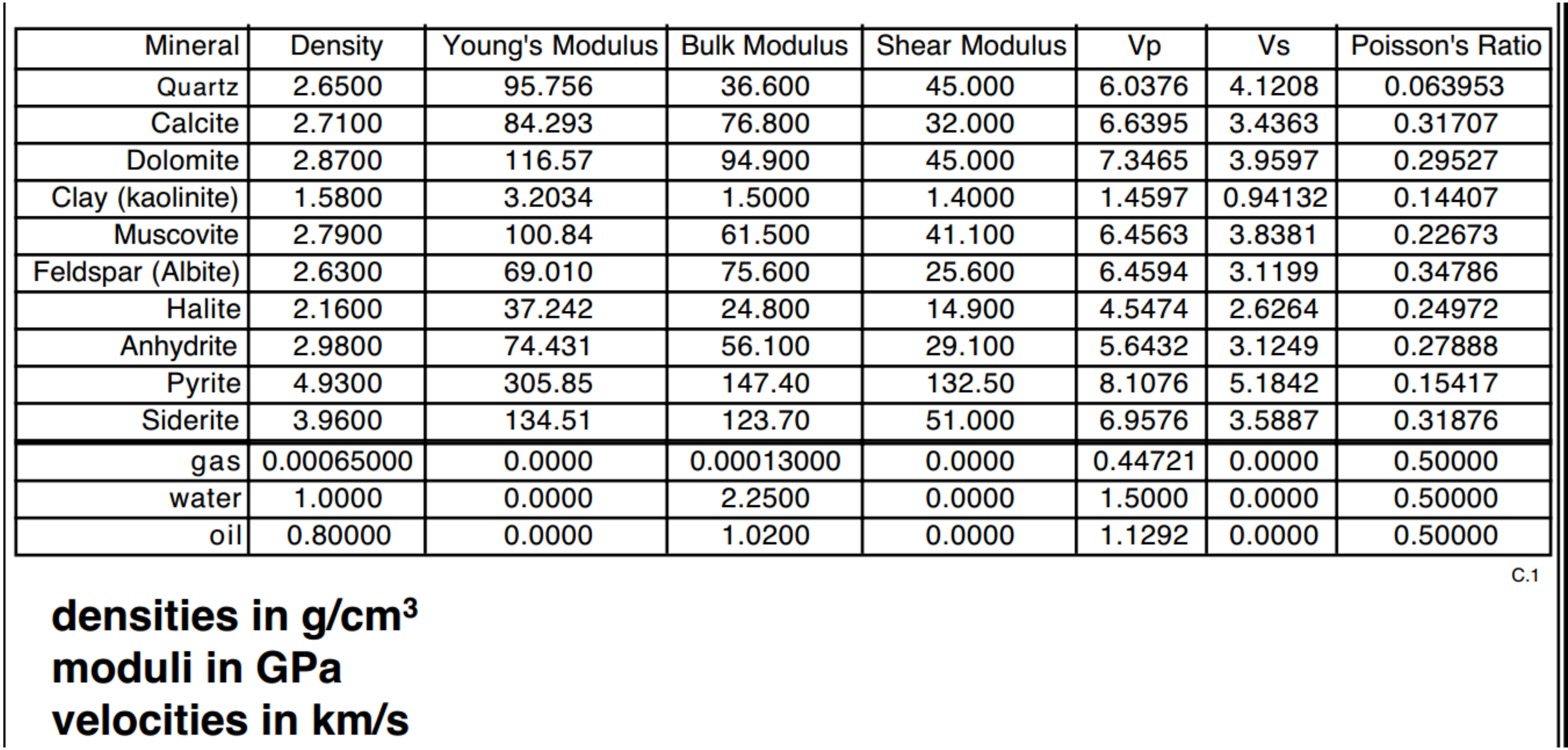
air: c_l=343 m/s

water: c_l=1433 m/s



Material	c_l (m/s)
Aluminum, shear - longitudinal wave	3100 - 6400
Beryllium	12890
Brass	3475
Brick	4176
Concrete	3200 - 3600
Copper	4600
Cork	366 - 518
Diamond	12000
Glass	3962
Glass, Pyrex	5640
Gold	3240
Granite	5950
Hardwood	3962
Iron	5130
Lead	1960 - 2160
Lucite	2680
Rubber, butyl	1830
Rubber	40 - 150
Silver	3650
Steel	6100
Steel, stainless	5790
Titanium	6070
Wood (hard)	3960
Wood	3300 - 3600

from:
https://pangea.stanford.edu/courses/gp262/Notes/5.Elasticity.pdf



Mineral	Density	Young's Modulus	Bulk Modulus	Shear Modulus	Vp	Vs	Poisson's Ratio
Quartz	2.6500	95.756	36.600	45.000	6.0376	4.1208	0.063953
Calcite	2.7100	84.293	76.800	32.000	6.6395	3.4363	0.31707
Dolomite	2.8700	116.57	94.900	45.000	7.3465	3.9597	0.29527
Clay (kaolinite)	1.5800	3.2034	1.5000	1.4000	1.4597	0.94132	0.14407
Muscovite	2.7900	100.84	61.500	41.100	6.4563	3.8381	0.22673
Feldspar (Albite)	2.6300	69.010	75.600	25.600	6.4594	3.1199	0.34786
Halite	2.1600	37.242	24.800	14.900	4.5474	2.6264	0.24972
Anhydrite	2.9800	74.431	56.100	29.100	5.6432	3.1249	0.27888
Pyrite	4.9300	305.85	147.40	132.50	8.1076	5.1842	0.15417
Siderite	3.9600	134.51	123.70	51.000	6.9576	3.5887	0.31876
gas	0.00065000	0.0000	0.00013000	0.0000	0.44721	0.0000	0.50000
water	1.0000	0.0000	2.2500	0.0000	1.5000	0.0000	0.50000
oil	0.80000	0.0000	1.0200	0.0000	1.1292	0.0000	0.50000

C.1

densities in g/cm^3
moduli in GPa
velocities in km/s

Dynamical equations of elastic continuum

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=\mu\nabla^2\mathbf{u}+\left(K+\frac{1}{3}\mu\right)\nabla\left(\nabla\cdot\mathbf{u}\right)+\rho\mathbf{f}$$

In absense of external force:

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=\mu\nabla^2\mathbf{u}+\left(K+\frac{1}{3}\mu\right)\nabla\left(\nabla\cdot\mathbf{u}\right)$$

Suppose: $\mathbf{u}=\mathbf{u}_l+\mathbf{u}_t$

where $\nabla\times\mathbf{u}_l=0$ and $\nabla\cdot\mathbf{u}_t=0$

$$\rho\frac{\partial^2\left(\mathbf{u}_l+\mathbf{u}_t\right)}{\partial t^2}=\mu\nabla^2\left(\mathbf{u}_l+\mathbf{u}_t\right)+\left(K+\frac{1}{3}\mu\right)\nabla\left(\nabla\cdot\mathbf{u}_l\right)$$

Dynamical equations of elastic continuum

Transverse component:

$$\rho \frac{\partial^2 \mathbf{u}_t}{\partial t^2} = \mu \nabla^2 \mathbf{u}_t \qquad \frac{\partial^2 \mathbf{u}_t}{\partial t^2} = \frac{\mu}{\rho} \nabla^2 \mathbf{u}_t \equiv c_t^2 \nabla^2 \mathbf{u}_t$$

Transverse wave velocity: $\mathrm{c}_t = \sqrt{\frac{\mu}{\rho}}$

Longitudinal component: $\rho \frac{\partial^2 \mathbf{u}_l}{\partial t^2} = \mu \nabla^2 \mathbf{u}_l + \left(K + \frac{1}{3}\mu \right) \nabla \left(\nabla \cdot \mathbf{u}_l \right)$

Note that the longitudinal wave has its displacement along its propagation direction x_p, so that $\mathbf{u}_l = \mathbf{u}_l \left(x_p \right) \equiv u_l(x_p)\hat{\mathbf{x}}_p$

$$\Rightarrow \rho \frac{\partial^2 u_l}{\partial t^2} = \mu \frac{\partial^2 u_l}{\partial x_p^2} + \left(K + \frac{1}{3}\mu \right) \frac{\partial^2 u_l}{\partial x_p^2} = \left(K + \frac{4}{3}\mu \right) \frac{\partial^2 u_l}{\partial x_p^2}$$

Dynamical equations of elastic continuum

Longitudinal component -- continued:

$$\rho \frac{\partial^2 u_l}{\partial t^2} = \left(K + \frac{4}{3}\mu \right) \frac{\partial^2 u_l}{\partial x_p^2}$$

$$\frac{\partial^2 u_l}{\partial t^2} = \left(\frac{K}{\rho} + \frac{4}{3}\frac{\mu}{\rho} \right) \frac{\partial^2 u_l}{\partial x_p^2} \equiv c_l^2 \frac{\partial^2 u_l}{\partial x_p^2}$$

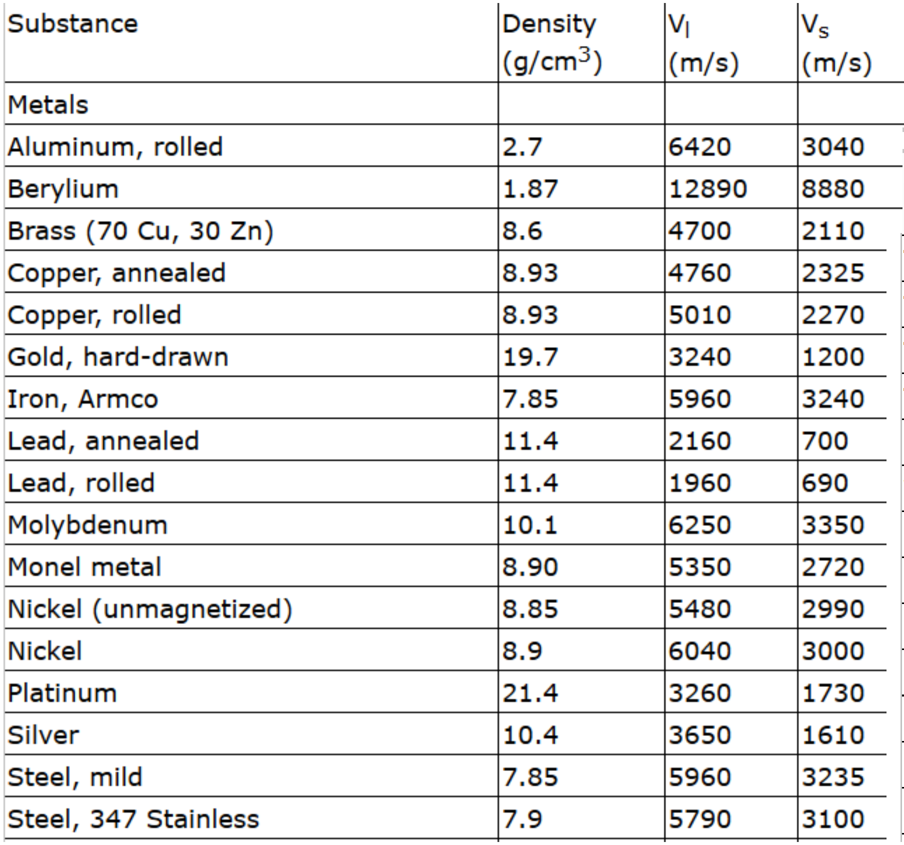
Longitudinal wave velocity: $\mathrm{c}_l = \sqrt{\frac{K}{\rho} + \frac{4}{3}\frac{\mu}{\rho}}$

Transverse component:

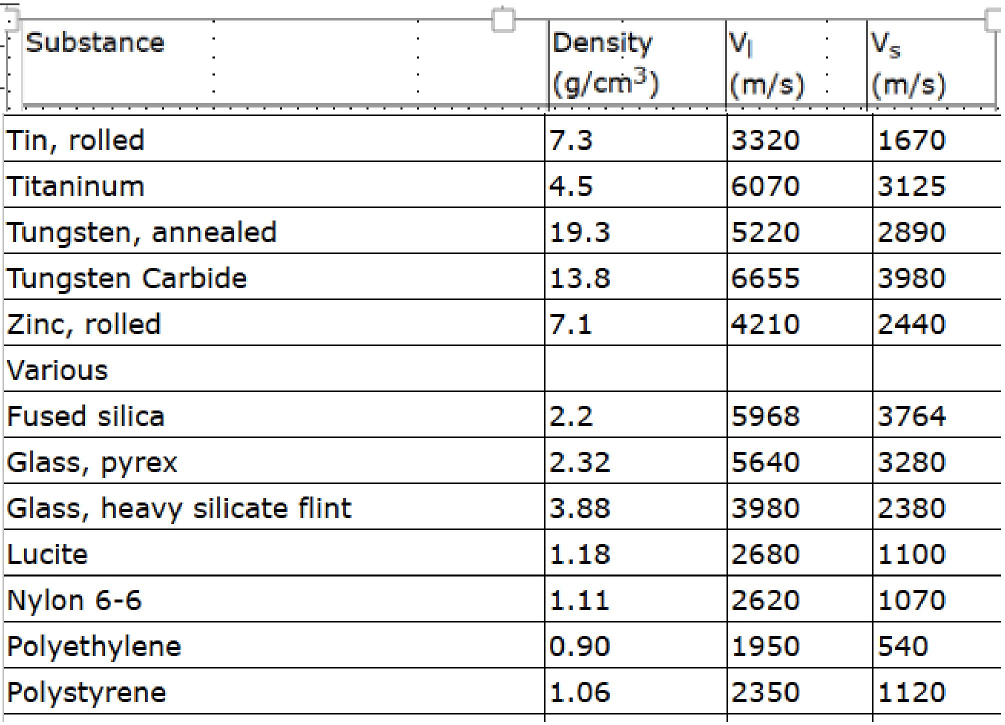
$$\rho \frac{\partial^2 \mathbf{u}_t}{\partial t^2} = \mu \nabla^2 \mathbf{u}_t \qquad \frac{\partial^2 \mathbf{u}_t}{\partial t^2} = \frac{\mu}{\rho} \nabla^2 \mathbf{u}_t \equiv c_t^2 \nabla^2 \mathbf{u}_t$$

Transverse wave velocity: $\mathrm{c}_t = \sqrt{\frac{\mu}{\rho}}$

Some values:



Substance	Density (g/cm^3)	V_l (m/s)	V_s (m/s)
Metals			
Aluminum, rolled	2.7	6420	3040
Berylium	1.87	12890	8880
Brass (70 Cu, 30 Zn)	8.6	4700	2110
Copper, annealed	8.93	4760	2325
Copper, rolled	8.93	5010	2270
Gold, hard-drawn	19.7	3240	1200
Iron, Armco	7.85	5960	3240
Lead, annealed	11.4	2160	700
Lead, rolled	11.4	1960	690
Molybdenum	10.1	6250	3350
Monel metal	8.90	5350	2720
Nickel (unmagnetized)	8.85	5480	2990
Nickel	8.9	6040	3000
Platinum	21.4	3260	1730
Silver	10.4	3650	1610
Steel, mild	7.85	5960	3235
Steel, 347 Stainless	7.9	5790	3100

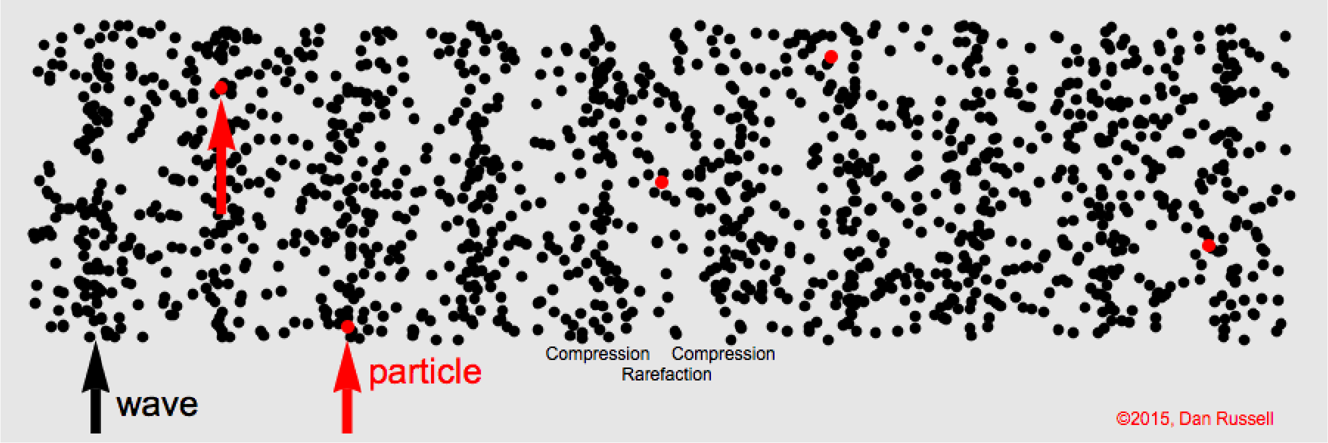


Substance	Density (g/cm^3)	V_l (m/s)	V_s (m/s)
Tin, rolled	7.3	3320	1670
Titaninum	4.5	6070	3125
Tungsten, annealed	19.3	5220	2890
Tungsten Carbide	13.8	6655	3980
Zinc, rolled	7.1	4210	2440
Various			
Fused silica	2.2	5968	3764
Glass, pyrex	2.32	5640	3280
Glass, heavy silicate flint	3.88	3980	2380
Lucite	1.18	2680	1100
Nylon 6-6	1.11	2620	1070
Polyethylene	0.90	1950	540
Polystyrene	1.06	2350	1120

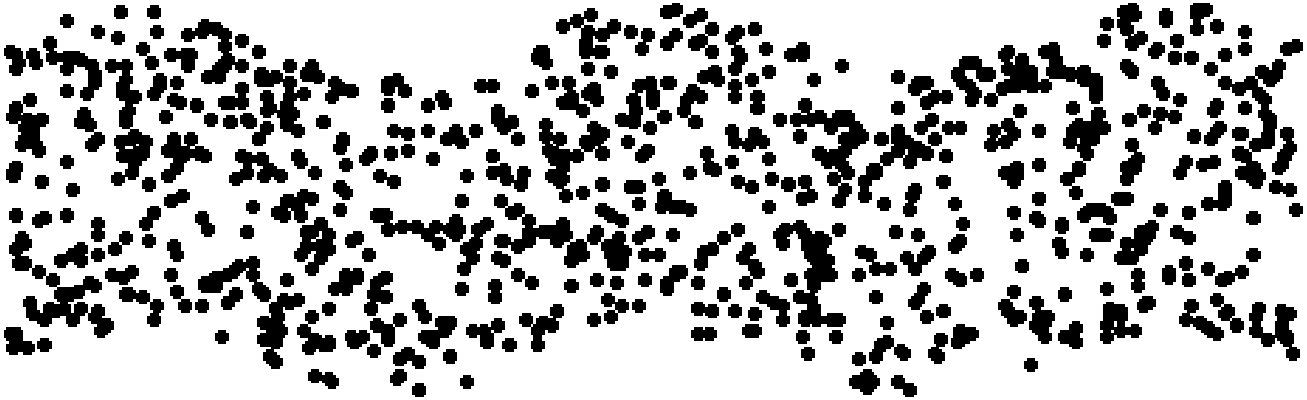
Animations from website:

https://www.acs.psu.edu/drussell/demos/waves/wavemotion.html

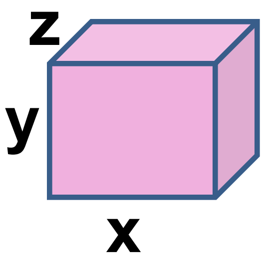
Longitudinal wave (p or "primary")



Transverse wave (s or "secondary")



Elasticity in solids



We imagine that near equilibrium, the solid can be described in terms of a potential function $\phi(\{\mathbf{R}\})$ where $\{\mathbf{R}\}$ represents the positions of each atom:

$$E = \sum_{\mathbf{R}} \phi(\{\mathbf{R}\}) + \frac{1}{2}\sum_{\mathbf{RR}'} (\mathbf{u}(\mathbf{R}) - \mathbf{u}(\mathbf{R}')) \cdot \nabla \phi(\{\mathbf{R}\}) + \frac{1}{4}\sum_{\mathbf{RR}'} ((\mathbf{u}(\mathbf{R}) - \mathbf{u}(\mathbf{R}')) \cdot \nabla)^2 \phi(\{\mathbf{R}\}) + \ldots$$

vanishes at equilibrium

$$\delta W^{\text{elastic}} = \frac{1}{4}\sum_{\substack{\mathbf{RR}' \\ ij}} (u_i(\mathbf{R}) - u_i(\mathbf{R}')) \frac{\partial^2 \phi(\{\mathbf{R}\})}{\partial u_i \partial u_j} (u_j(\mathbf{R}) - u_j(\mathbf{R}'))$$

$$\delta W^{\text{elastic}} = \frac{1}{4}\sum_{\substack{\mathbf{RR}' \\ ij}} \left(u_i(\mathbf{R}) - u_i(\mathbf{R}')\right) \frac{\partial^2 \phi(\{\mathbf{R}\})}{\partial u_i \partial u_j} \left(u_j(\mathbf{R}) - u_j(\mathbf{R}')\right)$$

Note that $\mathbf{u}(\mathbf{R}') \approx \mathbf{u}(\mathbf{R}') + (\mathbf{R}' - \mathbf{R}) \cdot \nabla \mathbf{u}(\mathbf{R})$

In terms of strain coeffients ϵ_{ij} :

$$\delta W^{\text{elastic}} = \frac{1}{2}\sum_{ijkl} \epsilon_{ij} c_{ijkl} \epsilon_{kl}$$

where coeffients c_{ijkl} are composed of permutations of $R_i \dfrac{\partial^2 \phi}{\partial u_j \partial u_k} R_l$

For the most general case c_{ijkl} have 21 distinct terms, for a cube there are only 3 unique terms.

Simplified notation:

$xx \rightarrow 1$

$yy \rightarrow 2$

$zz \rightarrow 3$

$yz \rightarrow 4$

$zx \rightarrow 5$

$xy \rightarrow 6$

For cubic crystals, the unique coefficients are:

$C_{11} = c_{xxxx}$ $\quad C_{12} = c_{xxyy}$ $\quad C_{44} = c_{yzyz}$

Some typical values (Ref. Ashcroft and Mermin (1976))

	C_{11} (GPa)	C_{12}(GPa)	C_{44}(GPa)
Na	7.0	6.1	4.5
Al	107	61	28
Fe	234	136	118
Si	166	64	80
NaCl	48.7	12.4	12.6