PHY 712 Electrodynamics
10-10:50 AM MWF in Olin 103

Lecture Notes for Lecture 15:

Start reading Chapter 6 (Sec. 6.1-6.4)

Note: Instead of following Sec. 6.5, we will introduce the Lienard-Wiéchert approach.

1. **Maxwell's full equations; effects of time varying fields and sources**
2. **Gauge choices and transformations**
3. **Green's function for vector and scalar potentials**

Course schedule for Spring 2025

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/13/2025	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/15/2025
2	Wed: 01/15/2025	Chap. 1	Electrostatic energy calculations	#2	01/17/2025
3	Fri: 01/17/2025	Chap. 1	Electrostatic energy calculations	#3	01/22/2025
	Mon: 01/20/2025	No Class	Martin Luther King Jr. Holiday		
4	Wed: 01/22/2025	Chap. 1	Electrostatic potentials and fields	#4	01/24/2025
5	Fri: 01/24/2025	Chap. 1 - 3	Poisson's equation in multiple dimensions		
6	Mon: 01/27/2025	Chap. 1 - 3	Brief introduction to numerical methods	#5	01/29/2025
7	Wed: 01/29/2025	Chap. 2 & 3	Image charge constructions	#6	01/31/2025
8	Fri: 01/31/2025	Chap. 2 & 3	Poisson equation in cylindrical geometries	#7	02/03/2025
9	Mon: 02/03/2025	Chap. 3 & 4	Spherical geometry and multipole moments	#8	02/05/2025
10	Wed: 02/05/2025	Chap. 4	Dipoles and Dielectrics	#9	02/07/2025
11	Fri: 02/07/2025	Chap. 4	Dipoles and Dielectrics	#10	02/10/2025
12	Mon: 02/10/2025	Chap. 5	Magnetostatics	#11	02/12/2025
13	Wed: 02/12/2025	Chap. 5	Magnetic dipoles and hyperfine interactions	#12	02/14/2025
14	Fri: 02/14/2025	Chap. 5	Magnetic materials and boundary value problems	#13	02/17/2025
15	Mon: 02/17/2025	Chap. 6	Maxwell's Equations	#14	02/19/2025
16	Wed: 02/19/2025	Chap. 6	Electromagnetic energy and forces		

PHY 712 – Problem Set #14

Assigned: 02/17/2025 Due: 02/19/2025

Read Chapter 6 (Sec. 6.1-6.4) in **Jackson**.

1. Evaluate the following integral over the Dirac delta function of a non-trivial argument. Explain your reasoning and show intermediate steps.

$$\int_{-10}^{10} \mathrm{e}^{-x^2} \delta(\sin(x)) dx.$$

Full electrodynamics with time varying fields and sources

Maxwell's equations

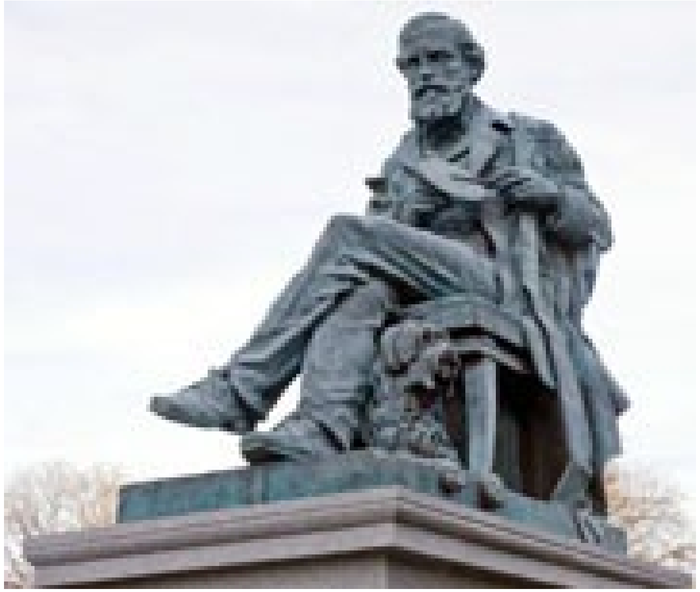


Image of statue of James Clerk-Maxwell (1831-1879) in Edinburgh

"From a long view of the history of mankind - seen from, say, ten thousand years from now - there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics"

Richard P Feynman

http://www.clerkmaxwellfoundation.org/

Maxwell's equations

Microscopic or vacuum form $(\mathbf{P}=0;\ \mathbf{M}=0)$:

Coulomb's law: $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\varepsilon_0 \mu_0}$$

Maxwell's equations

Coulomb's law: $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere-Maxwell's law: $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

Formulation of Maxwell's equations in terms of vector and scalar potentials – here focusing on the full **E** and **B** fields

$$\nabla \cdot \mathbf{B} = 0 \qquad \Rightarrow \; \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

$$\text{or} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 :$$

$$-\nabla^2 \Phi - \frac{\partial \left(\nabla \cdot \mathbf{A} \right)}{\partial t} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times \left(\nabla \times \mathbf{A} \right) + \frac{1}{c^2} \left(\frac{\partial \left(\nabla \Phi \right)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

General form for the scalar and vector potential equations:

$$-\nabla^2\Phi - \frac{\partial\left(\nabla\cdot\mathbf{A}\right)}{\partial t} = \rho/\varepsilon_0$$

$$\nabla\times\left(\nabla\times\mathbf{A}\right) + \frac{1}{c^2}\left(\frac{\partial\left(\nabla\Phi\right)}{\partial t} + \frac{\partial^2\mathbf{A}}{\partial t^2}\right) = \mu_0\mathbf{J}$$

Coulomb gauge form -- require $\nabla\cdot\mathbf{A}_C = 0$

$$-\nabla^2\Phi_C = \rho/\varepsilon_0$$

$$-\nabla^2\mathbf{A}_C + \frac{1}{c^2}\frac{\partial^2\mathbf{A}_C}{\partial t^2} + \frac{1}{c^2}\frac{\partial\left(\nabla\Phi_C\right)}{\partial t} = \mu_0\mathbf{J}$$

Note that it is useful to define the following:

$$\mathbf{J} = \mathbf{J}_l + \mathbf{J}_t \quad \text{with } \nabla\times\mathbf{J}_l = 0 \ \text{ and } \ \nabla\cdot\mathbf{J}_t = 0$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Coulomb gauge form - - require $\nabla \cdot \mathbf{A}_C = 0$

$$-\nabla^2 \Phi_C = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_C + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_C}{\partial t^2} + \frac{1}{c^2}\frac{\partial(\nabla \Phi_C)}{\partial t} = \mu_0 \mathbf{J}$$

Note that $\mathbf{J} = \mathbf{J}_l + \mathbf{J}_t$ with $\nabla \times \mathbf{J}_l = 0$ and $\nabla \cdot \mathbf{J}_t = 0$

Continuity equation for charge and current density:

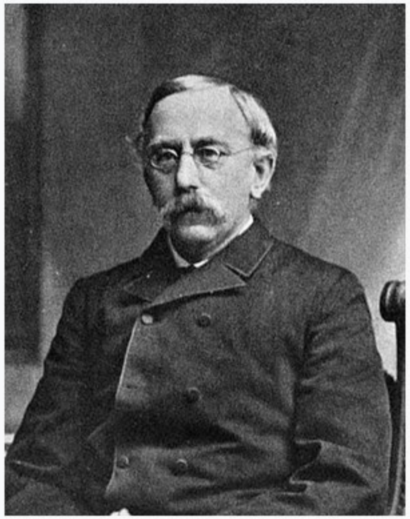
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}_l = 0 \qquad \Rightarrow \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}_l = -\varepsilon_0 \nabla \cdot \frac{\partial(\nabla \Phi_C)}{\partial t}$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial(\nabla \Phi_C)}{\partial t} = \varepsilon_0 \mu_0 \frac{\partial(\nabla \Phi_C)}{\partial t} = \mu_0 \mathbf{J}_l$$

$$-\nabla^2 \mathbf{A}_C + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_C}{\partial t^2} = \mu_0 \mathbf{J}_t$$

Another possible choice of gauge --

Ludvig Lorenz

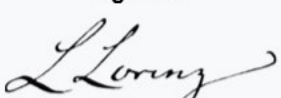


Born	18 January 1829 Helsingør, Denmark
Died	9 June 1891 (aged 62) Frederiksberg, Denmark
Resting place	Assistens Cemetery (Copenhagen), Denmark
Nationality	Danish
Known for	Wiedemann–Franz–Lorenz law Lorentz–Lorenz equation Lorenz gauge condition Lorenz–Mie theory

Scientific career

Fields	Physicist

Signature



Responsible for Lorenz Gauge

Hendrik Lorentz

ForMemRS

Lorentz in 1902

Born	Hendrik Antoon Lorentz 18 July 1853 Arnhem, Gelderland, Netherlands
Died	4 February 1928 (aged 74) Haarlem, North Holland, Netherlands
Alma mater	University of Leiden (BS, 1871; doctorate, 1875)
Known for	Postulating length contraction (1892) Formulating the Lorentz force law (1895) Proposing the Lorentz ether theory (1895) Explaining the Zeeman effect (1896) Introducing the Lorentz transformation (1899)

Responsible for Lorentz transformation

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Review of the general equations:

$$-\nabla^2\Phi - \frac{\partial\left(\nabla\cdot\mathbf{A}\right)}{\partial t} = \rho/\varepsilon_0$$

$$\nabla\times\left(\nabla\times\mathbf{A}\right) + \frac{1}{c^2}\left(\frac{\partial\left(\nabla\Phi\right)}{\partial t} + \frac{\partial^2\mathbf{A}}{\partial t^2}\right) = \mu_0\mathbf{J}$$

Lorenz gauge form -- require $\nabla\cdot\mathbf{A}_L + \frac{1}{c^2}\frac{\partial\Phi_L}{\partial t} = 0$

$$-\nabla^2\Phi_L + \frac{1}{c^2}\frac{\partial^2\Phi_L}{\partial t^2} = \rho/\varepsilon_0$$

$$-\nabla^2\mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} = \mu_0\mathbf{J}$$

Recall that

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Lorenz gauge form -- require $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2}\frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2}\frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

Alternate potentials: $\mathbf{A}'_L = \mathbf{A}_L + \nabla \Lambda$ and $\Phi'_L = \Phi_L - \frac{\partial \Lambda}{\partial t}$

Yields same physics provided that: $\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0$

Solution of Maxwell's equations in the Lorenz gauge

$$\nabla^2 \Phi_L - \frac{1}{c^2}\frac{\partial^2 \Phi_L}{\partial t^2} = -\rho / \varepsilon_0$$

$$\nabla^2 \mathbf{A}_L - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = -\mu_0 \mathbf{J}$$

Consider the general form of the 3-dimensional wave equation:

$$\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = -4\pi f$$

$\Psi(\mathbf{r},t) \Rightarrow$ wave field $\quad f(\mathbf{r},t) \Rightarrow$ source

Solution of Maxwell's equations in the Lorenz gauge -- continued

Let Ψ represent Φ, A_x, A_y, A_z Let f represent ρ, J_x, J_y, J_z

$$\nabla^2 \Psi(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2 \Psi(\mathbf{r},t)}{\partial t^2} = -4\pi f(\mathbf{r},t)$$

Green's function :

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Formal solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) + \int d^3r' \int dt' G(\mathbf{r},t;\mathbf{r}',t') f(\mathbf{r}',t')$$

Solution of Maxwell's equations in the Lorenz gauge -- continued

Determination of the form for the Green's function:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

For the case of isotropic boundary values at infinity:

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right)$$

Formal solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

Solution of Maxwell's equations in the Lorenz gauge -- continued

Analysis of the Green's function:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

"Proof" -- Fourier analysis in the time domain -- note that

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')}$$

Define:

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')}\ \tilde{G}(\mathbf{r},\mathbf{r}',\omega)$$

$$\Rightarrow \quad \left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{G}(\mathbf{r},\mathbf{r}',\omega) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

Solution of Maxwell's equations in the Lorenz gauge -- continued

Analysis of the Green's function (continued):

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\widetilde{G}(\mathbf{r},\mathbf{r}',\omega) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

For the case of isotropic boundary values at infinity:

$$\widetilde{G}(\mathbf{r},\mathbf{r}',\omega) = \widetilde{G}(\mathbf{r}-\mathbf{r}',\omega)$$

Further assuming that $\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega)$ is isotropic in $|\mathbf{r}-\mathbf{r}'| \equiv R$:

$$\left(\frac{1}{R}\frac{d^2}{dR^2}R + \frac{\omega^2}{c^2}\right)\widetilde{G}(\mathbf{r},\mathbf{r}',\omega) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

Solution: $\widetilde{G}(\mathbf{r},\mathbf{r}',\omega) = \frac{1}{R}e^{\pm i\omega R/c}$

Solution of Maxwell's equations in the Lorenz gauge -- continued

Analysis of the Green's function (continued):

$$\tilde{G}(\mathbf{r},\mathbf{r}',\omega) = \frac{1}{|\mathbf{r}-\mathbf{r}'|} e^{\pm i\omega|\mathbf{r}-\mathbf{r}'|/c}$$

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')}\, \tilde{G}(\mathbf{r},\mathbf{r}',\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')} \frac{1}{|\mathbf{r}-\mathbf{r}'|} e^{\pm i\omega|\mathbf{r}-\mathbf{r}'|/c}$$

$$= \frac{1}{|\mathbf{r}-\mathbf{r}'|}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega\left(t-t'\pm|\mathbf{r}-\mathbf{r}'|/c\right)}\right)$$

$$= \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t-t'\pm|\mathbf{r}-\mathbf{r}'|/c\right) = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-t\mp|\mathbf{r}-\mathbf{r}'|/c\right)$$

Solution of Maxwell's equations in the Lorenz gauge -- continued

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-\left(t \pm |\mathbf{r}-\mathbf{r}'|/c\right)\right)$$

Solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-\left(t-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

Solution of Maxwell's equations in the Lorenz gauge -- continued

Liènard-Wiechert potentials and fields --
Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz ***The Classical Theory of Fields***, Chapter 8.)

Consider the fields produced by the following source: a point charge q moving on a trajectory $R_q(t)$.

$$\text{Charge density: } \rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{R}_q(t))$$

$$\text{Current density: } \mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r}-\mathbf{R}_q(t)), \quad \text{where} \quad \dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}.$$

Solution of Maxwell's equations in the Lorenz gauge -- continued

$$\Phi(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\int d^3r'dt'\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}']|}\delta\Big(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\Big)$$

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0 c^2}\int\int d^3r'dt'\frac{\mathbf{J}(\mathbf{r}',\mathrm{t'})}{|\mathbf{r}-\mathbf{r}'|}\delta\Big(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\Big).$$

We performing the integrations over first d^3r' and then dt' making use of the fact that for any function of t',

$$\int_{-\infty}^{\infty} dt'\, f(t')\delta\Big(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\Big)=\frac{f(t_r)}{1-\dfrac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the ``retarded time'' is defined to be

$$t_r\equiv t-\frac{|\mathbf{r}-\ \mathbf{R}_q(t_r)|}{c}.$$

Solution of Maxwell's equations in the Lorenz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

Notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$$

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

Comment on Lienard-Wiechert potential results

$$\int_{-\infty}^{\infty} dt' \; f(t')\delta\left(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

Note that for any function $F(x)$:

$$\int_{-\infty}^{\infty} F(x)\delta(x - x_0)dx = F(x_0)$$

Now consider a function $p(x)$, for which $p(x_i) = 0$ for $i = 1,2,\cdots$

$$\int_{-\infty}^{\infty} F(x)\delta(p(x))dx = \int_{-\infty}^{\infty} F(x)\left(\sum_i \delta\left((x - x_i)\left.\frac{dp}{dx}\right|_{x_i}\right)\right)dx$$

$$= \sum_i \frac{F(x_i)}{\left|\frac{dp}{dx}\right|_{x_i}}$$

Comment on Lienard-Wiechert potential results -- continued

$$\int_{-\infty}^{\infty} dt' \; f(t')\delta\left(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

In this case we have: $\displaystyle\int_{-\infty}^{\infty} f(t')\delta\left(p(t')\right)dt' = \frac{f(t_r)}{\left|1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot\left(\mathbf{r}-\mathbf{R}_q(t')\right)}{c\left|\mathbf{r}-\mathbf{R}_q(t_r)\right|}\right|}$

where: $\displaystyle p(t') \equiv t' - \left(t - \frac{\left|\mathbf{r}-\mathbf{R}_q(t')\right|}{c}\right)$

$$\frac{dp(t')}{dt'} = 1 - \frac{\frac{d\mathbf{R}_q(t')}{dt'}\cdot\left(\mathbf{r}-\mathbf{R}_q(t')\right)}{c\left|\mathbf{r}-\mathbf{R}_q(t')\right|} \equiv 1 - \frac{\dot{\mathbf{R}}_q(t')\cdot\left(\mathbf{r}-\mathbf{R}_q(t')\right)}{c\left|\mathbf{r}-\mathbf{R}_q(t')\right|}$$

Summary of results for fields due to moving charge – Liénard Wiechert potentials

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{R - \dfrac{\mathbf{v}\cdot\ \mathbf{R}}{c}}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}}{R - \dfrac{\mathbf{v}\cdot\ \mathbf{R}}{c}}$$

Notation: $\mathbf{R} \equiv\ \mathbf{r} -\ \mathbf{R}_q(t_r)$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$$

$$t_r \equiv t - \frac{|\,\mathbf{r} -\ \mathbf{R}_q(t_r)\,|}{c}.$$