# PHY 711 Classical Mechanics and Mathematical Methods
# 10-10:50 AM MWF in Olin 103

## Notes on Lecture 40

### Review of topics covered in this course

1. Comment on numerical methods
2. Review of the Sturm-Liouville equation
3. Solving some example problems

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| **35** | Fri, 11/15/2024 | Chap. 12 | Viscous effects in hydrodynamics | #28 |
| **36** | Mon, 11/18/2024 | Chap. 12 | Viscous effects in hydrodynamics | #29 |
| **37** | Wed, 11/20/2024 | Chap. 13 | Elasticity | #30 |
| **38** | Fri, 11/22/2024 | Chap. 1-13 | Review | |
| **39** | Mon, 11/25/2024 | Chap. 1-13 | Review | |
| | Wed, 11/27/2024 | Thanksgiving | | |
| | Fri, 11/29/2024 | Thanksgiving | | |
| | Mon, 12/02/2024 | | Presentations 1 | |
| | Wed, 12/04/2024 | | Presentations 2 | |
| **40** | Fri, 12/06/2024 | Chap. 1-13 | Review | |

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Brief comment on numerical methods --

Consider a continuous function $x(t)$

A Taylor expansion in the neighborhood of $t$ :

$$x(t+h) = x(t) + h\frac{dx(t)}{dt} + \frac{1}{2}h^2\frac{d^2x(t)}{dt^2} + \frac{1}{3!}h^3\frac{d^3x(t)}{dt^3} + .....$$

Let $x_n \equiv nh$

$$x_{n+1} = x_n + hv_n + \frac{1}{2}h^2 a_n + .....$$

Here it is assumed that $h$ is small and $h^3 << h^2$

Example differential equation (one dimension);

$$\frac{d^2x}{dt^2} = f(t) \qquad \text{Let} \quad t = nh \qquad (n = 1, 2, 3\ldots)$$

$$x_n \equiv x(nh); \qquad f_n \equiv f(nh)$$

Euler's method:

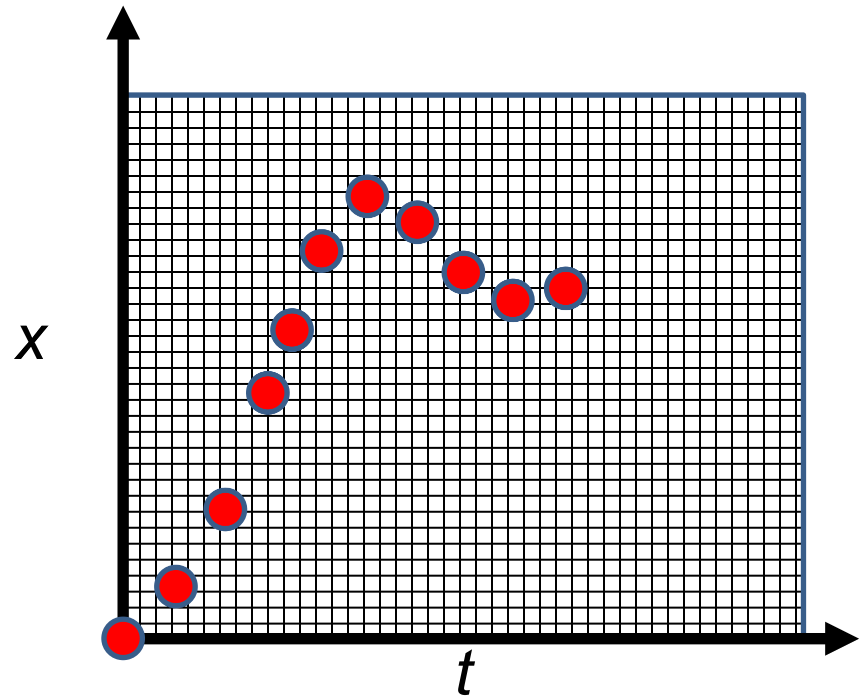
$$x_{n+1} = x_n + hv_n + \frac{1}{2}h^2 f_n$$

$$v_{n+1} = v_n + hf_n$$

Velocity Verlet algorithm:

$$x_{n+1} = x_n + hv_n + \frac{1}{2}h^2 f_n$$

$$v_{n+1} = v_n + \frac{1}{2}h\left(f_n + f_{n+1}\right)$$



Note that it is possible to check the magnitude of the terms that you are neglecting and estimate the error. Also, one needs to be careful of device-dependent restrictions. In general, it is useful to use scaled coordinates.

$$\text{For example,} \quad 1+1\times 10^{-15} = 1.000000000000001$$
$$= 1.0 \ \text{ for most devices.}$$

When you perform numerical work, you need to take care about your algorithms both in terms of software and hardware.

# Review of the Sturm-Liouville equations

Linear second-order ordinary differential equations
Sturm-Liouville equations

Inhomogenous problem: $$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

given functions

applied force

solution to be determined

When applicable, it is assumed that the form of the applied force is known.

Homogenous problem: $F(x)=0$

Why are Sturm-Liouville equations interesting?

A. They sound fancy?
B. They describe all second-order differential equations?
C. Another method of graduate student torture?
D. Several special functions are solutions to Sturm-Liouville equations?
E. They have several interesting properties?

# Examples of Sturm-Liouville eigenvalue equations --

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=0$$

Bessel functions: $0 \leq x \leq \infty$

$$\tau(x)=-x \quad v(x)=x \quad \sigma(x)=\frac{1}{x} \quad \lambda=\nu^2 \quad \varphi(x)=J_\nu(x)$$

Legendre functions: $-1 \leq x \leq 1$

$$\tau(x)=-\left(1-x^2\right) \quad v(x)=0 \quad \sigma(x)=1 \quad \lambda=l(l+1) \quad \varphi(x)=P_l(x)$$

Fourier functions: $0 \leq x \leq 1$

$$\tau(x)=1 \quad v(x)=0 \quad \sigma(x)=1 \quad \lambda=n^2\pi^2 \quad \varphi(x)=\sin(n\pi x)$$

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi_0(x)=0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi(x)=F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Orthogonality of eigenfunctions: $\int_a^b \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n$,

where $N_n\equiv\int_a^b\sigma(x)(f_n(x))^2\,dx.$

Completeness of eigenfunctions:

$$\sigma(x)\sum_n\frac{f_n(x)f_n(x')}{N_n}=\delta(x-x')$$

# Comment on orthogonality of eigenfunctions

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_m(x)=\lambda_m\sigma(x)f_m(x)$$

$$f_m(x)\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)-f_n(x)\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_m(x)$$

$$=\left(\lambda_n-\lambda_m\right)\sigma(x)f_n(x)f_m(x)$$

$$-\frac{d}{dx}\left(f_m(x)\tau(x)\frac{df_n(x)}{dx}-f_n(x)\tau(x)\frac{df_m(x)}{dx}\right)=\left(\lambda_n-\lambda_m\right)\sigma(x)f_n(x)f_m(x)$$

# Comment on orthogonality of eigenfunctions -- continued

$$-\frac{d}{dx}\left( f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right) = \left(\lambda_n - \lambda_m\right)\sigma(x) f_n(x) f_m(x)$$

Now consider integrating both sides of the equation in the interval

$a \leq x \leq b$:

$$-\left( f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right)\Bigg|_a^b = \left(\lambda_n - \lambda_m\right)\int_a^b dx\, \sigma(x) f_n(x) f_m(x)$$

Vanishes for various boundary conditions
at *x=a and x=b*

# Comment on orthogonality of eigenfunctions -- continued

$$-\left( f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right)\Bigg|_a^b = \left(\lambda_n - \lambda_m\right)\int_a^b dx\sigma(x) f_n(x) f_m(x)$$

Possible boundary values for Sturm-Liouville equations:

1. $f_m(a) = f_m(b) = 0$

2. $\tau(x)\frac{df_m(x)}{dx}\Big|_a = \tau(x)\frac{df_m(x)}{dx}\Big|_b = 0$

3. $f_m(a) = f_m(b)$ and $\frac{df_m(a)}{dx} = \frac{df_m(b)}{dx}$

In any of these cases, we can conclude that:

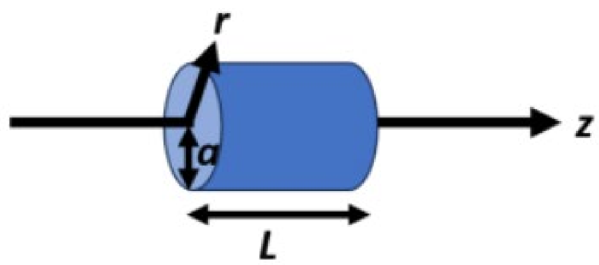
$$\int_a^b dx\sigma(x) f_n(x) f_m(x) = 0 \text{ for } \lambda_n \neq \lambda_m$$

For additional information about Sturm-Liouville equations, see Lectures 21,22,23

**PHY 711 – Homework # 27**

Assigned: 11/13/2024 Due: 11/18/2024

Read Chapter 11 of **Fetter and Walecka**.

1. 


A cylindrical solid material with cylindrical radius $a$ and length $L$ and thermal diffusivity $\kappa$ has a time-dependent cylindrically symmetric temperature profile $T(r, z, t)$. In these cylindrical coordinates, the material is contained within $0 \leq r \leq a$ and $0 \leq z \leq L$. In the absense of external heating, the temperature profile is is well-described by the equation of heat conduction

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T.$$

At $t \leq 0$, the material is prepared so that its temperature profile is given by

$$T(r, z, t \leq 0) = \begin{cases} 0 & \text{for} \quad r > a \quad \text{and/or} \quad z < 0 \quad z > L \\ A\cos(\pi z/L) & \text{for} \quad 0 \leq r \leq a \quad \text{and} \quad 0 \leq z \leq L, \end{cases}$$

where $A$ is a given constant. The cylindrical solid is placed in a thermally insulated container so that its temperature is well-described by the boundary conditions

$$\hat{\mathbf{n}} \cdot \nabla T(r, z, t) = 0$$

at all of its surfaces. Find an expression for the temperature profile of this system $T(r, z, t)$ for $t > 0$.

The diffusion (or heat conduction) equation for the temperature profile $T(\mathbf{r},t)$:

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T$$

For cylindrical coordinates -- $T(\mathbf{r},t) = T\left(r,\varphi,z,t\right)$

and the diffusion equation takes the form:

$$\frac{\partial T\left(r,\varphi,z,t\right)}{\partial t} = \kappa\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right)T\left(r,\varphi,z,t\right)$$

Partial differential equation:

$$\frac{\partial T(r,\varphi,z,t)}{\partial t} = \kappa\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right)T(r,\varphi,z,t)$$

Assume separable form: $T(r,\varphi,z,t) = R(r)\Phi(\varphi)Z(z)f(t)$

In this particular case, the $\varphi$ dependence is trivial, so that it is reasonable to assume that $T(r,\varphi,z,t) = T(r,z,t) = R(r)Z(z)f(t)$

Then $\dfrac{\partial T(r,z,t)}{\partial t} = R(r)Z(z)\dfrac{df(t)}{dt}$

$$\nabla^2 T(r,z,t) = Z(z)f(t)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)R(r) + R(r)f(t)\frac{d^2 Z(z)}{dz^2}$$

Summary:

$$\frac{\partial T(r,z,t)}{\partial t} = \kappa\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) T(r,z,t)$$

$$T(r,z,t) = R(r)Z(z)f(t)$$

Then $\dfrac{\partial T(r,z,t)}{\partial t} = R(r)Z(z)\dfrac{df(t)}{dt}$

$$\nabla^2 T(r,z,t) = Z(z)f(t)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)R(r) + R(r)f(t)\frac{d^2 Z(z)}{dz^2}$$

$$R(r)Z(z)\frac{df(t)}{dt} = \kappa\left(Z(z)f(t)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)R(r) + R(r)f(t)\frac{d^2 Z(z)}{dz^2}\right)$$

Divide both sides of the equation by $R(r)Z(z)f(t)$

$$\frac{1}{f(t)}\frac{df(t)}{dt} = \kappa\left(\frac{1}{R(r)}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)R(r) + \frac{1}{Z(z)}\frac{d^2 Z(z)}{dz^2}\right)$$

$$\frac{1}{f(t)}\frac{df(t)}{dt}=\kappa\left(\frac{1}{R(r)}\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)R(r)+\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}\right)$$

Suppose $\frac{df(t)}{dt}=-\lambda f(t)$ and $\frac{d^2Z(z)}{dz^2}=-\alpha^2Z(z)$

and $\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)R(r)=-\mu^2R(r)$

where $\lambda, \alpha,$ and $\mu$ are real numbers.

This will work if $\lambda=\kappa(\alpha^2+\mu^2)$

Solution of ordinary differential equation for the radial component:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\mu^2\right)R(r)=0$$

Recall that the regular solution of the Bessel equation of order 0 is a solution of the differential equation:

$$\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+1\right)J_0(x)=0$$

Therefore, $R(r)=CJ_0(\mu r)$ where $C$ is a constant

More generally, multiple solutions $\mu_n$ may be viable, in which case the solution has the form $R(r)=\sum_n C_n J_0(\mu_n r)$.

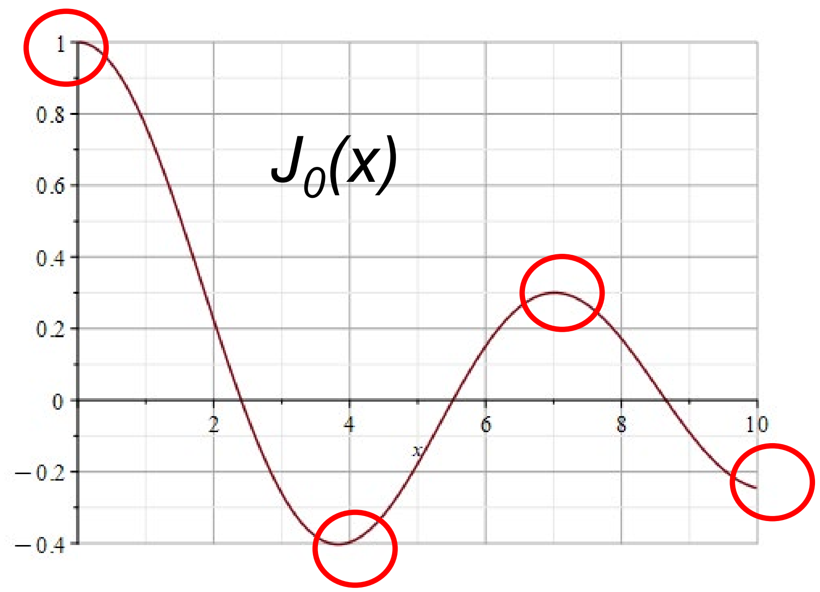
$$R(r) = \sum_n C_n J_0(\mu_n r)$$

Satisfies the radial differential equation, but does not satisfy boundary conditions

Need to find $\mu_n$ and $C_n$.

For boundary value at $r = a$

$$\left.\frac{dJ_0(\mu_n r)}{dr}\right|_{r=a} = 0$$




Define $\quad \frac{dJ_0(x'_n)}{dx} = 0$

$$\mu_n = \frac{x'_n}{a}$$

Note that the functions $J_0(\mu_n r)$ form a set of orthogonal functions over the range $0 \leq r \leq a.$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\mu_n^2\right)J_0(\mu_n r)=0$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\mu_m^2\right)J_0(\mu_m r)=0$$

$$J_0(\mu_m r)\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\right)J_0(\mu_n r)-J_0(\mu_n r)\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\right)J_0(\mu_m r)=$$

$$\left(\mu_m^2-\mu_n^2\right)J_0(\mu_n r)J_0(\mu_m r)$$

If $\mu_n = \mu_m,$ then the equality is trivial. If $\mu_n \neq \mu_m,$ the integrating both sides of the equation $0 \leq r \leq a$ implies that

$$\int_0^a dr\ rJ_0(\mu_n r)J_0(\mu_m r)=0$$

Solution of ordinary differential equation for the $z$ component:

$$\left(\frac{d^2}{dz^2}+\alpha^2\right)Z(z)=0 \text{ with boundary values } \left.\frac{dZ}{dz}\right|_{z=0}=0=\left.\frac{dZ}{dz}\right|_{z=L}$$

$$\Rightarrow Z(z)=\sum_k D_k \cos\left(\frac{k\pi z}{L}\right) \quad \text{where } \alpha_k=\frac{k\pi}{L}$$

In our case, because of the initial conditions, only the $k=1$ solution is present.

Full general solution:

$$T(r,z,t) = \sum_n C_n J_0(\mu_n r)\cos\left(\frac{\pi z}{L}\right) e^{-\lambda_n t}$$

$$\text{where } \lambda_n = \kappa\left(\mu_n^2 + \frac{\pi^2}{L^2}\right) = \kappa\left(\frac{x'^2_n}{a^2} + \frac{\pi^2}{L^2}\right)$$

# Finishing up --

$$T(r,z,t)=\sum_n C_n J_0(\mu_n r)\cos\left(\frac{\pi z}{L}\right)e^{-\lambda_n t}\ \text{ with } \lambda_n=\kappa\left(\mu_n^2+\frac{\pi^2}{L^2}\right)$$

$$\text{At } t=0, \quad 0\le r\le a, \quad 0\le z\le L$$

$$T(r,z,0)=A\cos\left(\frac{\pi z}{L}\right)=\sum_n C_n J_0(\mu_n r)\cos\left(\frac{\pi z}{L}\right)$$

Full general solution:

$$T(r,z,t) = \sum_n C_n J_0(\mu_n r) \cos\left(\frac{\pi z}{L}\right) e^{-\lambda_n t}$$

where $\lambda_n = \kappa\left(\mu_n^2 + \frac{\pi^2}{L^2}\right)$

and where $C_n = A \frac{\int_0^a dr\ r J_0(\mu_n r)}{\int_0^a dr\ r J_0^2(\mu_n r)}$

# Advice about problem solving such as coupled differential equations

A particle of mass $m$ and charge $q$ is subjected to a vector potential $\mathbf{A}(\mathbf{r}, t) = -(E_0 ct + B_0 x)\hat{\mathbf{z}}$. In this case, the scalar potential is zero: $\Phi(\mathbf{r}, t) = 0$. (Note that we are using the cgs Gaussian units of your text book.) Here $E_0$ denotes a constant electric field amplitude and $B_0$ denotes a constant magnetic field amplitude. The initial particle position is $\mathbf{r}(t = 0) = 0$ and the initial particle velocity is $\dot{\mathbf{r}}(t = 0) = 0$.

a. Determine the Lagrangian $L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$ which describes the particle's motion.

b. Write the Euler-Lagrange equations for this system.

c. Find and evaluate the constants of motion for this system.

d. Find the particle trajectories $x(t)$, $y(t)$, $z(t)$ by solving the equations and imposing the given initial conditions.

2. A particle of mass $m$ and charge $q$ moves in three-dimensional space in the presence of a constant electric field of strength $E_0$ and a constant magnetic field of strength $B_0$ in cgs Gaussian units. Initially, $x(t = 0) = y(t = 0) = z(t = 0) = 0$ and $\dot{x}(t = 0) = \dot{y}(t = 0) = \dot{z}(t = 0) = 0$ The Lagrangian for this system in Cartesian coordinates is given by

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + qE_0 z - \frac{q}{c}B_0 x\dot{z},$$

where $c$ denotes the speed of light in vacuum.

(a) Find the Hamiltonian of this system in canonical form.

(b) Using either the Hamiltonian or Lagrangian formalisms, find the equations of motion for this system.

(c) Solve the equations of motion using your knowledge of the initial values.

(d) Comment on whether or not your solution makes physical sense.

1. $L(x,y,z,\dot{x},\dot{y},\dot{z}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - qE_0 ct\dot{z} - \frac{q}{c}B_0 x\dot{z}$

2. $L(x,y,z,\dot{x},\dot{y},\dot{z}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) + qE_0 z - \frac{q}{c}B_0 x\dot{z}$

Initial conditions for both #1 and #2: $\mathbf{r}(t)=0$ and $\dot{\mathbf{r}}(t)=0$

What can you say about these two cases?

Solution:

$$z(t)=\frac{E_0 c}{B_0}\frac{mc}{qB_0}\left(1-\cos\left(\frac{qB_0}{mc}t\right)\right) \qquad x(t)=\frac{E_0 c}{B_0}\left(\frac{mc}{qB_0}\sin\left(\frac{qB_0}{mc}t\right)-t\right)$$

Do these solutions satisfy the equations?
Do these solutions satisfy the initial conditions?