



# PHY 711 Classical Mechanics and Mathematical Methods

10-10:50 AM MWF in Olin 103

## Discussion of Lecture 7 – Chap. 3&6 (F&W)

### Hamiltonian formalism

1. Constructing the Hamiltonian
2. Hamilton's canonical equation
3. Examples

## Your questions

### From Thomas

Why are Poisson brackets useful?

### From Julia

On slide 14, you said that  $H$  "often" represents the mechanical energy of the system. Does this mean that in some cases it represents something else? If so, what else can it mean?

# Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/26/2024		Introduction and overview	<a href="#">#1</a>
2	Wed, 8/28/2024	Chap. 3(17)	Calculus of variation	<a href="#">#2</a>
3	Fri, 8/30/2024	Chap. 3(17)	Calculus of variation	<a href="#">#3</a>
4	Mon, 9/02/2024	Chap. 3	Lagrangian equations of motion	<a href="#">#4</a>
5	Wed, 9/04/2024	Chap. 3 & 6	Lagrangian equations of motion	<a href="#">#5</a>
6	Fri, 9/06/2024	Chap. 3 & 6	Lagrangian equations of motion	<a href="#">#6</a>
7	Mon, 9/09/2024	Chap. 3 & 6	Lagrangian to Hamiltonian formalism	<a href="#">#7</a>
8	Wed, 9/11/2024	Chap. 3 & 6	Phase space	
9	Fri, 9/13/2024	Chap. 3 & 6	Canonical Transformations	



# PHY 711 -- Assignment #7

Assigned: 9/09/2024 Due: 9/16/2024

Continue reading Chapter 3 and 6 in **Fetter & Walecka**.

This problem considers the same system as Problem #5.

- a. Starting from the Lagrangian used to analyze Problem #5, determine the corresponding Hamiltonian in canonical form.
- b. Determine and solve the canonical equations of motion, comparing the resulting trajectories with those obtained by working Problem #5.

## Lagrangian picture

For independent generalized coordinates  $q_\sigma(t)$ :

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$  Second order differential equations for  $q_\sigma(t)$

## Switching variables – Legendre transformation

Define:  $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

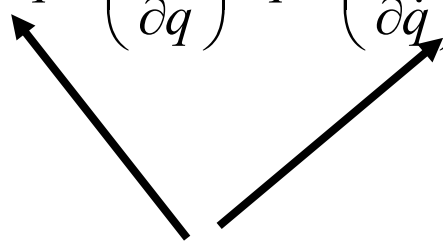
$$H = \sum_{\sigma} \dot{q}_\sigma p_\sigma - L \quad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_{\sigma} \left( \dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

# Application of the Legendre transformation for the Lagrangian and Hamiltonian

$L(q, \dot{q}, t)$  and  $H(q, p, t)$

suppose  $H(q, p, t) = \dot{q}p - L(q, \dot{q}, t)$

$$dH = \dot{q}dp + pd\dot{q} - \left(\frac{\partial L}{\partial q}\right) dq - \left(\frac{\partial L}{\partial \dot{q}}\right) d\dot{q} - \left(\frac{\partial L}{\partial t}\right) dt = \left(\frac{\partial H}{\partial q}\right) dq + \left(\frac{\partial H}{\partial p}\right) dp + \left(\frac{\partial H}{\partial t}\right) dt$$


Note that these two terms cancel if  $p = \frac{\partial L}{\partial \dot{q}}$

$$\Rightarrow dH = \dot{q}dp - \left(\frac{\partial L}{\partial q}\right) dq - \left(\frac{\partial L}{\partial t}\right) dt = \left(\frac{\partial H}{\partial q}\right) dq + \left(\frac{\partial H}{\partial p}\right) dp + \left(\frac{\partial H}{\partial t}\right) dt$$

The analysis on the following slides is a generalization to multiple dimensions  $q_\sigma$  and  $p_\sigma$  ....

# Hamiltonian picture – continued

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \quad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left( \dot{q}_\sigma dp_\sigma + \cancel{p_\sigma d\dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \cancel{\frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma} \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left( \dot{q}_\sigma dp_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma \right) - \frac{\partial L}{\partial t} dt$$

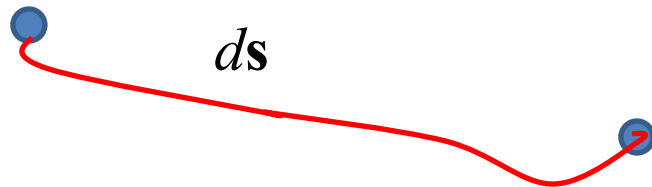
$$dH = \sum_\sigma \left( \frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$$

$$\frac{\partial L}{\partial q_\sigma} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = - \frac{\partial H}{\partial q_\sigma}$$

$$\frac{\partial L}{\partial t} = - \frac{\partial H}{\partial t}$$

# Direct application of Hamiltonian's principle using the Hamiltonian function --



Generalized coordinates :

$$q_{\sigma}(\{x_i\})$$

Define -- Lagrangian:  $L \equiv T - U$

$$L = L(\{q_{\sigma}\}, \{\dot{q}_{\sigma}\}, t)$$

$$\Rightarrow \text{Minimization integral: } S = \int_{t_i}^{t_f} L(\{q_{\sigma}\}, \{\dot{q}_{\sigma}\}, t) dt$$

Expressed in terms of Hamiltonian:

$$H = H(\{q_{\sigma}(t)\}, \{p_{\sigma}(t)\}, t)$$

$$H = \sum_{\sigma} \dot{q}_{\sigma} p_{\sigma} - L \quad \Rightarrow \quad L = \sum_{\sigma} \dot{q}_{\sigma} p_{\sigma} - H(\{q_{\sigma}(t)\}, \{p_{\sigma}(t)\}, t)$$



# Hamilton's principle continued: Minimization integral:

$$S = \int_{t_i}^{t_f} \left( \sum_{\sigma} \dot{q}_{\sigma} p_{\sigma} - H(\{q_{\sigma}(t)\}, \{p_{\sigma}(t)\}, t) \right) dt$$

$$\delta S = \int_{t_i}^{t_f} \left( \sum_{\sigma} \left( \dot{q}_{\sigma} \delta p_{\sigma} + \delta \dot{q}_{\sigma} p_{\sigma} - \frac{\partial H}{\partial q_{\sigma}} \delta q_{\sigma} - \frac{\partial H}{\partial p_{\sigma}} \delta p_{\sigma} \right) \right) dt = 0$$

$$\Rightarrow \dot{q}_{\sigma} = \frac{\partial H}{\partial p_{\sigma}}$$

**Canonical equations**

$$\Rightarrow \dot{p}_{\sigma} = -\frac{\partial H}{\partial q_{\sigma}}$$

Detail:

$$\int_{t_i}^{t_f} \left( \sum_{\sigma} (\delta \dot{q}_{\sigma} p_{\sigma}) \right) dt = \int_{t_i}^{t_f} \left( \sum_{\sigma} \left( \frac{d(\delta q_{\sigma} p_{\sigma})}{dt} - \delta q_{\sigma} \dot{p}_{\sigma} \right) \right) dt = \sum_{\sigma} \delta q_{\sigma} p_{\sigma} \Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} \left( \sum_{\sigma} (\delta q_{\sigma} \dot{p}_{\sigma}) \right) dt$$

# More comments about “details”

Detail:

$$\int_{t_i}^{t_f} \left( \sum_{\sigma} (\delta \dot{q}_{\sigma} p_{\sigma}) \right) dt = \int_{t_i}^{t_f} \left( \sum_{\sigma} \left( \frac{d(\delta q_{\sigma} p_{\sigma})}{dt} - \delta q_{\sigma} \dot{p}_{\sigma} \right) \right) dt = \sum_{\sigma} \delta q_{\sigma} p_{\sigma} \Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} \left( \sum_{\sigma} (\delta q_{\sigma} \dot{p}_{\sigma}) \right) dt$$



Vanishes because  
 $\delta q(t_f) = \delta q(t_i)$  due to  
the premise of  
Hamilton's principle.

In the Hamiltonian formulation --

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$$

$$\Rightarrow \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}$$

Why are these equations known as the “canonical equations”?

- a. Because they are beautiful.
- b. The term is meant to elevate their importance to the level of the music of J. S. Bach
- c. To help you remember them
- d. No good reason; it is just a name

# Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function :  $L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$
2. Compute generalized momenta :  $p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$
3. Construct Hamiltonian expression :  $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$
4. Form Hamiltonian function :  $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$
5. Analyze canonical equations of motion :

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \quad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

What happens when you miss a step in the recipe?

- a. No big deal
- b. Big deal – can lead to shame and humiliation  
(or at least wrong analysis)

## Lagrangian picture

For independent generalized coordinates  $q_\sigma(t)$ :

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \Rightarrow \quad \text{Second order differential equations for } q_\sigma(t)$$

## Hamiltonian picture

For independent generalized coordinates  $q_\sigma(t)$  and momenta  $p_\sigma(t)$ :

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \quad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \quad \Rightarrow \quad \text{Two first order differential equations}$$

## Constants of the motion in Hamiltonian formalism

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \quad \Rightarrow \text{constant } q_\sigma \quad \text{if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \quad \Rightarrow \text{constant } p_\sigma \quad \text{if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left( \frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

$$\frac{dH}{dt} = \sum_\sigma (-\dot{p}_\sigma \dot{q}_\sigma + \dot{q}_\sigma \dot{p}_\sigma) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

$$\Rightarrow \text{constant } H \quad \text{if } \frac{\partial H}{\partial t} = 0$$

What is the physical meaning of a constant  $H$ ?

Comment -- Whenever you find a constant of the motion, it is helpful for analyzing the trajectory. In this case,  $H$  often represents the mechanical energy of the system so that constant  $H$  implies that energy is conserved.

Example 1: one-dimensional potential:

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z)$$

$$p_x = m\dot{x} \quad p_y = m\dot{y} \quad p_z = m\dot{z}$$

$$H = m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2 - \left( \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z) \right)$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + V(z)$$

Constants:  $\bar{p}_x, \bar{p}_y, \bar{H}$  (using bar to indicate constant)

Equations of motion:  $\frac{dz}{dt} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \quad \frac{dp_z}{dt} = - \frac{dV}{dz}$



## Example 2: Motion in a central potential

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$

$$p_r = m\dot{r} \quad p_\phi = mr^2\dot{\phi}$$

$$\begin{aligned} H &= m\dot{r}^2 + mr^2\dot{\phi}^2 - \left(\frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)\right) \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + V(r) \end{aligned}$$

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r)$$

Constants:  $\bar{p}_\phi, \bar{H}$

Equations of motion:

$$\frac{dr}{dt} = \frac{p_r}{m} \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial r} = \frac{\bar{p}_\phi^2}{mr^3} - \frac{\partial V}{\partial r}$$

## Other examples

Lagrangian for symmetric top with Euler angles  $\alpha, \beta, \gamma$ :

$$L = L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 (\dot{\alpha} \cos \beta + \dot{\gamma})^2 - Mgh \cos \beta$$

$$p_\alpha = I_1 \dot{\alpha} \sin^2 \beta + I_3 (\dot{\alpha} \cos \beta + \dot{\gamma}) \cos \beta$$

$$p_\beta = I_1 \dot{\beta}$$

$$p_\gamma = I_3 (\dot{\alpha} \cos \beta + \dot{\gamma})$$

$$H = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 (\dot{\alpha} \cos \beta + \dot{\gamma})^2 + Mgh \cos \beta$$

$$H = \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + \frac{p_\beta^2}{2I_1} + \frac{p_\gamma^2}{2I_3} + Mgh \cos \beta$$

Constants:  $\bar{p}_\alpha, \bar{p}_\gamma, \bar{H}$

## Other examples

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + \dot{y}x)$$


$$p_x = m\dot{x} - \frac{q}{2c} B_0 y$$

$$p_y = m\dot{y} + \frac{q}{2c} B_0 x$$

$$p_z = m\dot{z}$$

$$H = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Canonical form

$$H = \frac{\left( p_x + \frac{q}{2c} B_0 y \right)^2}{2m} + \frac{\left( p_y - \frac{q}{2c} B_0 x \right)^2}{2m} + \frac{p_z^2}{2m}$$


Constants:  $\bar{p}_z, \bar{H}$

## Canonical equations of motion for constant magnetic field:

$$H = \frac{\left( p_x + \frac{q}{2c} B_0 y \right)^2}{2m} + \frac{\left( p_y - \frac{q}{2c} B_0 x \right)^2}{2m} + \frac{p_z^2}{2m}$$

Constants:  $\bar{p}_z, \bar{H}$

$$\frac{dx}{dt} = \frac{p_x + \frac{q}{2c} B_0 y}{m} \quad \frac{dy}{dt} = \frac{p_y - \frac{q}{2c} B_0 x}{m}$$

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial x} = \frac{qB_0}{2mc} \left( p_y - \frac{q}{2c} B_0 x \right)$$

$$\frac{dp_y}{dt} = -\frac{\partial H}{\partial y} = -\frac{qB_0}{2mc} \left( p_x + \frac{q}{2c} B_0 y \right)$$

Canonical equations of motion for constant magnetic field  
-- continued:

$$\frac{dx}{dt} = \frac{p_x + \frac{q}{2c} B_0 y}{m} \quad \frac{dy}{dt} = \frac{p_y - \frac{q}{2c} B_0 x}{m}$$

$$\frac{dp_x}{dt} = \frac{qB_0}{2mc} \left( p_y - \frac{q}{2c} B_0 x \right) = \frac{qB_0}{2c} \frac{dy}{dt}$$

$$\frac{dp_y}{dt} = -\frac{qB_0}{2mc} \left( p_x + \frac{q}{2c} B_0 y \right) = -\frac{qB_0}{2c} \frac{dx}{dt}$$

$$\frac{d^2 x}{dt^2} = \frac{\dot{p}_x}{m} + \frac{q}{2mc} B_0 \dot{y} = \frac{qB_0}{mc} \frac{dy}{dt}$$

$$\frac{d^2 y}{dt^2} = \frac{\dot{p}_y}{m} - \frac{q}{2mc} B_0 \dot{x} = -\frac{qB_0}{mc} \frac{dx}{dt}$$

$$\frac{d^2 x}{dt^2} = \frac{qB_0}{mc} \frac{dy}{dt}$$

$$\frac{d^2 y}{dt^2} = -\frac{qB_0}{mc} \frac{dx}{dt}$$

Are these results equivalent to the results of the Lagrangian analysis?

- a. Yes
- b. No



General treatment of particle of mass  $m$  and charge  $q$  moving in 3 dimensions in an potential  $U(\mathbf{r})$  as well as electromagnetic scalar and vector potentials  $\Phi(\mathbf{r}, t)$  and  $\mathbf{A}(\mathbf{r}, t)$ :

Lagrangian: 
$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2} m \dot{\mathbf{r}}^2 - U(\mathbf{r}) - q\Phi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

Hamiltonian: 
$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + \frac{q}{c} \mathbf{A}(\mathbf{r}, t)$$

$$\begin{aligned} H(\mathbf{r}, \mathbf{p}, t) &= \mathbf{p} \cdot \dot{\mathbf{r}} - L(\mathbf{r}, \dot{\mathbf{r}}, t) \\ &= \frac{1}{2m} \left( \mathbf{p} - \frac{q}{c} \mathbf{A}(\mathbf{r}, t) \right)^2 + U(\mathbf{r}) + q\Phi(\mathbf{r}, t) \end{aligned}$$



Some details:  $L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2} m \dot{\mathbf{r}}^2 - U(\mathbf{r}) - q\Phi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$

Hamiltonian:  $\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m \dot{\mathbf{r}} + \frac{q}{c} \mathbf{A}(\mathbf{r}, t)$

$$H(\mathbf{r}, \mathbf{p}, t) = \mathbf{p} \cdot \dot{\mathbf{r}} - L(\mathbf{r}, \dot{\mathbf{r}}, t)$$

$$= \left( m \dot{\mathbf{r}} + \frac{q}{c} \mathbf{A}(\mathbf{r}, t) \right) \cdot \dot{\mathbf{r}} - \left( \frac{1}{2} m \dot{\mathbf{r}}^2 - U(\mathbf{r}) - q\Phi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t) \right)$$

$$= \frac{1}{2} m \dot{\mathbf{r}}^2 + U(\mathbf{r}) + q\Phi(\mathbf{r}, t)$$

$$H(\mathbf{r}, \mathbf{p}, t) = \frac{1}{2m} \left( \mathbf{p} - \frac{q}{c} \mathbf{A}(\mathbf{r}, t) \right)^2 + U(\mathbf{r}) + q\Phi(\mathbf{r}, t)$$



**Canonical form**



# Other properties of Hamiltonian formalism – Poisson brackets:

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \Rightarrow \text{constant } q_\sigma \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \Rightarrow \text{constant } p_\sigma \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left( \frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

Similarly for an arbitrary function :  $F = F(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$\frac{dF}{dt} = \sum_\sigma \left( \frac{\partial F}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial F}{\partial t} = \sum_\sigma \left( \frac{\partial F}{\partial q_\sigma} \frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma} \frac{\partial H}{\partial q_\sigma} \right) + \frac{\partial F}{\partial t}$$

## Poisson brackets -- continued:

For an arbitrary function :  $F = F(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$\frac{dF}{dt} = \sum_{\sigma} \left( \frac{\partial F}{\partial q_{\sigma}} \dot{q}_{\sigma} + \frac{\partial F}{\partial p_{\sigma}} \dot{p}_{\sigma} \right) + \frac{\partial F}{\partial t} = \sum_{\sigma} \left( \frac{\partial F}{\partial q_{\sigma}} \frac{\partial H}{\partial p_{\sigma}} - \frac{\partial F}{\partial p_{\sigma}} \frac{\partial H}{\partial q_{\sigma}} \right) + \frac{\partial F}{\partial t}$$

Define :

$$[F, G]_{PB} \equiv \sum_{\sigma} \left( \frac{\partial F}{\partial q_{\sigma}} \frac{\partial G}{\partial p_{\sigma}} - \frac{\partial F}{\partial p_{\sigma}} \frac{\partial G}{\partial q_{\sigma}} \right) = -[G, F]_{PB}$$

So that : 
$$\frac{dF}{dt} = [F, H]_{PB} + \frac{\partial F}{\partial t}$$

## Poisson brackets -- continued:

$$[F, G]_{PB} \equiv \sum_{\sigma} \left( \frac{\partial F}{\partial q_{\sigma}} \frac{\partial G}{\partial p_{\sigma}} - \frac{\partial F}{\partial p_{\sigma}} \frac{\partial G}{\partial q_{\sigma}} \right) = -[G, F]_{PB}$$

Examples:

$$\begin{aligned} [x, x]_{PB} &= 0 & [x, p_x]_{PB} &= 1 & [x, p_y]_{PB} &= 0 \\ [L_x, L_y]_{PB} &= L_z \end{aligned}$$

## Liouville theorem

Let  $D \equiv$  density of particles in phase space :

$$\frac{dD}{dt} = [D, H]_{PB} + \frac{\partial D}{\partial t} = 0$$



For next time