# PHY 711 Classical Mechanics and Mathematical Methods
# 10-10:50 AM MWF in Olin 103

## Notes on Lecture 39

### Review of topics covered in this course

1. Basic mathematical tools
2. Mathematical methods
3. Physical concepts

| | | | | |
|---|---|---|---|---|
| **35** | Fri, 11/15/2024 | Chap. 12 | Viscous effects in hydrodynamics | #28 |
| **36** | Mon, 11/18/2024 | Chap. 12 | Viscous effects in hydrodynamics | #29 |
| **37** | Wed, 11/20/2024 | Chap. 13 | Elasticity | #30 |
| **38** | Fri, 11/22/2024 | Chap. 1-13 | Review | |
| **39** | Mon, 11/25/2024 | Chap. 1-13 | Review | |
| | Wed, 11/27/2024 | Thanksgiving | | |
| | Fri, 11/29/2024 | Thanksgiving | | |
| | Mon, 12/02/2024 | | Presentations 1 | |
| | Wed, 12/04/2024 | | Presentations 2 | |
| **40** | Fri, 12/06/2024 | Chap. 1-13 | Review | |

**PHY 711 Presentation Schedule**
**Fall 2024**

**Monday 12/02/2024**

| | Presenter Name | Topic |
|---|---|---|
| **10:00-10:24** | Thomas Myers | Velocity Dependent forces in lagrangian mechanics |
| **10:26-10:50** | Bhargava Ramachandrappa | Lambert W function and the Damped Harmonic Oscillator Problem |

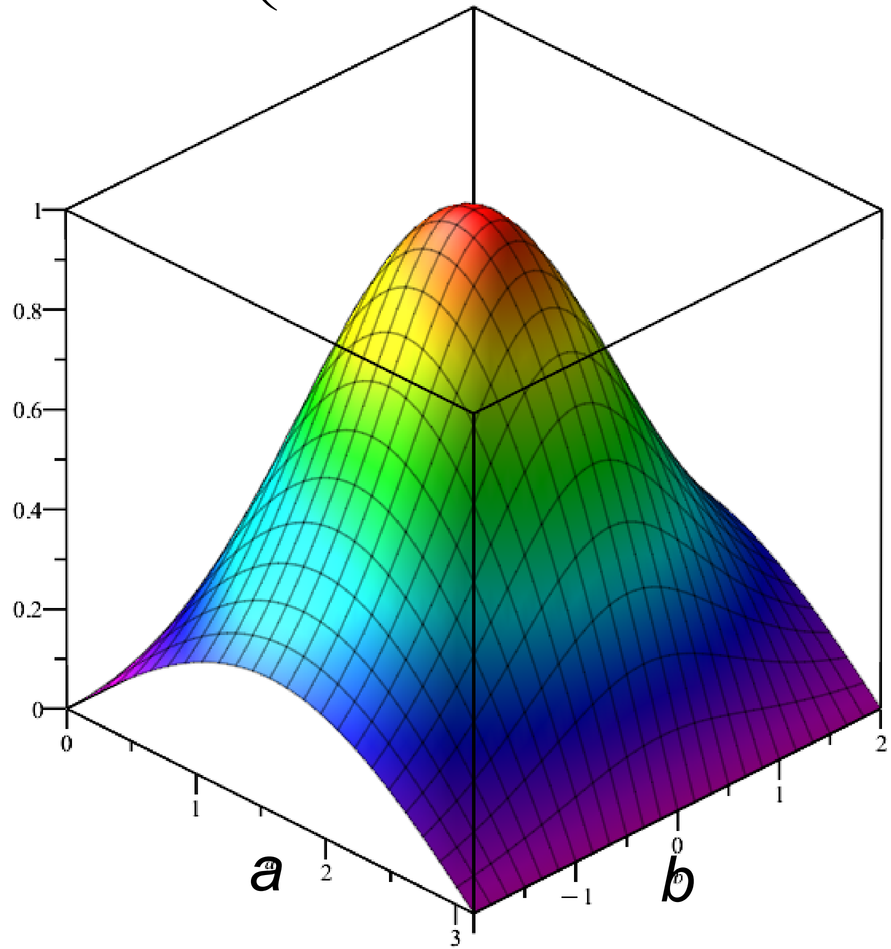
**Wednesday 12/04/2024**

| | Presenter Name | Topic |
|---|---|---|
| **10:00-10:24** | Julia Radtke | |
| **10:26-10:50** | Conall O'Leary | |

Some comments on basic functional dependencies and partial and total derivatives --

-- consider $f(a,b)$

$$\frac{\partial f}{\partial a} \equiv \lim_{da \to 0}\left(\frac{f(a+da,b)-f(a,b)}{da}\right) \equiv \left.\frac{\partial f}{\partial a}\right|_b$$




$$df = \left(\frac{\partial f}{\partial a}\right)_b da + \left(\frac{\partial f}{\partial b}\right)_a db$$

Note that in the case, *a* and *b* are independent variables.

Now, consider the case where $a$ and $b$ are not independent, but $b \equiv b(a)$.

$\Rightarrow f = f(a, b(a))$.

Partial derivatives are still defined as before:

$$\frac{\partial f}{\partial a} \equiv \lim_{da \to 0} \left( \frac{f(a+da,b) - f(a,b)}{da} \right) \equiv \left. \frac{\partial f}{\partial a} \right|_b$$

$$\frac{\partial f}{\partial b} \equiv \lim_{db \to 0} \left( \frac{f(a,b+db) - f(a,b)}{db} \right) \equiv \left. \frac{\partial f}{\partial b} \right|_a$$

$$\frac{\partial^2 f}{\partial a \partial b} = \frac{\partial^2 f}{\partial b \partial a}$$

We often have the situation where $a,\ b,$ and $f$ all depend on another variable, $t$. $\Rightarrow f = f(a(t), b(t); t).$

Now, the total time derivative is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial a}\frac{da}{dt} + \frac{\partial f}{\partial b}\frac{db}{dt} + \frac{\partial f}{\partial t} \qquad \text{Here } \frac{da}{dt} = \frac{\partial a}{\partial t}$$

because $a(t)$ implies $t$ dependence only.

This becomes more complicated when $b(t) \equiv \frac{da}{dt}(t) \equiv \dot{a}(t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial a}\frac{da}{dt} + \frac{\partial f}{\partial \dot{a}}\frac{d\dot{a}}{dt} + \frac{\partial f}{\partial t}$$

Compare $\frac{\partial}{\partial a}\left(\frac{df}{dt}\right)$ and $\frac{d}{dt}\left(\frac{\partial f}{\partial a}\right)$

Compare $\frac{\partial}{\partial \dot{a}}\left(\frac{df}{dt}\right)$ and $\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{a}}\right)$

# Changing functional dependencies – Legendre transformations

Suppose we have a function $f(x,y)$

$$f(x,y) \;\Rightarrow\; df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

Define: $u \equiv \left(\frac{\partial f}{\partial x}\right)_y$ and $v \equiv \left(\frac{\partial f}{\partial y}\right)_x$ $\Rightarrow df = udx + vdy$

Now suppose we want to construct a related function $g(u,y)$

$$g(u,y) \;\Rightarrow\; dg = \left(\frac{\partial g}{\partial u}\right)_y du + \left(\frac{\partial g}{\partial y}\right)_u dy$$

Legendre suggests we try : $g(u,y) = f(x,y) - ux$

$$dg = \left(\frac{\partial g}{\partial u}\right)_y du + \left(\frac{\partial g}{\partial y}\right)_u dy = df - udx - xdu = -xdu + vdy$$

# Changing functional dependencies – Legendre transformations

Summary -- $f(x,y) \Rightarrow df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$

Define: $u \equiv \left(\frac{\partial f}{\partial x}\right)_y$ and $v \equiv \left(\frac{\partial f}{\partial y}\right)_x \Rightarrow df = udx + vdy$

Related function (thanks to Legendre) : $g(u,y) = f(x,y) - ux$

$$dg = \left(\frac{\partial g}{\partial u}\right)_y du + \left(\frac{\partial g}{\partial y}\right)_u dy = df - udx - xdu = -xdu + vdy$$

$$\Rightarrow -x = \left(\frac{\partial g}{\partial u}\right)_y \qquad v = \left(\frac{\partial g}{\partial y}\right)_u = \left(\frac{\partial f}{\partial y}\right)$$

$$\Rightarrow g(u,y)$$

# Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function : $L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$
2. Compute generalized momenta : $p_\sigma \equiv \dfrac{\partial L}{\partial \dot{q}_\sigma}$
3. Construct Hamiltonian expression : $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$
4. Form Hamiltonian function : $H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$
5. Analyze canonical equations of motion :

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

Note that the equations of motion should yield equivalent trajectories for the Lagrangian and Hamiltonian formulations.

# Review of mathematical methods

## Some useful identities for vectors and vector operators

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\nabla \times \nabla \psi = 0$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

$$\nabla \cdot (\psi \mathbf{a}) = \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a}$$

$$\nabla \times (\psi \mathbf{a}) = \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

## Vector relations for spherical polar coordinates

$$\nabla\psi = \hat{\mathbf{r}}\frac{\partial\psi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right]$$
$$+ \hat{\boldsymbol{\theta}}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right)\right] + \hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]$$

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$$

$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$$

$$\hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$$

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \cos\theta\cos\phi\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \cos\theta\sin\phi\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{\partial}{\partial\theta}$$

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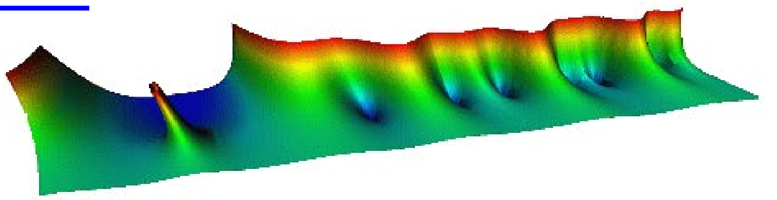


# NIST Digital Library of Mathematical Functions

*Project News*

2018-09-15 *DLMF Update; Version 1.0.20*
2018-06-22 *DLMF Update; Version 1.0.19*
2018-06-22 *Philip J. Davis, A&S Author, dies at age 95*
2018-03-27 *DLMF Update; Version 1.0.18*
*More news*

# Example – special functions

## 10 Bessel Functions
## Bessel and Hankel Functions

10.1 Special Notation 10.3 Graphics

## §10.2 Definitions

***Contents***



### §10.2(i) Bessel's Equation

10.2.1
$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+(z^2-\nu^2)w=0.$$

This differential equation has a regular singularity at $z=0$ with indices $\pm\nu$, and an irregular singularity at $z=\infty$ of rank 1; compare §§2.7(i) and 2.7(ii).

### §10.2(ii) Standard Solutions

#### Bessel Function of the First Kind

10.2.2
$$J_\nu(z)=(\tfrac{1}{2}z)^\nu\sum_{k=0}^{\infty}(-1)^k\frac{(\frac{1}{4}z^2)^k}{k!\,\Gamma(\nu+k+1)}.$$

This solution of (10.2.1) is an analytic function of $z\in\mathbb{C}$, except for a branch point at $z=0$ when $\nu$ is not an integer. The *principal branch* of $J_\nu(z)$ corresponds to the principal value of $(\frac{1}{2}z)^\nu$ (§4.2(iv)) and is analytic in the $z$-plane cut along the interval $(-\infty,0]$.

## Complex numbers

$$i \equiv \sqrt{-1} \qquad i^2 = -1$$

Define $z = x + iy$

$$|z|^2 = zz^* = (x+iy)(x-iy) = x^2 + y^2$$

Polar representation

$$z = \rho(\cos\phi + i\sin\phi) = \rho\, e^{i\phi}$$

## Functions of complex variables

$$f(z) = \Re(f(z)) + i\Im(f(z)) \equiv u(x,y) + iv(x,y)$$

## Derivatives: Cauchy-Riemann equations

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(z)}{\partial x} + i\frac{\partial v(z)}{\partial x} \qquad \frac{\partial f(z)}{i\partial y} = \frac{\partial u(z)}{i\partial y} + i\frac{\partial v(z)}{i\partial y} = \frac{\partial v(z)}{\partial y} - i\frac{\partial u(z)}{\partial y}$$

Argue that $\frac{df}{dz} = \frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{i\partial y} \quad \Rightarrow \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y}$ and $\frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y}$

# Analytic function

$f(z)$ is analytic if it is:

- ∘ continuous
- ∘ single valued
- ∘ its first derivative satisfies Cauchy-Rieman conditions

➔A closed integral of an analytic function is zero.

However:

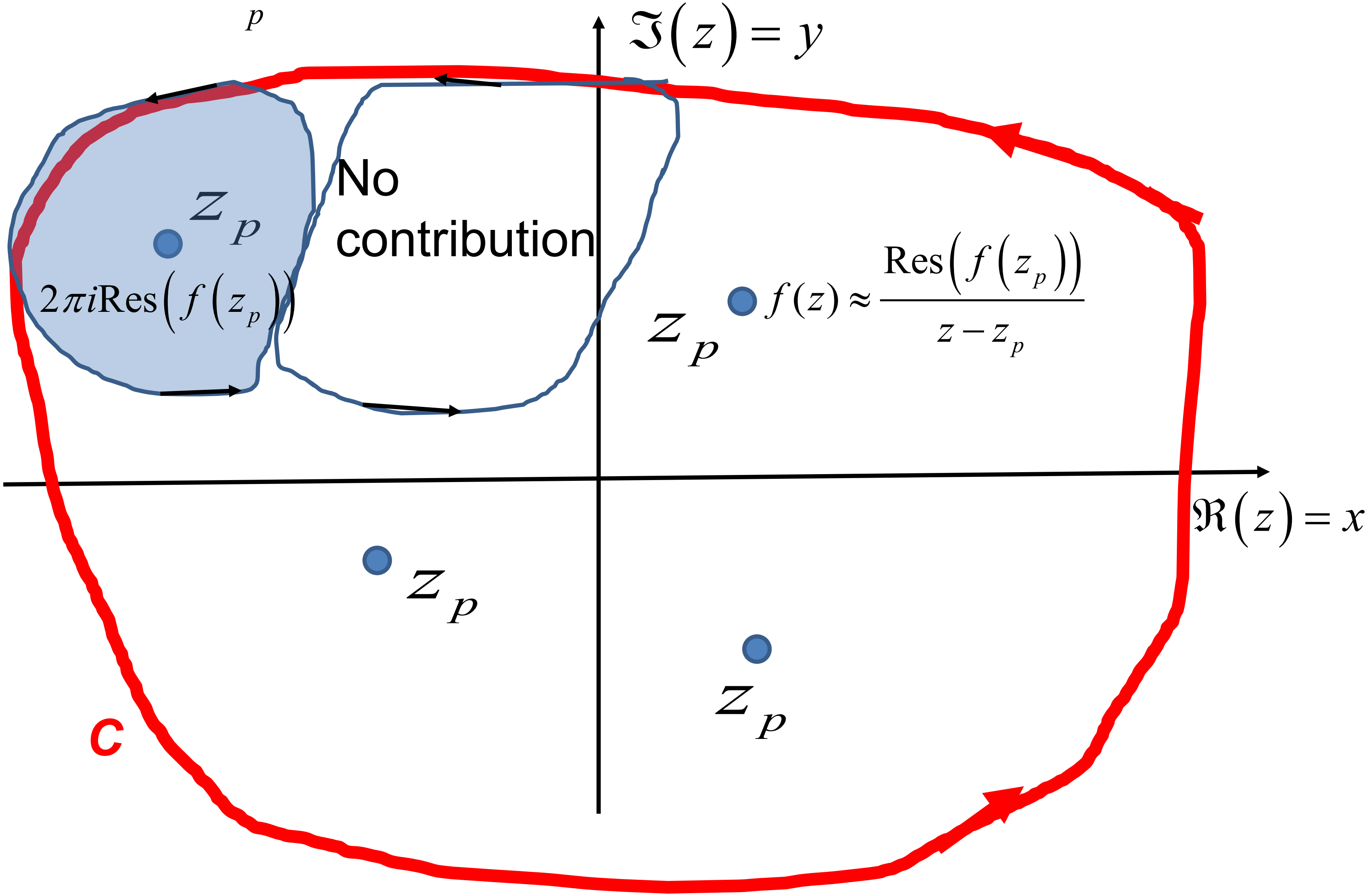
Behavior of $f(z) = \frac{1}{z^n}$ about the point $z = 0$:

For an integer $n$, consider

$$\oint \frac{1}{z^n} dz = \int_0^{2\pi} \frac{\rho e^{i\phi} i \, d\phi}{\rho^n e^{in\phi}} = \rho^{1-n} \int_0^{2\pi} e^{i(1-n)\phi} i \, d\phi = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

# Contour integration methods --

$$\oint_C f(z)dz = 2\pi i \sum_p \text{Res}\left(f(z_p)\right)$$



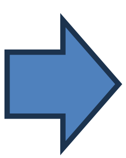
General formula for determining residue:

Suppose that in the neighborhood of $z_p$, $f(z) \approx \frac{g(z)}{\left(z-z_p\right)^m} \underset{z\to z_p}{\equiv} \frac{\text{Res}\left(f(z_p)\right)}{z-z_p}$

Since $g(z)$ is analytic near $z_p$, we can make a Taylor exansion about $z_p$ :

$$g(z) \approx g(z_p) + \left(z-z_p\right)\frac{dg(z_p)}{dz} + ... + \frac{\left(z-z_p\right)^{m-1}}{(m-1)!}\frac{d^{m-1}g(z_p)}{dz^{m-1}} + ....$$

$$\Rightarrow \text{Res}\left(f(z_p)\right) = \lim_{z\to z_p}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}\left(\left(z-z_p\right)^m f(z)\right)}{dz^{m-1}}\right\}$$



$$\oint_C f(z)dz = 2\pi i \sum_p \text{Res}\left(f(z_p)\right)$$

# Fourier transforms --

Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega\, F(\omega)\, e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, f(t)\, e^{i\omega t}$$

Check:

$$f(t) = \int_{-\infty}^{\infty} d\omega \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} dt'\, f(t')\, e^{i\omega t'} \right) e^{-i\omega t}$$

$$f(t) = \int_{-\infty}^{\infty} dt'\, f(t') \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\;\, e^{i\omega(t'-t)} \right) = \int_{-\infty}^{\infty} dt'\, f(t')\, \delta(t'-t)$$

Note: The location of the 2π factor varies among texts.

Properties of Fourier transforms -- Parseval's theorem:

$$\int_{-\infty}^{\infty} dt \left( f(t) \right)^* f(t) = 2\pi \int_{-\infty}^{\infty} d\omega \left( F(\omega) \right)^* F(\omega)$$

Check: $$\int_{-\infty}^{\infty} dt \left( f(t) \right)^* f(t) = \int_{-\infty}^{\infty} dt \left( \left( \int_{-\infty}^{\infty} d\omega F(\omega) e^{i\omega t} \right)^* \int_{-\infty}^{\infty} d\omega' F(\omega') e^{i\omega' t} \right)$$

$$= \int_{-\infty}^{\infty} d\omega F^*(\omega) \int_{-\infty}^{\infty} d\omega' F(\omega') \int_{-\infty}^{\infty} dt\ e^{i(\omega' - \omega)t}$$

$$= \int_{-\infty}^{\infty} d\omega F^*(\omega) \int_{-\infty}^{\infty} d\omega' F(\omega')\, 2\pi\delta(\omega' - \omega)$$

$$= 2\pi \int_{-\infty}^{\infty} d\omega F^*(\omega) F(\omega)$$

# Doubly discrete Fourier Transforms

## Doubly periodic functions

$$\omega \rightarrow \frac{2\pi\nu}{T} \qquad t \rightarrow \frac{\mu T}{2N+1} \qquad (N,\ \nu,\ \text{and}\ \mu\ \text{integers})$$

$$\tilde{f}_\mu = \frac{1}{2N+1}\sum_{\nu=-N}^{N} \tilde{F}_\nu e^{-i2\pi\nu\mu/(2N+1)}$$

$$\tilde{F}_\nu = \sum_{\mu=-N}^{N} \tilde{f}_\mu e^{i2\pi\nu\mu/(2N+1)}$$

➔Fast Fourier Transforms (FFT)

Notions of eigenvalues and eigenvectors

In the context of linear algebra --

Eigenvalue properties of matrices $\qquad \mathbf{M}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$

Hermitian matrix: $\quad \mathbf{H}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha \qquad H_{ij} = H^*_{ji}$

Theorem for Hermitian matrices:

$$\lambda_\alpha \text{ have real values and } \mathbf{y}_\alpha^H \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$$

Unitary matrix: $\quad \mathbf{U}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha \qquad \mathbf{U}^H\mathbf{U} = \mathbf{I}$

$$|\lambda_\alpha| = 1 \text{ and } \mathbf{y}_\alpha^H \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$$

In the context of Sturm-Liouville differential equations --

# Notions of eigenvalues and eigenvectors -- continued

Sturm Liouville differential equations, in terms of given functions $\tau(x)$, $v(x)$, and $\sigma(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Orthogonality of eigenfunctions: $\int_a^b \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n$,

where $N_n \equiv \int_a^b \sigma(x)(f_n(x))^2 dx.$

Calculus of variation – a method to find a function (*y(x))* which optimizes a particular integral relationship.

$$\text{For} \quad f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right),$$

$$\text{a necessary condition to extremize} \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx :$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)_{x,y}\right] = 0$$

Euler-Lagrange equation

# Lagrangian in the presence of electromagnetic forces

Lagrangian: (using cartesian coordinates)

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U_{mech} - U_{EM}$$

$$T = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad U_{EM} = q\Phi(\mathbf{r}, t) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

where $\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \qquad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U_{mech} - q\Phi(\mathbf{r}, t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

# Digression on tools for solving ordinary differential equations – Method of Frobenius

https://mathshistory.st-andrews.ac.uk/Biographies/Frobenius/

Ferdinand Georg Frobenius

**Born:** 26 October 1849
Berlin-Charlottenburg, Prussia (now Germany)

**Died:** 3 August 1917
Berlin, Germany

**Summary:** Georg Frobenius combined results from the theory of algebraic equations, geometry, and number theory, which led him to the study of abstract groups, the representation theory of groups and the character theory of groups. He also developed methods for solving linear differential equations.

Simple example of ordinary differential equation:

Solutions of the differential equation: $\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right)f(r)=0$

Frobenius method for finding solutions near $r=0$:

Guess series solution form: $f(r)=\sum_{n=0} A_n r^{s+n}$

Evaluate: $\hat{O}f(r)=\sum_{n=0} A_n \hat{O} r^{s+n}=0$ for each power of $r^{s+m}$ to find

relationships between coefficients $A_m$

and the condition for non-trivial $A_0$.

Example (thanks to F. B. Hildebrand):

$$\hat{O}=2r\frac{d^2}{dr^2}+(1-2r)\frac{d}{dr}-1$$

$$\sum_{n=0} A_n \hat{O} r^{s+n}=0=\sum_{n=0} A_n\left((s+n)(2s+2n-1)r^{s+n-1}-(2s+2n+1)r^{s+n}\right)$$

Condition for non-trivial $A_0$: $s(2s-1)=0$

# Continued --

Example (thanks to F. B. Hildebrand):

$$\hat{O} = 2r\frac{d^2}{dr^2} + (1-2r)\frac{d}{dr} - 1$$

$$\sum_{n=0} A_n \hat{O} r^{s+n} = 0 = \sum_{n=0} A_n \left( (s+n)(2s+2n-1) r^{s+n-1} - (2s+2n+1) r^{s+n} \right)$$

Condition for non-trivial $A_0$ : $\quad s(2s-1) = 0$

First solution: $\quad s = 0$

Coefficient of $r^m$ : $\quad A_{m+1}(2m+1)(m+1) - A_m(2m+1) = 0$

$$f_1(r) = A_0\left(1 + r + \frac{r^2}{2} + \frac{r^3}{3!} + ...\right) = A_0 e^r$$

Second solution: $\quad s = \frac{1}{2}$

Coefficient of $r^m$ : $\quad A_{m+1}(2m+3)(m+1) - A_m 2(m+1) = 0$

$$f_2(r) = A_0 r^{1/2}\left(1 + \frac{2}{3}r + \frac{2^2}{3\cdot 5}r^2 + \frac{2^3}{3\cdot 5\cdot 7}r^3 ....\right)$$

(infinite series, converges slowly)

Simple example of ordinary differential equation:

Solutions of the differential equation: $$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right)f(r)=0$$

We can use the Frobenius method for this example; in this case the series truncates.

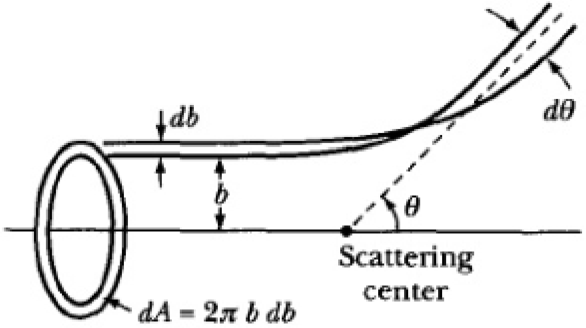
Mechanics topics

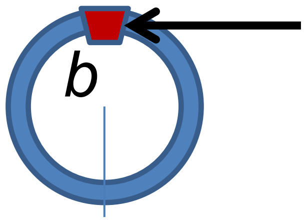
- Scattering theory
- Non-inertial reference frames
- Lagrangian mechanics
- Hamiltonian mechanics
- Liouville theorem
- Rigid body motion
- Normal modes of oscillation about equilibrium
- Wave motion
- Fluid mechanics (ideal or including viscosity; linear and nonlinear)
- Heat conduction
- Elasticity

Note: The following review slides are necessarily brief. Please refer to the original lecture slides for details. Please email: natalie@wfu.edu with any corrections/suggestions

Scattering theory

**Note:** The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.




$d\varphi b db$

Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in $\phi$

# Lagrangian mechanics

Given the Lagrangian function: $L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) \equiv T - U,$

The physical trajectories of the generalized coordinates $\{q_\sigma(t)\}$

Are those which minimize the action: $S = \int L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt$

Euler-Lagrange equations:

$$\sum_\sigma \left( \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right)\delta q_\sigma = 0 \qquad \Rightarrow \text{for each } \sigma: \quad \left( \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$

For the case that there both mechanical and electromagnetic contributions in terms of electric and magnetic fields:

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} \qquad \mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$$

$$L = T - U_{\text{mech}} - q\Phi(\mathbf{r},t) + \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

Example of solving coupled equations

Lagrangian equations of motion for a Lorentz force

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0 \left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\left( m\dot{x} - \frac{q}{2c} B_0 y \right) - \frac{q}{2c} B_0 \dot{y} = 0 \qquad \Rightarrow m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0$$

$$\frac{d}{dt}\left( m\dot{y} + \frac{q}{2c} B_0 x \right) + \frac{q}{2c} B_0 \dot{x} = 0 \qquad \Rightarrow m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \qquad \Rightarrow m\ddot{z} = 0$$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$m\ddot{x} = +\frac{q}{c} B_0 \dot{y}$$

$$m\ddot{y} = -\frac{q}{c} B_0 \dot{x}$$

$$m\ddot{z} = 0$$

Need to find $z(t), x(t), y(t)$.

In this case, the initial conditions are

$$z(0) = 0, x(0) = 0, y(0) = 0 \qquad \dot{z}(0) = 0, \dot{x}(0) = U_0, \dot{y}(0) = 0$$

# Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function: $L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$
2. Compute generalized momenta: $p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$
3. Construct Hamiltonian expression: $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$
4. Form Hamiltonian function: $H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$
5. Analyze canonical equations of motion:

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

Important tool for analyzing Lagrangian and/or Hamiltonian systems -- finding constants of the motion

In Lagrangian formulation --

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right) \qquad \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Note that if $\frac{\partial L}{\partial q_\sigma} = 0$, then $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} = 0 \qquad \Rightarrow \frac{\partial L}{\partial \dot{q}_\sigma} =$ (constant)

Additionally: $$\frac{d}{dt}\left( L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma \right) = \frac{\partial L}{\partial t}$$

For $\frac{\partial L}{\partial t} = 0 \qquad \Rightarrow L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma = -E$ (constant)

# Constants of the motion in the Hamiltonian formulation

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \Rightarrow \text{constant } \; q_\sigma \; \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \qquad \Rightarrow \text{constant } \; p_\sigma \; \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left( \frac{\partial H}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

$$\frac{dH}{dt} = \sum_\sigma \left( -\dot{p}_\sigma \dot{q}_\sigma + \dot{q}_\sigma \dot{p}_\sigma \right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

$$\Rightarrow \text{ constant } \; H \; \text{ if } \; \frac{\partial H}{\partial t} = 0$$

Question – Why use this fancy formalism when simple conservation of energy or momentum intuitively apply?

a. You should use your intuition whenever possible.
b. You should never trust your intuition.
c. The equations should be consistent with correct intuitive solutions and also reveal additional solutions (perhaps beyond intuition)

# Liouville's Theorem (1838)

The density of representative points in phase space corresponding to the motion of a system of particles remains constant during the motion.

Denote the density of particles in phase space: $D = D\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$

$$\frac{dD}{dt} = \sum_\sigma \left( \frac{\partial D}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial D}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial D}{\partial t}$$

According to Liouville's theorem: $\frac{dD}{dt} = 0$

# Rigid body motion

Moment of inertia tensor :

$$\overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left( \mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right) \qquad \text{(dyad notation)}$$

In a reference frame attached to the object, there are 3 moments of inertia and 3 distinct principal axes

Representation of rotational kinetic energy:

$$T\left(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}\right) = \frac{1}{2} I_1 \tilde{\omega}_1^2 + \frac{1}{2} I_2 \tilde{\omega}_2^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2$$

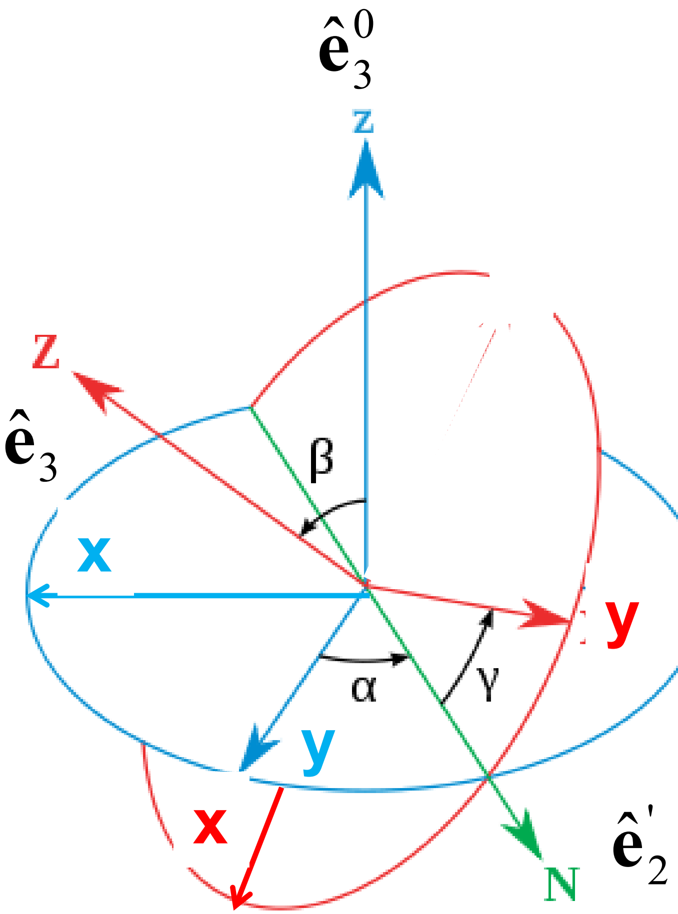
$$= \frac{1}{2} I_1 \left[ \dot{\alpha}\left(-\sin\beta\cos\gamma\right) + \dot{\beta}\sin\gamma \right]^2$$

$$+ \frac{1}{2} I_2 \left[ \dot{\alpha}\left(\sin\beta\sin\gamma\right) + \dot{\beta}\cos\gamma \right]^2$$

$$+ \frac{1}{2} I_3 \left[ \dot{\alpha}\cos\beta + \dot{\gamma} \right]^2$$

# Euler's transformation between body fixed and inertial reference frames

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

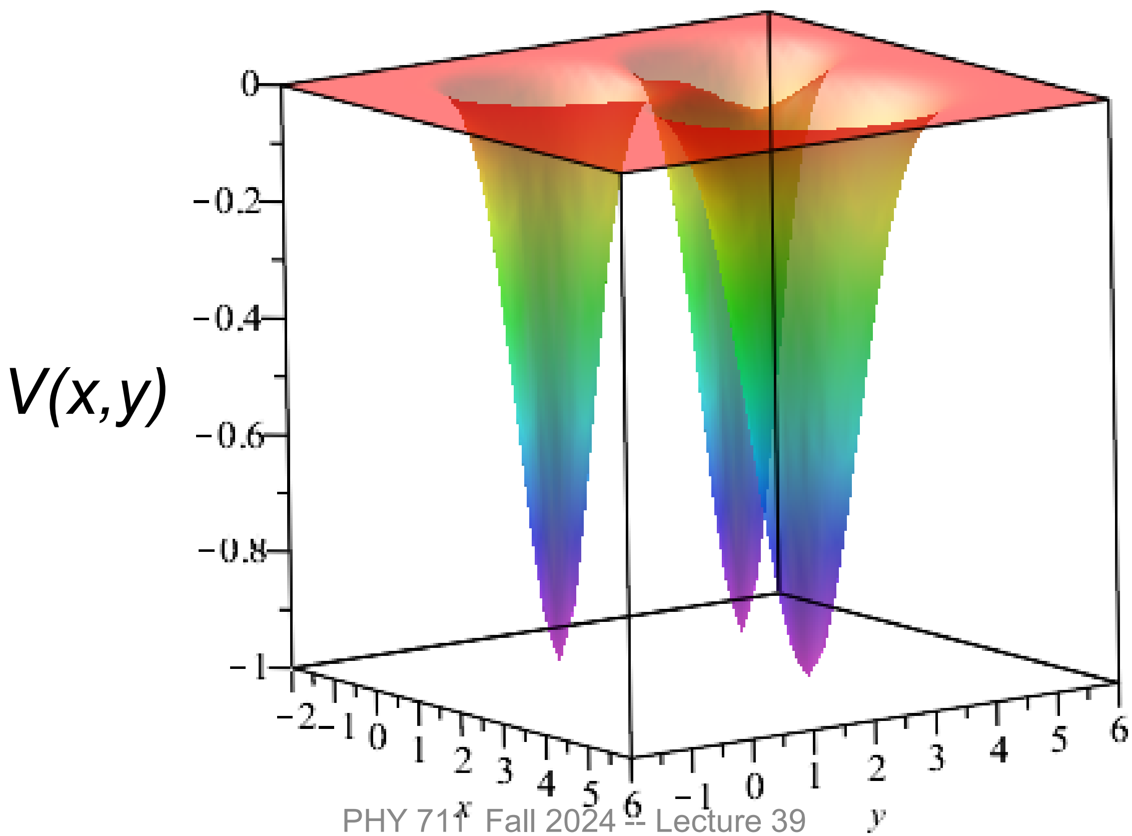



$$\begin{aligned}\tilde{\boldsymbol{\omega}} &= \left[\dot{\alpha}\left(-\sin\beta\cos\gamma\right)+\dot{\beta}\sin\gamma\right]\hat{\mathbf{e}}_1 \\ &+\left[\dot{\alpha}\left(\sin\beta\sin\gamma\right)+\dot{\beta}\cos\gamma\right]\hat{\mathbf{e}}_2 \\ &+\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]\hat{\mathbf{e}}_3\end{aligned}$$

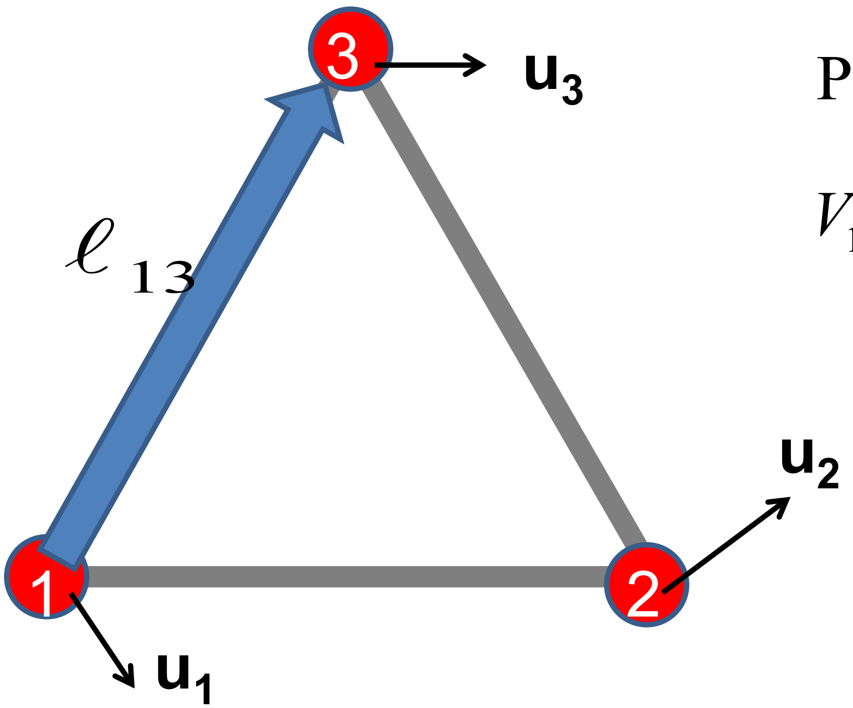
# Normal modes of vibration -- potential in 2 and more dimensions

$$V(x,y) \approx V(x_{eq}, y_{eq}) + \tfrac{1}{2}\left(x - x_{eq}\right)^2 \left.\frac{\partial^2 V}{\partial x^2}\right|_{x_{eq}, y_{eq}}$$

$$+ \tfrac{1}{2}\left(y - y_{eq}\right)^2 \left.\frac{\partial^2 V}{\partial y^2}\right|_{x_{eq}, y_{eq}} + \left(x - x_{eq}\right)\left(y - y_{eq}\right) \left.\frac{\partial^2 V}{\partial x \partial y}\right|_{x_{eq}, y_{eq}}$$



# Example – normal modes of a system with the symmetry of an equilateral triangle -- continued




$$\boldsymbol{\ell}_{13} = |\boldsymbol{\ell}_{13}|\left(\frac{1}{2}\hat{\mathbf{x}} + \frac{\sqrt{3}}{2}\hat{\mathbf{y}}\right)$$

Potential contribution for spring 13:

$$V_{13} = \frac{1}{2}k\left(|\boldsymbol{\ell}_{13} + \mathbf{u}_3 - \mathbf{u}_1| - |\boldsymbol{\ell}_{13}|\right)^2$$

$$\approx \frac{1}{2}k\left(\frac{\boldsymbol{\ell}_{13}\cdot(\mathbf{u}_3 - \mathbf{u}_1)}{|\boldsymbol{\ell}_{13}|}\right)^2$$

$$\approx \frac{1}{2}k\left(\frac{1}{2}(u_{x3} - u_{x1}) + \frac{\sqrt{3}}{2}(u_{y3} - u_{y1})\right)^2$$

# Example – normal modes of a system with the symmetry of an equilateral triangle -- continued

Potential contributions: $V = V_{12} + V_{13} + V_{23}$

$$\approx \frac{1}{2}k\left(\frac{\ell_{12}\cdot(\mathbf{u}_2-\mathbf{u}_1)}{|\ell_{12}|}\right)^2 + \frac{1}{2}k\left(\frac{\ell_{13}\cdot(\mathbf{u}_3-\mathbf{u}_1)}{|\ell_{13}|}\right)^2$$

$$+\frac{1}{2}k\left(\frac{\ell_{23}\cdot(\mathbf{u}_3-\mathbf{u}_2)}{|\ell_{23}|}\right)^2$$

$$\approx \frac{1}{2}k\left(u_{x2}-u_{x1}\right)^2$$

$$+\frac{1}{2}k\left(\frac{1}{2}(u_{x3}-u_{x1})+\frac{\sqrt{3}}{2}(u_{y3}-u_{y1})\right)^2$$

$$+\frac{1}{2}k\left(\frac{1}{2}(u_{x2}-u_{x3})-\frac{\sqrt{3}}{2}(u_{y2}-u_{y3})\right)^2$$

# Example – normal modes of a system with the symmetry of an equilateral triangle -- continued

$$\frac{k}{m}\begin{bmatrix} \frac{5}{4} & -1 & -\frac{1}{4} & \frac{1}{4}\sqrt{3} & 0 & -\frac{1}{4}\sqrt{3} \\ -1 & \frac{5}{4} & -\frac{1}{4} & 0 & -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} & 0 \\ \frac{1}{4}\sqrt{3} & 0 & -\frac{1}{4}\sqrt{3} & \frac{3}{4} & 0 & -\frac{3}{4} \\ 0 & -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} & 0 & \frac{3}{4} & -\frac{3}{4} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} & 0 & -\frac{3}{4} & -\frac{3}{4} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \\ u_{y1} \\ u_{y2} \\ u_{y3} \end{bmatrix} = \omega^2 \begin{bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \\ u_{y1} \\ u_{y2} \\ u_{y3} \end{bmatrix}$$

## Discrete particle interactions → continuous media → The wave equation

Initial value solutions $\mu(x,t)$ to the wave equation; attributed to D'Alembert:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = \varphi(x) \text{ and } \frac{\partial \mu}{\partial t}(x,0) = \psi(x)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\big(\varphi(x-ct) + \varphi(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

Mechanical motion of fluids

Newton's equations for fluids

Use Euler formulation; following "particles" of fluid

$$\text{Variables:} \quad \text{Density} \quad \rho(x,y,z,t)$$

$$\text{Pressure} \quad p(x,y,z,t)$$

$$\text{Velocity} \quad \mathbf{v}(x,y,z,t)$$

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v} \cdot \nabla\right)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \underbrace{\frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla\left(\nabla \cdot \mathbf{v}\right)}_{\text{Viscosity contributions}}$$

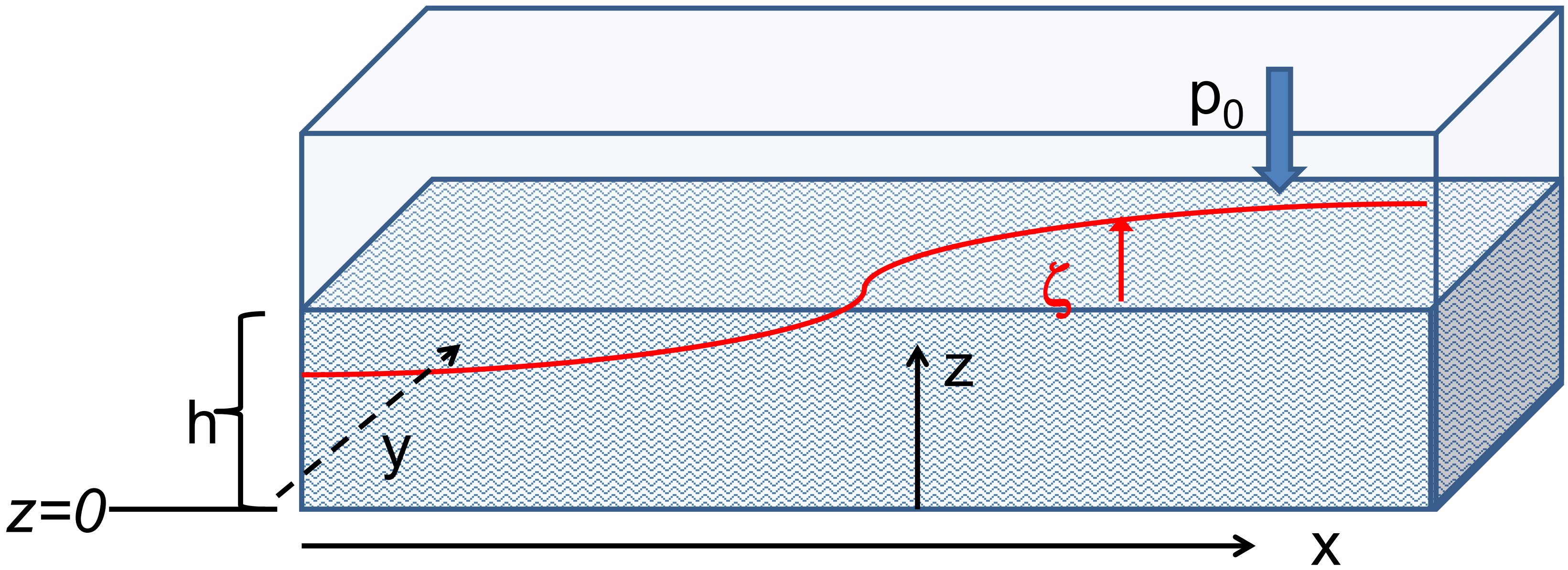
Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v}\right) = 0$$

Viscosity contributions

# Fluid mechanics of incompressible fluid plus surface

## Non-linear effects in surface waves:




Dominant non-linear effects $\Rightarrow$ soliton solutions

$$\zeta(x,t) = \eta_0 \operatorname{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\frac{x-ct}{2h}\right) \qquad \eta_0 = \text{constant}$$

$$\text{where } c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1+\frac{\eta_0}{2h}\right)$$