PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion of Lecture 6 -- Chap. 3 & 6 (F &W)

Details and extensions of Lagrangian mechanics

1. More about constraints and Lagrange multipliers.
2. Constants of the motion
3. Conserved quantities
4. Legendre transformations

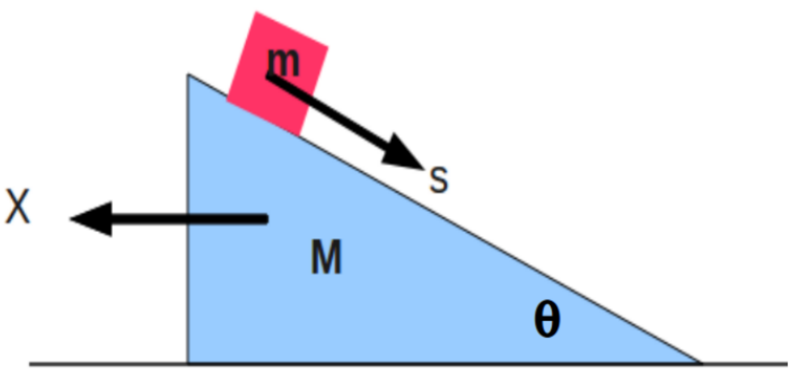
Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/26/2024		Introduction and overview	#1
2	Wed, 8/28/2024	Chap. 3(17)	Calculus of variation	#2
3	Fri, 8/30/2024	Chap. 3(17)	Calculus of variation	#3
4	Mon, 9/02/2024	Chap. 3	Lagrangian equations of motion	#4
5	Wed, 9/04/2024	Chap. 3 & 6	Lagrangian equations of motion	#5
6	Fri, 9/06/2024	Chap. 3 & 6	Lagrangian equations of motion	#6
7	Mon, 9/09/2024	Chap. 3 & 6	Lagrangian to Hamiltonian formalism	
8	Wed, 9/11/2024	Chap. 3 & 6	Phase space	
9	Fri, 9/13/2024	Chap. 3 & 6	Canonical Transformations	

PHY 711 -- Assignment #6

Assigned: 9/06/2024 Due: 9/09/2024



a. The figure above shows a box of mass m sliding on the frictionless surface of an inclined plane (angle θ). The inclined plane itself has a mass M and is supported on a horizontal frictionless surface. Write down the Lagrangian for this system in terms of the generalized coordinates X and s and the fixed constants of the system (θ, m, M, and g) and determine the equations of motion. (Note that X and s represent the lengths of vectors whose directions are related by the angle θ.)

b. Assume that initially positions $X(t=0)=0$, $s(t=0)=0$ and that their corresponding velocities are also 0. Find the trajectories $X(t)$ and $s(t)$ for $t > 0$.

From previous lecture -- motion with constraints --

Comments on generalized coordinates:

$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Here we have assumed that the generalized coordinates q_σ are independent. Now consider the possibility that the coordinates are related through constraint equations of the form:

$$\text{Lagrangian}: \quad L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\text{Constraints}: \quad f_j = f_j\left(\{q_\sigma(t)\}, t\right) = 0$$

Lagrange multipliers

$$\text{Modified Euler-Lagrange equations}: \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$$

Some details --

Lagrangian: $L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$

Constraints: $f_j = f_j\left(\{q_\sigma(t)\}, t\right) = 0$

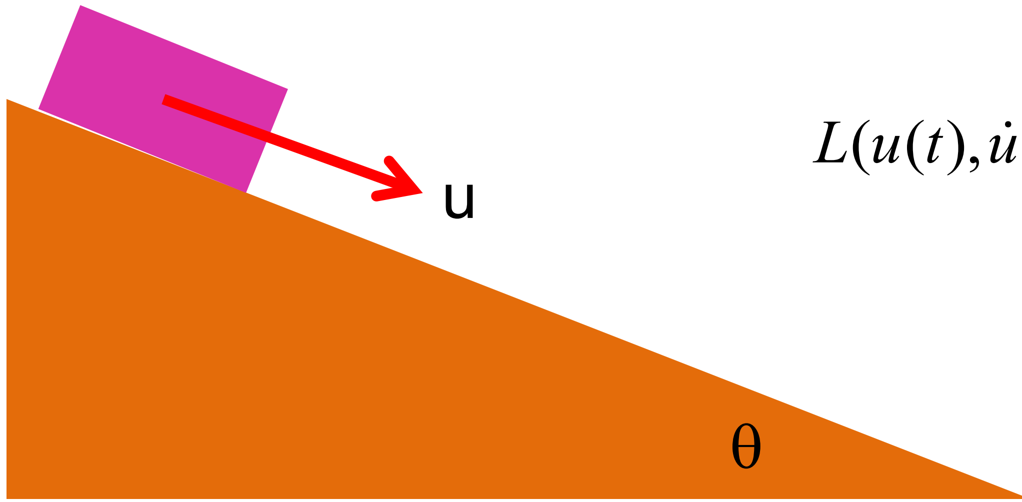
Modified Euler - Lagrange equations: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$

This amounts to modifying our optimization problem --

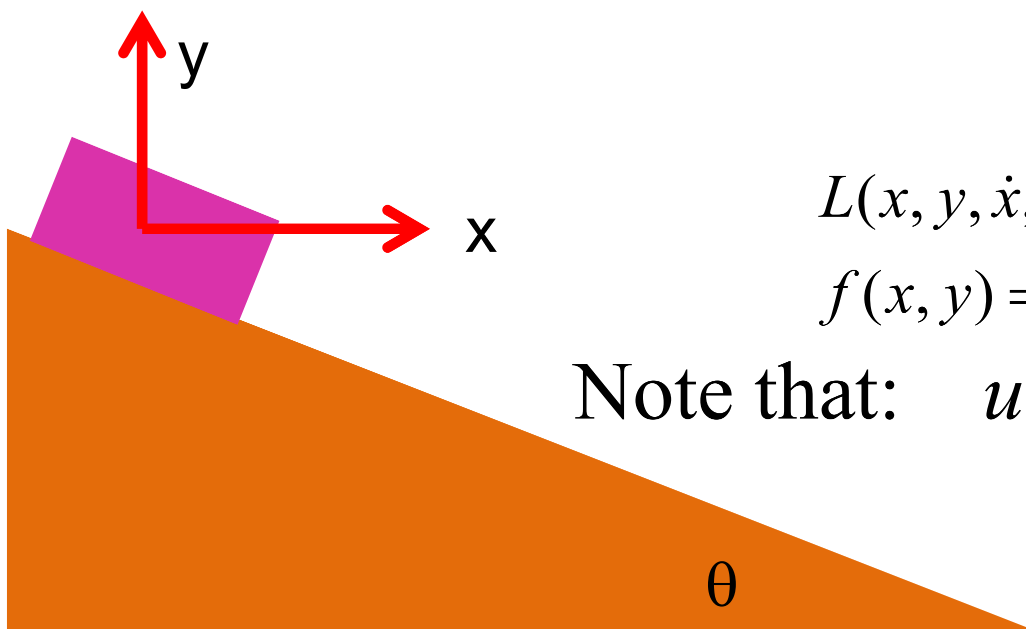
$\delta S = 0$ and for each *i*: $\delta f_i = 0$

$\Rightarrow \delta W \equiv \delta(S + \sum_i \lambda_i f_i) = 0,$ introducing the new constants λ_i.

Simple example:



$$L(u(t),\dot{u}(t)) = \tfrac{1}{2} m\dot{u}^2 + mgu\sin\theta$$



$$L(x, y, \dot{x}, \dot{y}) = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$f(x, y) = \sin\theta\, x + \cos\theta\, y = 0$$

Note that: $u = x\cos\theta - y\sin\theta$

Case 1:

$$L(u(t),\dot{u}(t)) = \tfrac{1}{2}m\dot{u}^2 + mgu\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = 0 = m\ddot{u} - mg\sin\theta = 0$$

$$\Rightarrow \ddot{u} = g\sin\theta$$

Case 2:

$$L(x,y,\dot{x},\dot{y}) = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$f(x,y) = \sin\theta\ x + \cos\theta\ y = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \lambda\frac{\partial f}{\partial x} = 0 = m\ddot{x} + \lambda\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \lambda\frac{\partial f}{\partial y} = 0 = m\ddot{y} + mg + \lambda\cos\theta$$

$$\sin\theta\ \ddot{x} + \cos\theta\ \ddot{y} = 0$$

$$\Rightarrow \lambda = -mg\cos\theta$$

$$\left(\cos\theta\ \ddot{x} - \sin\theta\ \ddot{y}\right) = g\sin\theta$$

Force of constraint; normal to incline

Which method would you use to solve the problem?

Case 1

Case 2

Rational for Lagrange multipliers

Recall Hamilton's principle:

$$S = \int_{t_i}^{t_f} L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right) dt$$

$$\delta S = 0 = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma \right) dt$$

With constraints: $f_j = f_j\left(\{q_\sigma(t)\}, t\right) = 0$

Variations δq_σ are no longer independent.

$$\delta f_j = 0 = \sum_\sigma \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma \quad \text{at each } t$$

$\Rightarrow$ Add 0 to Euler-Lagrange equations in the form:

$$\sum_j \lambda_j \sum_\sigma \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma$$

Example --

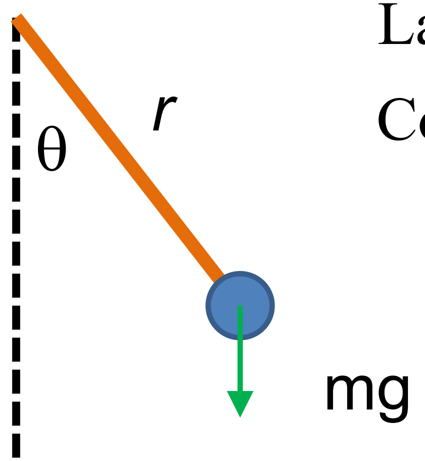
Euler-Lagrange equations with constraints:

$$\text{Lagrangian}: \quad L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\text{Constraints}: \quad f_j = f_j\left(\{q_\sigma(t)\}, t\right) = 0$$

$$\text{Modified Euler-Lagrange equations}: \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$$

Example:



$$\text{Lagrangian}: \quad L = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$$

$$\text{Constraints}: \quad f = r - \ell = 0$$

Example continued:

$$\text{Lagrangian:} \quad L = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$$

$$\text{Constraints:} \quad f = r - \ell = 0$$

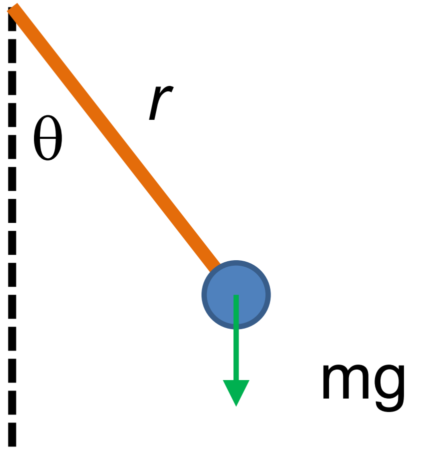
$$\frac{d}{dt}m\dot{r} - mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$

$$\frac{d}{dt}mr^2\dot{\theta} + mgr\sin\theta = 0$$

$$\dot{r} = 0 = \ddot{r} \qquad\qquad r = \ell$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

$$\Rightarrow \quad \lambda = m\ell\dot{\theta}^2 + mg\cos\theta$$



Another example:

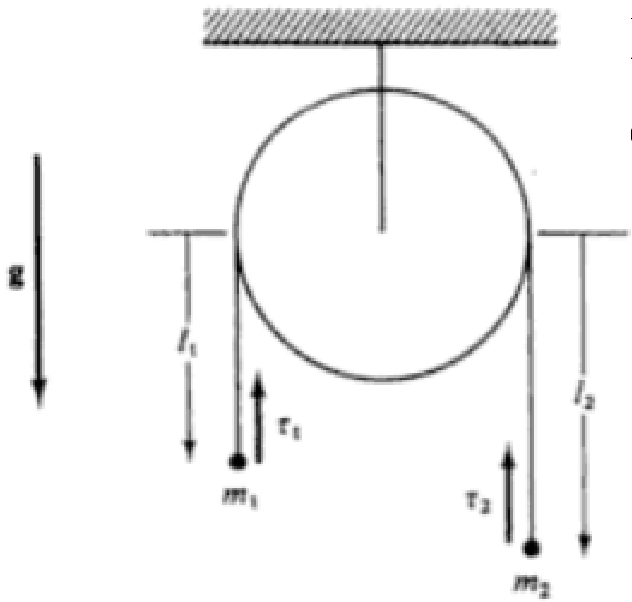


Figure 19.1 Atwood's machine.

$$\text{Lagrangian}: \quad L = \tfrac{1}{2} m_1 \dot{\ell}_1^2 + \tfrac{1}{2} m_2 \dot{\ell}_2^2 + m_1 g \ell_1 + m_2 g \ell_2$$

$$\text{Constraints}: \quad f = \ell_1 + \ell_2 - \ell = 0$$

$$\frac{d}{dt} m_1 \dot{\ell}_1 - m_1 g + \lambda = 0$$

$$\frac{d}{dt} m_2 \dot{\ell}_2 - m_2 g + \lambda = 0$$

$$\dot{\ell}_1 + \dot{\ell}_2 = 0 = \ddot{\ell}_1 + \ddot{\ell}_2$$

$$\Rightarrow \lambda = \frac{2 m_1 m_2}{m_1 + m_2} g$$

$$\ddot{\ell}_1 = -\ddot{\ell}_2 = \frac{m_1 - m_2}{m_1 + m_2} g$$

Summary of Lagrangian formalism (without constraints)

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Note that if $\dfrac{\partial L}{\partial q_\sigma} = 0$, then $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_\sigma} = 0$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_\sigma} = \text{ (constant)}$$

Comment -- Note that in deriving these equations we have assumed that there are only conservative forces (no friction) acting on this system. The equations are easily modified to take constraints, including those from static friction, into account. What is harder to treat is dynamic friction which is typically modeled by velocity dependent dissipative forces. However, some tricks for this have been developed such as described in the textbook by Herbert Goldstein.

Examples of constants of the motion:

Example 1: one-dimensional potential:

$$L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)$$

$$\Rightarrow \frac{d}{dt} m\dot{x} = 0 \qquad \Rightarrow m\dot{x} \equiv p_x \quad \text{(constant)}$$

$$\Rightarrow \frac{d}{dt} m\dot{y} = 0 \qquad \Rightarrow m\dot{y} \equiv p_y \quad \text{(constant)}$$

$$\Rightarrow \frac{d}{dt} m\dot{z} = -\frac{\partial V}{\partial z}$$

Examples of constants of the motion:

Example 2: Motion in a central potential

$$L = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

$$\Rightarrow \frac{d}{dt} mr^2\dot{\theta} = 0 \qquad \Rightarrow mr^2\dot{\theta} \equiv p_\theta \;\; \text{(constant)}$$

$$\Rightarrow \frac{d}{dt} m\dot{r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r} = \frac{{p_\theta}^2}{mr^3} - \frac{\partial V}{\partial r}$$

Recall alternative form of Euler-Lagrange equations:

Starting from:

$$L = L\Big(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\Big)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Also note that: $$\frac{dL}{dt} = \sum_\sigma \frac{\partial L}{\partial q_\sigma}\dot{q}_\sigma + \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\ddot{q}_\sigma + \frac{\partial L}{\partial t}$$

$$= \sum_\sigma \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma + \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\ddot{q}_\sigma + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt}\left(\sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right) = \frac{\partial L}{\partial t}$$

Additional constant of the motion:

If $\frac{\partial L}{\partial t} = 0;$

then: $\frac{d}{dt}\left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma \right) = \frac{\partial L}{\partial t} = 0$

$$\Rightarrow L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma = -E \quad \text{(constant)}$$

Example 1: one-dimensional potential:

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)$$

$$\Rightarrow \frac{d}{dt}\left(\tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z) - m\dot{x}^2 - m\dot{y}^2 - m\dot{z}^2\right) = 0$$

$$\Rightarrow -\left(\tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + V(z)\right) = -E \quad \text{(constant)}$$

For this case, we also have $m\dot{x} \equiv p_x$ and $m\dot{y} \equiv p_y$

$$\Rightarrow E = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \tfrac{1}{2} m\dot{z}^2 + V(z)$$

Summary from previous slide

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)$$ ➔3 variable functions

$$E = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \tfrac{1}{2} m\dot{z}^2 + V(z)$$ p_x, p_y, E constant

➔1 variable function

Why might this be useful?

Additional constant of the motion -- continued:

If $\frac{\partial L}{\partial t}=0;$

$$\text{then}: \quad \frac{d}{dt}\left(L-\sum_{\sigma}\frac{\partial L}{\partial \dot{q}_{\sigma}}\dot{q}_{\sigma}\right)=\frac{\partial L}{\partial t}=0$$

$$\Rightarrow L-\sum_{\sigma}\frac{\partial L}{\partial \dot{q}_{\sigma}}\dot{q}_{\sigma}=-E \quad \text{(constant)}$$

Example 2: Motion in a central potential

$$L=\tfrac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)-V(r)$$

➔2 variable functions

$$\Rightarrow \frac{d}{dt}\left(\tfrac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)-V(r)-m\dot{r}^2-mr^2\dot{\theta}^2\right)=0$$

$$\Rightarrow -\left(\tfrac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+V(r)\right)=-E \quad \text{(constant)}$$

For this case, we also have $mr^2\dot{\theta}\equiv p_{\theta}$

$$\Rightarrow E=\frac{p_{\theta}^2}{2mr^2}+\tfrac{1}{2}m\dot{r}^2+V(r)$$

➔1 variable function

Other examples

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{\partial L}{\partial z} = 0 \qquad \Rightarrow m\dot{z} = p_z \quad \text{(constant)}$$

$$E = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma - L$$

$$= m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$- \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\Rightarrow E = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{p_z^2}{2m}$$

Other examples

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q}{c} B_0 \dot{x} y$$

$$\frac{\partial L}{\partial z} = 0 \qquad \Rightarrow m\dot{z} = p_z \quad \text{(constant)}$$

$$\frac{\partial L}{\partial x} = 0 \qquad \Rightarrow m\dot{x} = p_x \quad \text{(constant)}$$

$$E = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma - L$$

$$= m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q}{c} B_0 \dot{x} y$$

$$- \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{c} B_0 \dot{x} y$$

$$\Rightarrow E = \tfrac{1}{2} m\dot{y}^2 + \frac{p_x^2}{2m} + \frac{p_z^2}{2m}$$

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\Big(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\Big)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Mathematical transformations for continuous functions of several variables & Legendre transforms:

Simple change of variables:

$$z(x,y) \quad \Leftrightarrow \quad x(y,z)???$$

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y,z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\text{But}: \quad \left(\frac{\partial x}{\partial y}\right)_z = -\frac{\left(\partial z / \partial y\right)_x}{\left(\partial z / \partial x\right)_y}$$ Assuming $dz=0$.

Note on notation for partial derivatives

$$z(x,y) \quad \Rightarrow \quad dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

hold *y* fixed.

hold *x* fixed.

Simple change of variables -- continued:

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y,z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\Rightarrow \left(\frac{\partial x}{\partial y}\right)_z = -\frac{\left(\partial z / \partial y\right)_x}{\left(\partial z / \partial x\right)_y} \qquad \Rightarrow \left(\frac{\partial x}{\partial z}\right)_y = \frac{1}{\left(\partial z / \partial x\right)_y}$$

Simple change of variables -- continued:

Example:

$$z(x,y) = e^{x^2+y}$$

$$x(y,z) = (\ln z - y)^{1/2}$$

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y,z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\left(\frac{\partial x}{\partial y}\right)_z \stackrel{?}{=} -\frac{(\partial z/\partial y)_x}{(\partial z/\partial x)_y} \qquad \left(\frac{\partial x}{\partial z}\right)_y \stackrel{?}{=} \frac{1}{(\partial z/\partial x)_y}$$

$$-\frac{1}{2(\ln z - y)^{1/2}} \stackrel{\checkmark}{=} -\frac{e^{x^2+y}}{2xe^{x^2+y}} \qquad \frac{1}{2z(\ln z - y)^{1/2}} \stackrel{\checkmark}{=} \frac{1}{2xe^{x^2+y}}$$

Now that we see that these transformations are possible, we should ask the question why we might want to do this?

An example comes from thermodynamics where we have various interdependent variables such as temperature T, pressure P, volume V, etc. etc. Often a measurable property can be specified as a function of two of those, while the other variables are also dependent on those two. For example we might specify T and P while the volume will be V(T,P). Or we might specify T and V while the pressure will be P(T,V).

Other examples from thermo --
For thermodynamic functions:

$$\text{Internal energy:} \qquad U = U(S,V)$$

$$dU = TdS - PdV$$

$$dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV$$

$$\Rightarrow T = \left(\frac{\partial U}{\partial S}\right)_V \qquad P = -\left(\frac{\partial U}{\partial V}\right)_S$$

$$\text{Enthalpy:} \qquad H = H(S,P) = U + PV$$

$$dH = dU + PdV + VdP = TdS + VdP = \left(\frac{\partial H}{\partial S}\right)_P dS + \left(\frac{\partial H}{\partial P}\right)_S dP$$

$$\Rightarrow T = \left(\frac{\partial H}{\partial S}\right)_P \qquad V = \left(\frac{\partial H}{\partial P}\right)_S$$

Name	Potential	Differential Form
Internal energy	$E(S,V,N)$	$dE = TdS - PdV + \mu dN$
Entropy	$S(E,V,N)$	$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN$
Enthalpy	$H(S,P,N) = E + PV$	$dH = TdS + VdP + \mu dN$
Helmholtz free energy	$F(T,V,N) = E - TS$	$dF = -SdT - PdV + \mu dN$
Gibbs free energy	$G(T,P,N) = F + PV$	$dG = -SdT + VdP + \mu dN$
Landau potential	$\Omega(T,V,\mu) = F - \mu N$	$d\Omega = -SdT - PdV - Nd\mu$

Mathematical transformations for continuous functions of several variables & Legendre transforms continued:

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$\text{Let} \quad u \equiv \left(\frac{\partial z}{\partial x}\right)_y \quad \text{and} \quad v \equiv \left(\frac{\partial z}{\partial y}\right)_x$$

Define new function

$$w(u,y) \quad \Rightarrow \quad dw = \left(\frac{\partial w}{\partial u}\right)_y du + \left(\frac{\partial w}{\partial y}\right)_u dy$$

$$\text{For} \quad w = z - ux, \qquad dw = dz - udx - xdu = udx + vdy - udx - xdu$$

$$dw = -xdu + vdy$$

$$\Rightarrow \left(\frac{\partial w}{\partial u}\right)_y = -x \qquad \left(\frac{\partial w}{\partial y}\right)_u = \left(\frac{\partial z}{\partial y}\right)_x = v$$

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

Hamiltonian picture – continued

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \qquad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$