PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin103

Lecture notes for Lecture 3
Chapter 3.17 of F&W

More about the calculus of variations

1. Review examples – Area of lamp shade
2. Brachistochrone problem
3. Calculus of variation with constraints

Your questions

From **Thomas** –

On slide 21 is the time for the blue path infinite? That doesn't seem right to me. On 30 there is the integral constraint, how can one plug the integral constraint into y(x) to help find the constant lambda, K and a?

From **Connal**

On slide 26 there is the mathematical trick. How can we ensure that these infinitesimal changes (delta) in the composite function are zero. Is this due to a fixed Lambda?

From **Julia**

how do I know when the alternate Euler-Lagrange equation is easier than the regular Euler-Lagrange equation? Are there specific examples for when one would be much better than the other?

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/26/2024		Introduction and overview	#1
2	Wed, 8/28/2024	Chap. 3(17)	Calculus of variation	#2
3	Fri, 8/30/2024	Chap. 3(17)	Calculus of variation	#3
4	Mon, 9/02/2024	Chap. 3	Lagrangian equations of motion	
5	Wed, 9/04/2024	Chap. 3 & 6	Lagrangian equations of motion	
6	Fri, 9/06/2024	Chap. 3 & 6	Lagrangian equations of motion	

Note that Monday is not a holiday for us…

PHY 711 – Assignment #3

Assigned: 08/30/2024 Due: 09/02/2024

This exercise is designed to illustrate the differences between partial and total derivatives.

1. Consider an arbitrary function of the form $f = f(q, \dot{q}, t)$, where it is assumed that $q = q(t)$ and $\dot{q} \equiv dq/dt$.

   (a) Write a formal expression for $\frac{df}{dt}$ in terms of an arbitrary form of $f = f(q, \dot{q}, t)$ and an arbitrary function $q(t)$.

   (b) Now suppose that

   $$f(q, \dot{q}, t) = q\dot{q}^2 t, \quad \text{where} \quad q(t) = e^{-t/\tau}.$$

   Here τ is a constant. Evaluate df/dt as a function of t using the expression you derived in part (a)..

   (c) Now find the expression for f as an explicit function of t ($f(t)$) and then take its time derivative directly to check your previous result.

Summary of the method of calculus of variation --

Consider a family of functions $y(x)$, with the end points $y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral function

$$L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \int_{x_i}^{x_f} f\left(y(x), \frac{dy}{dx}; x\right) dx.$$

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

$\delta L = 0 \quad \Rightarrow$ Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Example: Find minimum curve between points -- $y(0)=0; y(1)=1$

$$L=\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$

Solution:

$$\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=K \qquad \frac{dy}{dx}=K'\equiv\frac{K}{\sqrt{1-K^2}}$$

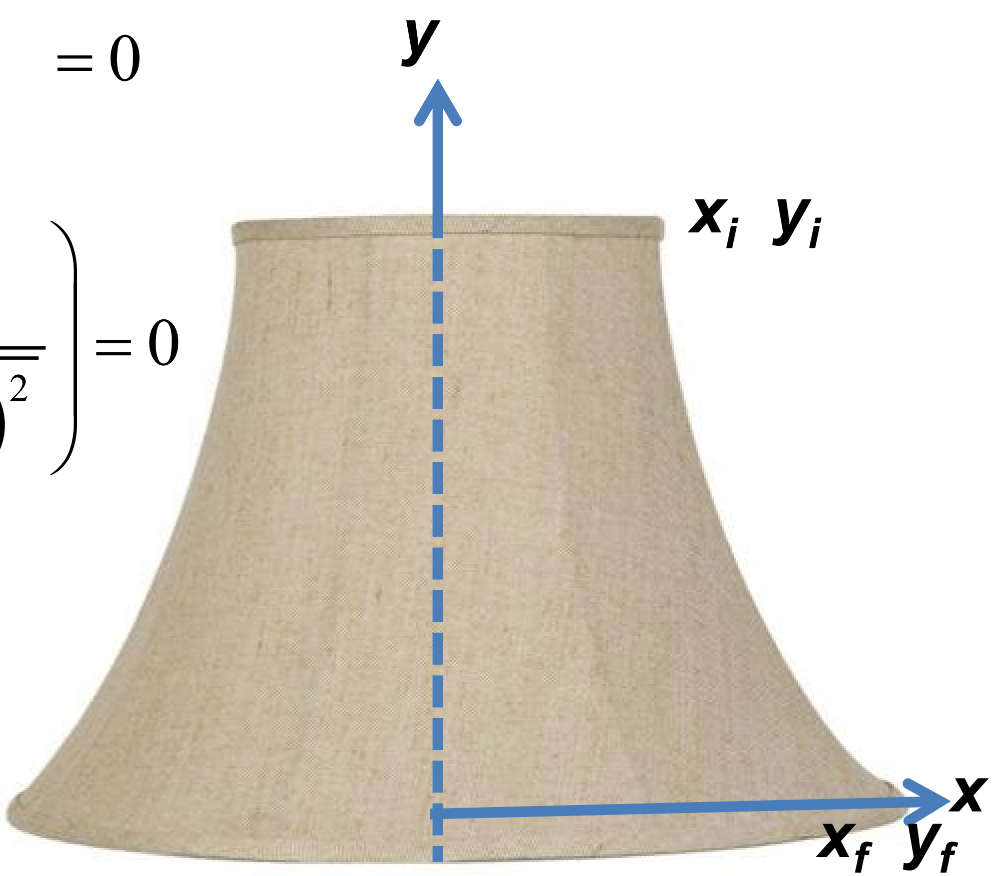
$$\Rightarrow y(x)=K'x+C \qquad y(x)=x$$

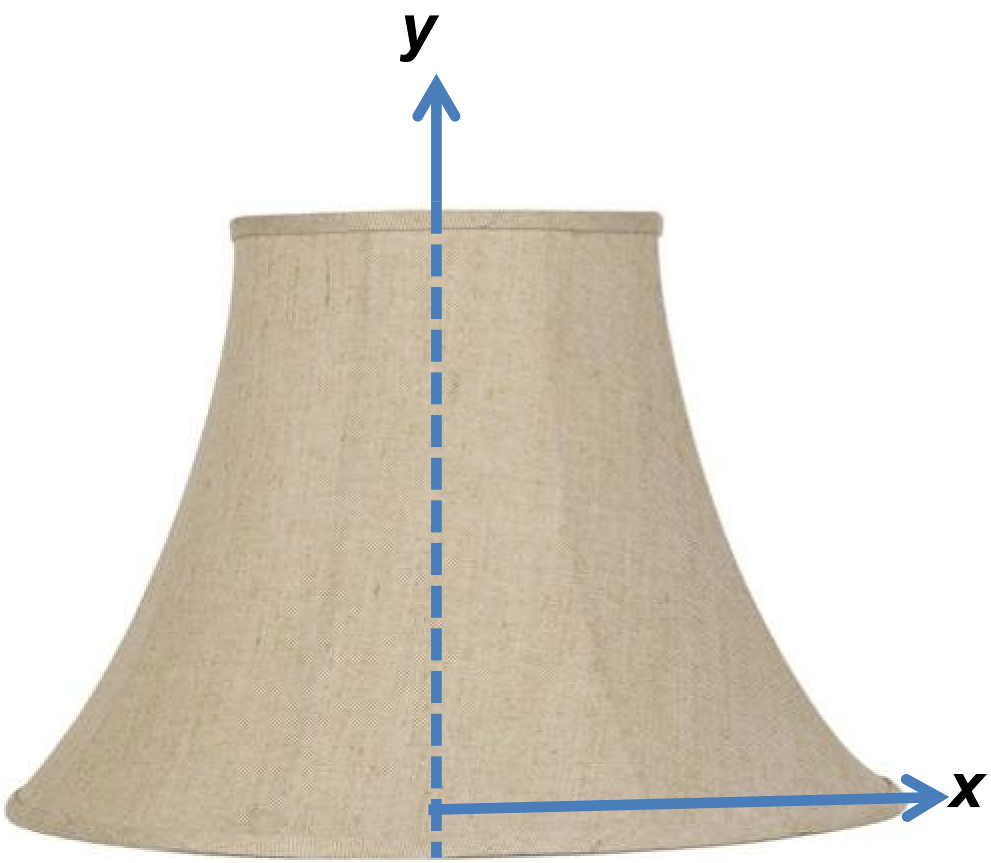
Another example: Lamp shade shape $y(x)$

$$A = 2\pi \int_{x_i}^{x_f} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) = x\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

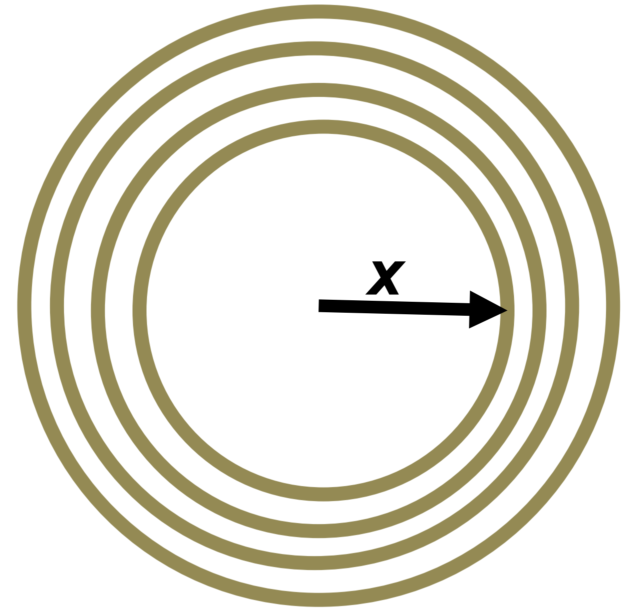
$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}\right) = 0$$





Top view



$$A = 2\pi \int_{x_i y_i}^{x_f y_f} x\sqrt{(dx)^2 + (dy)^2}$$

$$= 2\pi \int_{x_i}^{x_f} x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Lamp shade area

$2\pi x dL$ where $dL = \sqrt{(dx)^2 + (dy)^2}$

$$-\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$

$$\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}=K_1$$

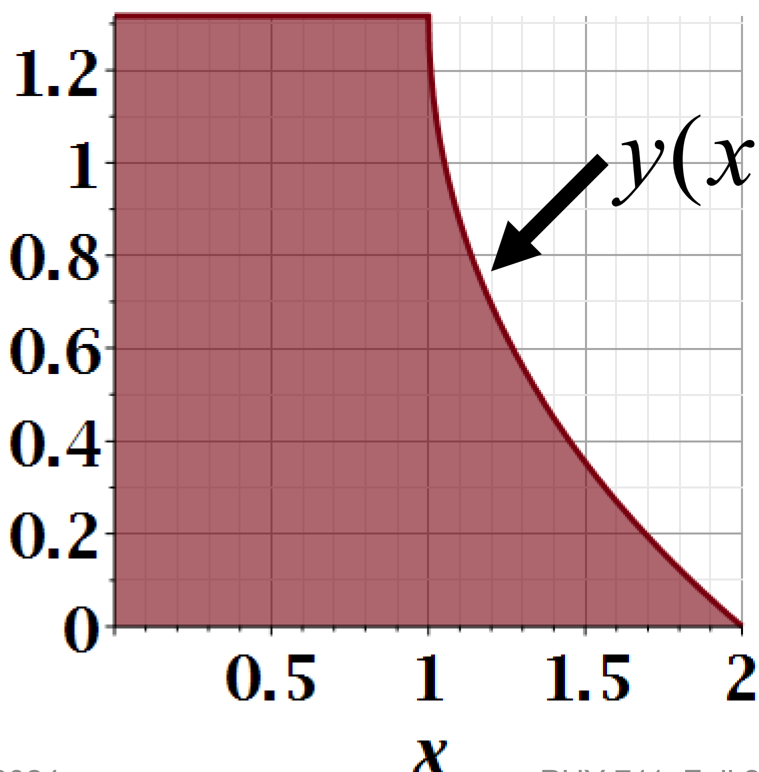
$$\frac{dy}{dx}=-\frac{1}{\sqrt{\left(\frac{x}{K_1}\right)^2-1}}$$

$$\Rightarrow y(x)=K_2-K_1\ln\left(\frac{x}{K_1}+\sqrt{\frac{x^2}{K_1^2}-1}\right)$$

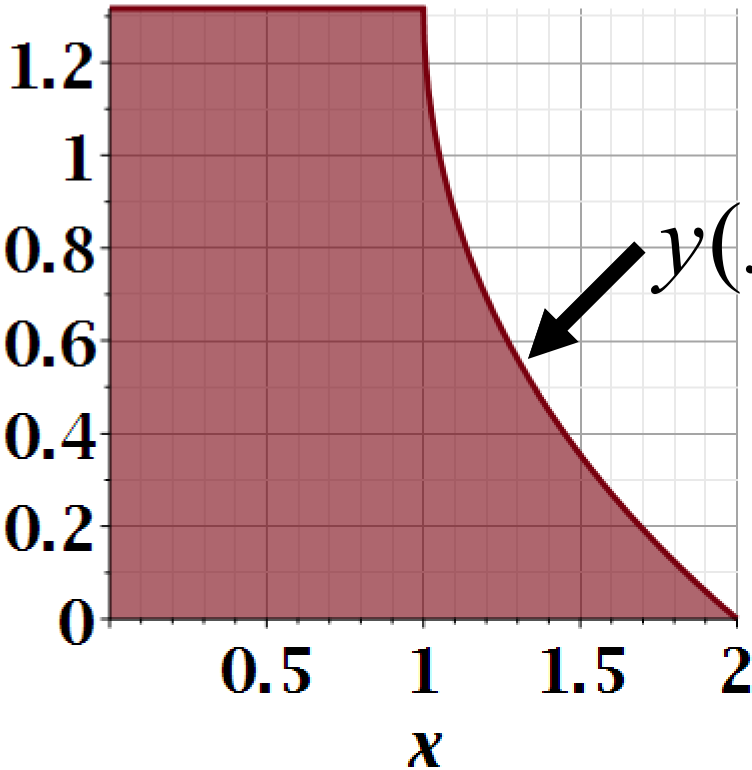
General form of solution --

$$y(x) = K_2 - K_1 \ln\left(\frac{x}{K_1} + \sqrt{\frac{x^2}{K_1^2} - 1} \right)$$

Suppose $K_1 = 1$ and $K_2 = \ln\left(2+\sqrt{3}\right)$



$$y(x) = \ln\left(\frac{2+\sqrt{3}}{x + \sqrt{x^2 - 1}} \right)$$



$$y(x) = \ln\left(\frac{2+\sqrt{3}}{x+\sqrt{x^2-1}}\right)$$

$$A = 2\pi\int_1^2 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 15.02014144$$

(according to Maple)

Review : for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

Euler-Lagrange equation

Note that for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Alternate Euler-Lagrange equation

A few more steps --

$$\text{Note that for} \quad f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right),$$

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

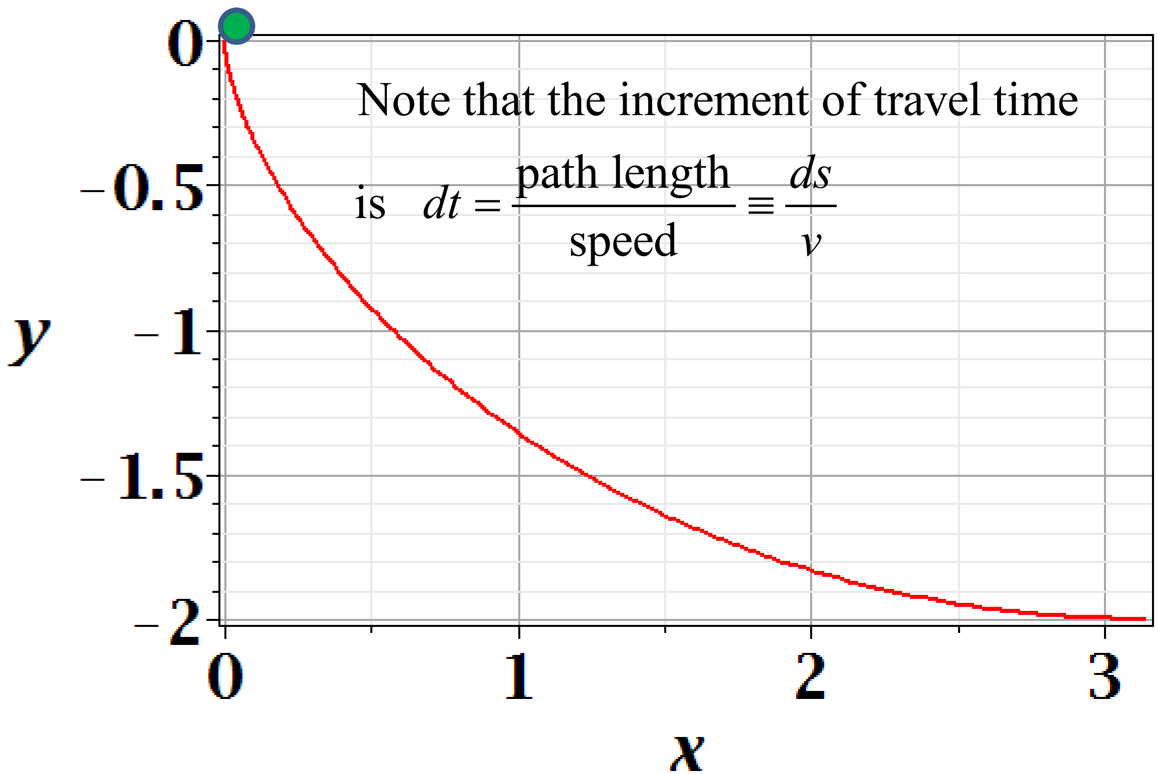
$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial\left(dy/dx\right)}\frac{dy}{dx}\right)\right) + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial\left(dy/dx\right)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Brachistochrone problem: (solved by Newton in 1696)

http://mathworld.wolfram.com/BrachistochroneProblem.html



Note that the increment of travel time is $dt = \frac{\text{path length}}{\text{speed}} \equiv \frac{ds}{v}$

A particle of weight *mg* travels frictionlessly down a path of shape *y(x)*. What is the shape of the path *y(x)* that minimizes the travel time from $y(0)=0$ to $y(\pi)=-2$?

$$E = \frac{1}{2}mv^2 + mgy \qquad \text{with } y(t=0)=0 \text{ and } \dot{y}(t=0)=0$$

With this choice of initial conditions, $E = 0$

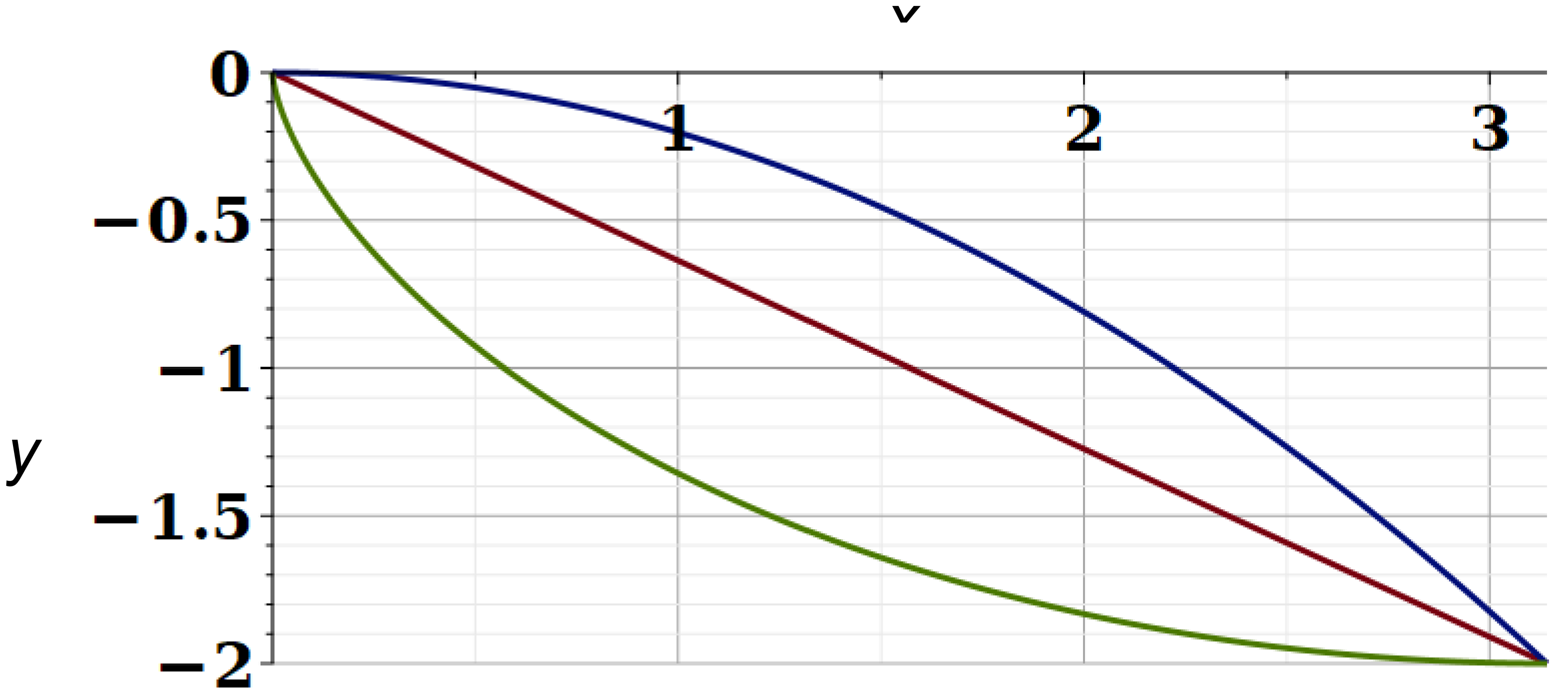
Note that the increment of travel time

is $$dt = \frac{\text{path length}}{\text{speed}} \equiv \frac{ds}{v}$$

Alternatively --

$$v = \frac{ds}{dt} \qquad \Rightarrow dt = \frac{ds}{v}$$

Vote for your favorite path



Which gives the shortest time?

a. Green
b. Red
c. Blue

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}}dx \quad \text{because} \quad \tfrac{1}{2}mv^2 = -mgy$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = 0$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

Note that for the original form of Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0,$$

differential equation is more complicated:

$$-\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y^3}} - \frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

$$f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=\left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right)=0 \qquad -y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

Question – why this choice? Answer – because the answer will be more beautiful. (Be sure that was not my cleverness.)

$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

$$\frac{dy}{dx}=-\sqrt{\frac{2a}{-y}-1}$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=dx$$

Let $y=-2a\sin^2\frac{\theta}{2}=a(\cos\theta-1)$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=\frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta}{\sqrt{\frac{2a}{2a\sin^2\frac{\theta}{2}}-1}}=dx$$

$$x=\int_0^\theta a(1-\cos\theta')d\theta'=a(\theta-\sin\theta)$$

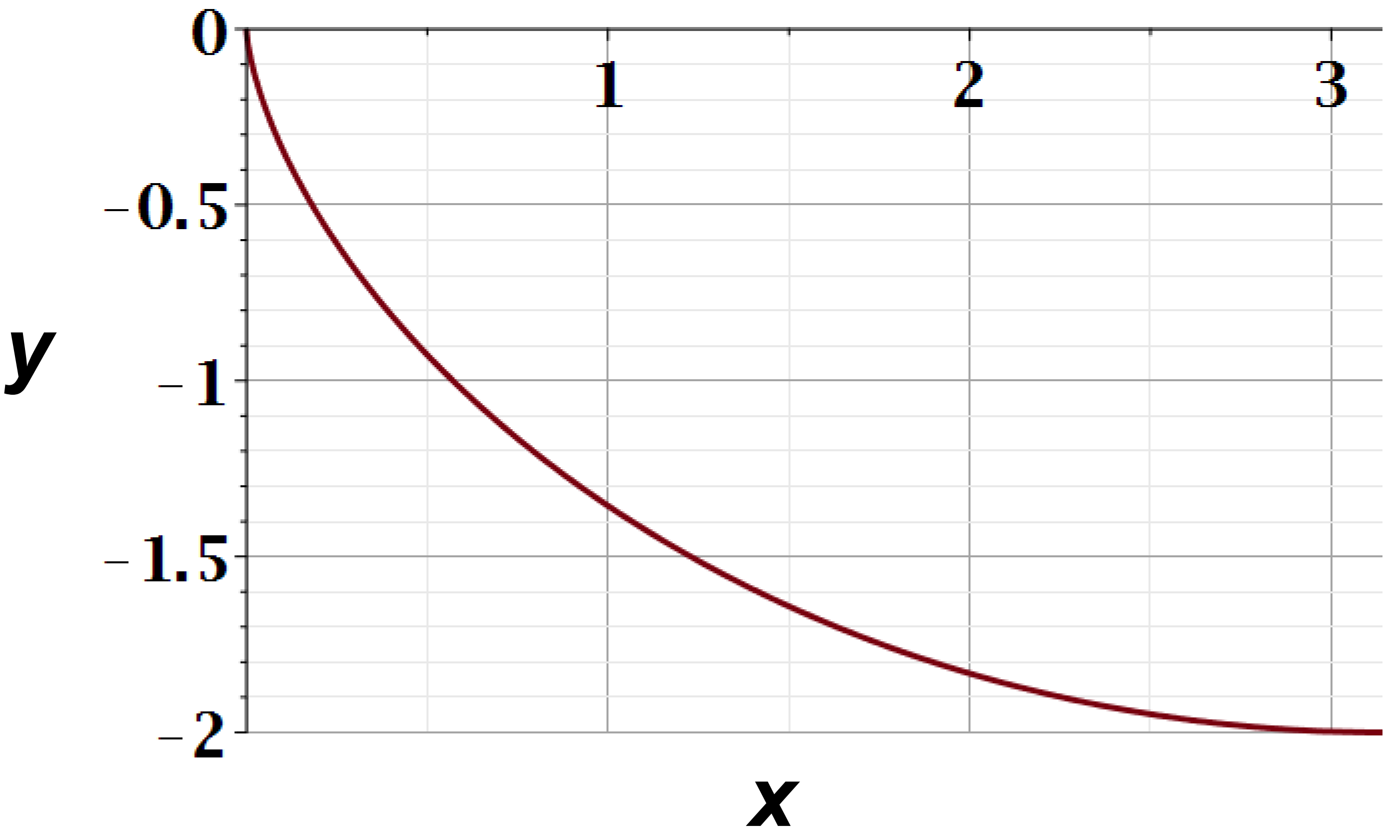
Parametric equations for Brachistochrone:

$$x=a(\theta-\sin\theta)$$

$$y=a(\cos\theta-1)$$

Parametric plot --

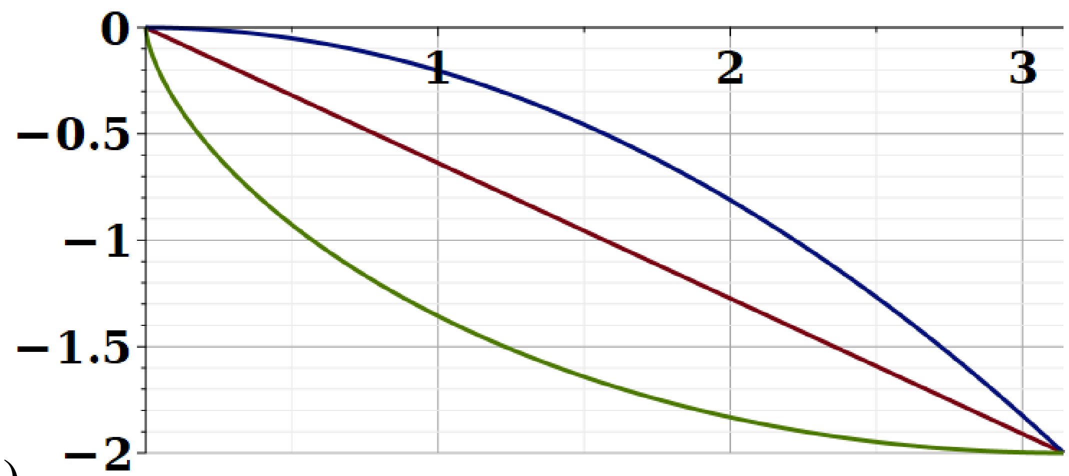
plot([theta-sin(theta), cos(theta)-1, theta = 0 .. Pi])



Checking the results

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}}\,dx$$

units of $\frac{1}{\sqrt{(2g)}}$; $(0,0) \rightarrow (\pi,-2)$



T=4.4429 $x = \theta - \sin\theta \quad y = \cos\theta - 1$

T=5.2668 $y(x) = -2x/\pi$

T=infinite $y(x) = -2x^2/\pi^2$

Summary of the method of calculus of variation --

Consider a family of functions $y(x)$, with the end points $y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral function

$$I\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\int_{x_i}^{x_f} f\left(y(x),\frac{dy}{dx};x\right)dx.$$

Find the function $y(x)$ which extremizes $I\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$.

$\delta I = 0 \quad \Rightarrow$ Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Euler-Lagrange equation:

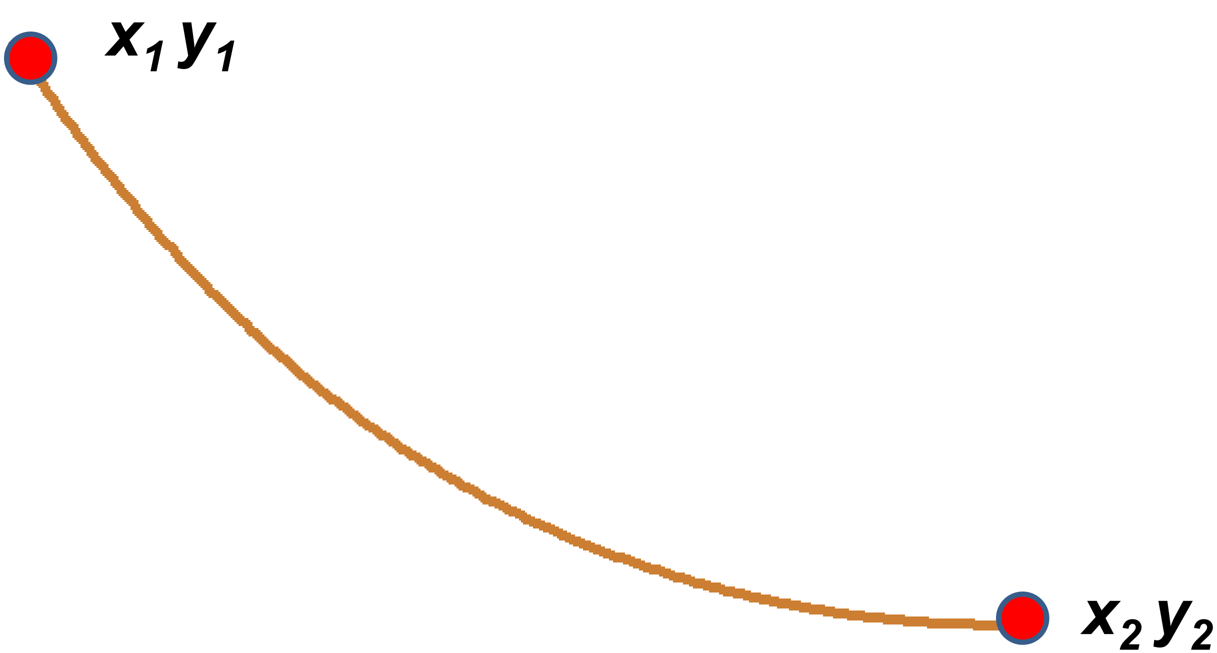
$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

Alternate Euler-Lagrange equation:

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Another example optimization problem:

Determine the shape *y(x)* of a rope of length L and mass density ρ hanging between two points



Example from internet --

Potential energy of hanging rope:

$$E = \rho g \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Length of rope:

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Define a composite function to minimize:

$$W \equiv E + \lambda L$$

Lagrange multiplier

$$\delta W = 0 = \delta E + \lambda \delta L \text{ for fixed } \lambda$$

is a very clever mathematical trick to help solve the minimization and constraint at the same time.

$$W = \int_{x_1}^{x_2} (\rho g y + \lambda)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$f\left(\left\{y, \frac{dy}{dx}\right\}\right) = (\rho g y + \lambda)\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow (\rho g y + \lambda)\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right) = K$$

$$\left(\rho g y+\lambda\right)\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}-\frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)=K$$

$$\left(\rho g y+\lambda\right)\left(\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)=K$$

$$y(x)=\frac{1}{\rho g}\left(-\lambda+K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

$$y(x) = \frac{1}{\rho g}\left(-\lambda + K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

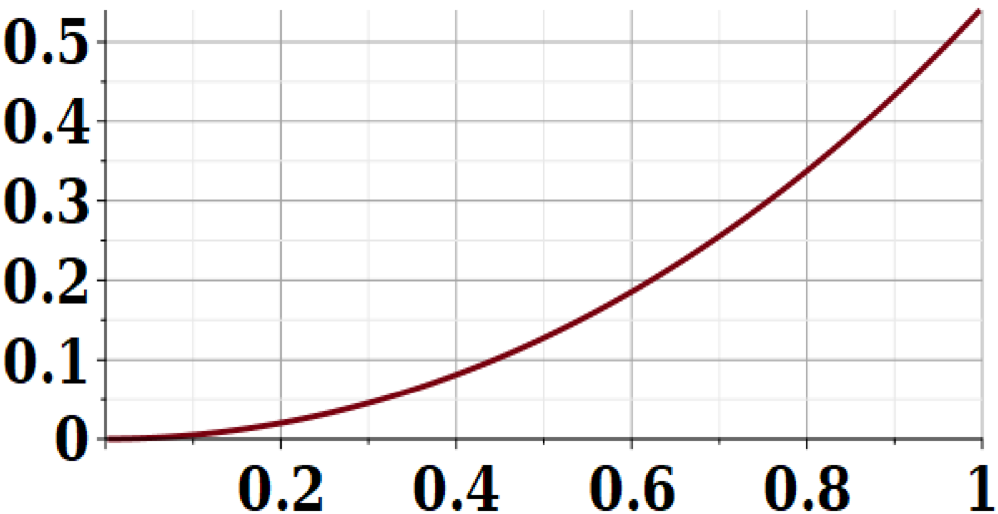
Integration constants: K, a, λ

Constraints: $y(x_1) = y_1$

$$y(x_2) = y_2$$

$$\int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = L$$

$$y(x) = \frac{1}{\rho g}\left(-\lambda + K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$



Summary of results

For the class of problems where we need to perform an extremization on an integral form:

$$I=\int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx \qquad\qquad \delta I=0$$

A necessary condition is the Euler-Lagrange equations:

$$\left(\frac{\partial f}{\partial y}\right)-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)\right]=0$$

or $$\frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=\left(\frac{\partial f}{\partial x}\right)$$