PHY 712 Electrodynamics
10-10:50 AM MWF in Olin 103

Notes for Lecture 16:

Finish reading Chapter 6 (Sec. 6.6-6.10 in JDJ) (some sections covered in less detail)

1. **Some details of Liénard-Wiechert results**
2. **Energy density and flux associated with electromagnetic fields**
3. **Time harmonic fields**

Tentative schedule for next several weeks --

12	Mon: 02/10/2025	Chap. 5	Magnetostatics	#11	02/12/2025
13	Wed: 02/12/2025	Chap. 5	Magnetic dipoles and hyperfine interactions	#12	02/14/2025
14	Fri: 02/14/2025	Chap. 5	Magnetic materials and boundary value problems	#13	02/17/2025
15	Mon: 02/17/2025	Chap. 6	Maxwell's Equations	#14	02/19/2025
16	Wed: 02/19/2025	Chap. 6	Electromagnetic energy and forces	#15	02/21/2025
17	Fri: 02/21/2025	Chap. 7	Electromagnetic plane waves		
18	Mon: 02/24/2025	Chap. 7	Electromagnetic plane waves		
19	Wed: 02/26/2025	Chap. 7	Optical effects of refractive indices		
20	Fri: 02/28/2025				
21	Mon: 03/03/2025				
22	Wed: 03/05/2025				
23	Fri: 03/07/2025		Review		
	Mon: 03/10/2025	No class	*Spring Break*		
	Wed: 03/12/2025	No class	*Spring Break*		
	Fri: 03/14/2025	No class	*Spring Break*		
	Mon: 03/17/2025	No class	*Take-home exam*		
	Wed: 03/19/2025	No class	*Take-home exam*		
	Fri: 03/21/2025	No class	*Take-home exam*		

PHY 712 – Problem Set #15

Assigned: 02/19/2025 Due: 02/21/2025

Complete reading Chapter 6 in **Jackson**.

1. Consider the $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ fields derived in class for a point particle of charge q moving on a trajectory $\mathbf{R}_q(t')$, keeping only the terms that depend on the particle's acceleration
$$\dot{\mathbf{v}} \equiv \frac{d^2\mathbf{R}_q(t_r)}{dt_r^2},$$
where t_r denotes the retarded time. Using the short hand notation used in the lecture notes, show that
$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r},t)}{cR}.$$

Slide from Lecture 15

Solution of Maxwell's equations in the Lorentz gauge

Let Ψ represent Φ, A_x, A_y, A_z Let f represent ρ, J_x, J_y, J_z

$$\nabla^2 \Psi(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2 \Psi(\mathbf{r},t)}{\partial t^2} = -4\pi f(\mathbf{r},t)$$

Green's function:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Formal solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) + \int d^3r' \int dt' G(\mathbf{r},t;\mathbf{r}',t') f(\mathbf{r}',t')$$

Operationally, $G(\mathbf{r},t;\mathbf{r}',t')$ is the inverse of the differential $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)$

Checking:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) + \int d^3r' \int dt' G(\mathbf{r},t;\mathbf{r}',t') f(\mathbf{r}',t')$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi(\mathbf{r},t) = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi_{f=0}(\mathbf{r},t) + \int d^3r' \int dt' \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r},t;\mathbf{r}',t') f(\mathbf{r}',t')$$

$$= \quad 0 \quad + \int d^3r' \int dt' \left(-4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')\right) f(\mathbf{r}',t')$$

$$= \quad -4\pi f(\mathbf{r},t)$$

For the case of isotropic boundary values at infinity:

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right)$$

"Proof" involved several steps which we can review at a later time.

Solution of Maxwell's equations in the Lorentz gauge – Review from previous lecture --

Liénard-Wiechert potentials and fields --

Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz ***The Classical Theory of Fields***, Chapter 8.)

Consider the fields produced by the following source: a point charge q moving on a trajectory $R_q(t)$.

$$\text{Charge density: } \rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{R}_q(t))$$

$$\text{Current density: } \mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r}-\mathbf{R}_q(t)), \quad \text{where} \quad \dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\Phi(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\int d^3r'dt'\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}']|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right)$$

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0 c^2}\int\int d^3r'dt'\frac{\mathbf{J}(\mathbf{r}',\mathrm{t}')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right).$$

We performing the integrations over first d^3r' and then dt' making use of the fact that for any function of t',

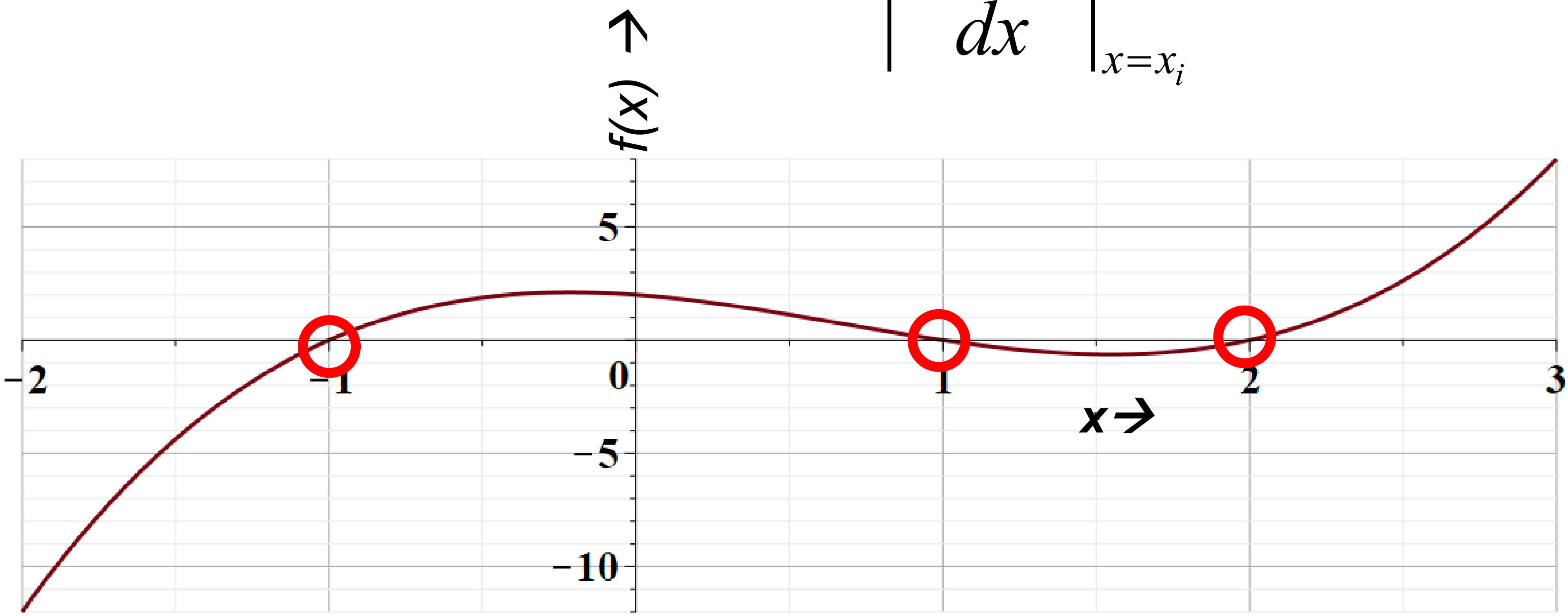
$$\int_{-\infty}^{\infty} dt'\, f(t')\delta\left(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\right)=\frac{f(t_r)}{1-\frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the ``retarded time'' is defined to be

$$t_r\equiv t-\frac{|\mathbf{r}-\ \mathbf{R}_q(t_r)|}{c}.$$

Comment about delta functions -- See Pg. 26 in **Jackson**

$$\int_{-\infty}^{\infty} dx\ \Psi(x)\delta(f(x)) = \sum_i \frac{\Psi(x_i)}{\left|\frac{df(x)}{dx}\right|_{x=x_i}}$$



$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\int d^3r'dt'\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}']|}\delta\big(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\big)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0 c^2}\int\int d^3r'dt'\frac{\mathbf{J}(\mathbf{r}',\mathrm{t'})}{|\mathbf{r}-\mathbf{r}'|}\delta\big(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\big).$$

Charge density: $\rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{R}_q(t))$

Current density: $\mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r}-\mathbf{R}_q(t))$, where $\dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}$.

We performing the integrations over first d^3r':

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\int dt'\frac{1}{|\mathbf{r}-\mathbf{R}_q(t')|}\delta\big(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\big)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\int dt'\left(\frac{\frac{d\mathbf{R}_q(t')}{dt'}}{|\mathbf{r}-\mathbf{R}_q(t')|}\right)\delta\big(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\big).$$

Some details --

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\int dt' \frac{1}{|\mathbf{r}-\mathbf{R}_q(t')|}\delta\Big(t' - (t - |\mathbf{r}-\mathbf{R}_q(t')|/c)\Big)$$

Define $t_r \equiv t - |\mathbf{r}-\mathbf{R}_q(t_r)|/c$

$$\left.\frac{\partial\Big(t' - (t - |\mathbf{r}-\mathbf{R}_q(t')|/c)\Big)}{\partial t'}\right|_{t'=t_r} = 1 - \left.\frac{\dot{\mathbf{R}}_q(t')\cdot\Big(\mathbf{r}-\mathbf{R}_q(t')\Big)}{|\mathbf{r}-\mathbf{R}_q(t')|c}\right|_{t'=t_r}$$

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{R}_q(t_r)| - \dot{\mathbf{R}}_q(t_r)\cdot\Big(\mathbf{r}-\mathbf{R}_q(t_r)\Big)/c}$$

Similarly, $\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\int dt' \left(\frac{\frac{d\mathbf{R}_q(t')}{dt'}}{|\mathbf{r}-\mathbf{R}_q(t')|}\right)\delta\Big(t' - (t - |\mathbf{r}-\mathbf{R}_q(t')|/c)\Big).$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\dot{\mathbf{R}}_q(t_r)}{|\mathbf{r}-\mathbf{R}_q(t_r)| - \dot{\mathbf{R}}_q(t_r)\cdot\Big(\mathbf{r}-\mathbf{R}_q(t_r)\Big)/c}$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

Notation:

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c} \qquad \mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$$

$$R \equiv \left|\mathbf{r} - \mathbf{R}_q(t_r)\right| \qquad \mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

In order to find the electric and magnetic fields, we need to evaluate

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$$

The trick of evaluating these derivatives is that the retarded time t_r depends on position **r** and on itself. We can show the following results using the shorthand notation:

$$\nabla t_r = -\frac{\mathbf{R}}{c\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)} \quad \text{and} \quad \frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)}.$$

Some details --

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$$

$$\frac{\partial t_r}{\partial t} = 1 + \frac{\left(\mathbf{r} - \mathbf{R}_q(t_r)\right) \cdot \frac{d\mathbf{R}_q(t_r)}{dt_r}}{c\,|\mathbf{r} - \mathbf{R}_q(t_r)|} \frac{\partial t_r}{\partial t}$$

Using notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r) \quad \mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$

$$R \equiv \left|\mathbf{r} - \mathbf{R}_q(t_r)\right|$$

$$\Rightarrow \frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)}$$

Similarly -- $\nabla t_r = -\frac{\mathbf{R}}{c\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)}$

After a few steps --

$$-\nabla\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\mathbf{R}\left(1-\frac{v^2}{c^2}\right)-\frac{v}{c}\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)+\mathbf{R}\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right],$$

$$-\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t} = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\frac{\mathbf{v}R}{c}\left(\frac{v^2}{c^2}-\frac{\mathbf{v}\cdot\mathbf{R}}{Rc}-\frac{\dot{\mathbf{v}}\cdot R}{c^2}\right)-\frac{\dot{\mathbf{v}}R}{c^2}\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)\right].$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\left(1-\frac{v^2}{c^2}\right)+\left(\mathbf{R}\times\left\{\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right].$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1-\frac{v^2}{c^2}+\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right)-\frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]=\frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{cR}$$

Famous result for point charge moving on an arbitrary trajectory $\mathbf{R}_q(t')$

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c} \qquad \mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$$

$$R \equiv \left|\mathbf{r} - \mathbf{R}_q(t_r)\right| \qquad \mathbf{v} \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \qquad \dot{\mathbf{v}} \equiv \frac{d^2\mathbf{R}_q(t_r)}{dt_r^{\,2}}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3} \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right].$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right] = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{cR}$$

Back to general case --

Maxwell's equations

$$\text{Coulomb's law:} \qquad \nabla \cdot \mathbf{D} = \rho_{free}$$

$$\text{Ampere-Maxwell's law:} \qquad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$$

$$\text{Faraday's law:} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\text{No magnetic monopoles:} \qquad \nabla \cdot \mathbf{B} = 0$$

Energy analysis of electromagnetic fields and sources

Rate of work done on source $\mathbf{J}(\mathbf{r},t)$ by electromagnetic field:

$$\frac{dW_{mech}}{dt} \equiv \frac{dE_{mech}}{dt} = \int d^3r \;\; \mathbf{E} \cdot \mathbf{J}_{free}$$

Expressing source current in terms of fields it produces:

$$\frac{dW_{mech}}{dt} = \int d^3r \;\; \mathbf{E} \cdot \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right)$$

Energy analysis of electromagnetic fields and sources - continued

$$\frac{dW_{mech}}{dt} = \int d^3r \;\; \mathbf{E}\cdot\mathbf{J}_{free} = \int d^3r \;\; \mathbf{E}\cdot\left(\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}\right)$$

$$= -\int d^3r \;\; \left(\nabla\cdot\left(\mathbf{E}\times\mathbf{H}\right) + \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)$$

Let $\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$ "Poynting vector"

$u \equiv \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}\right)$ energy density

$$\Rightarrow \frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J}_{free}$$

Assuming that $\mathbf{D}=\epsilon\mathbf{E}$ and that $\mathbf{B}=\mu\mathbf{H}$

Energy analysis of electromagnetic fields and sources - continued

$$\frac{dE_{mech}}{dt} \equiv \int d^3r \;\; \mathbf{E}\cdot\mathbf{J}_{free}$$

Electromagnetic energy density: $u \equiv \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}\right)$

$$E_{field} \equiv \int d^3r \;\; u(\mathbf{r},t)$$

Poynting vector: $\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$

From the previous energy analysis: $\frac{\partial u}{\partial t}+\nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J}_{free}$

$$\Rightarrow \frac{dE_{mech}}{dt}+\frac{dE_{field}}{dt} = -\int d^3r \;\; \nabla\cdot\mathbf{S}(\mathbf{r},t) = -\oint d^2r \; \hat{\mathbf{r}}\cdot\mathbf{S}(\mathbf{r},t)$$

Momentum analysis of electromagnetic fields and sources

$$\frac{d\mathbf{P}_{mech}}{dt} \equiv \int d^3r \quad \left(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\right)$$

Follows by analogy with Lorentz force:

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$$

$$\mathbf{P}_{field} = \varepsilon_0 \int d^3r \quad \left(\mathbf{E}\times\mathbf{B}\right)$$

Expression for vacuum fields:

$$\left(\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt}\right)_i = \sum_j \int d^3r \, \frac{\partial T_{ij}}{\partial r_j}$$

Maxwell stress tensor:

$$T_{ij} \equiv \varepsilon_0 \left(E_i E_j + c^2 B_i B_j - \delta_{ij} \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B}\right)\right)$$

Summary -- By considering a complete system involving self-contained sources and fields, we examined the energy and force relationships and introduce energy and force equivalents of the electromagnetic fields

Electromagnetic energy density: $u \equiv \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}\right)$

Poynting vector: $\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$

Differential relationship: $\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J}_{free}$

Maxwell stress tensor (for vacuum case):

$$T_{ij} \equiv \varepsilon_0\left(E_iE_j + c^2B_iB_j - \delta_{ij}\frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B}\right)\right)$$

Integral relationships:

$$\frac{dE_{mech}}{dt} \equiv \int d^3r \;\; \mathbf{E}\cdot\mathbf{J}_{free}$$

$$E_{field} \equiv \int d^3r \;\; u(\mathbf{r},t)$$

$$\Rightarrow \frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt} = -\int d^3r \;\; \nabla\cdot\mathbf{S}(\mathbf{r},t) = -\oint d^2r \; \hat{\mathbf{r}}\cdot\mathbf{S}(\mathbf{r},t)$$

$$\left(\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt}\right)_i = \sum_j \int d^3r \, \frac{\partial T_{ij}}{\partial r_j}$$

Comment on treatment of time-harmonic fields

Fourier transformation in time domain :

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, \widetilde{\mathbf{E}}(\mathbf{r},\omega)\, e^{-i\omega t}$$

$$\widetilde{\mathbf{E}}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} dt\, \mathbf{E}(\mathbf{r},t)\, e^{i\omega t}$$

Note that $\mathbf{E}(\mathbf{r},t)$ is real $\Rightarrow \widetilde{\mathbf{E}}(\mathbf{r},\omega) = \widetilde{\mathbf{E}}^{*}(\mathbf{r},-\omega)$

These relations and the notion of the superposition principle, lead to the common treatment:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} \right) \equiv \frac{1}{2}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{E}}^{*}(\mathbf{r},\omega)e^{i\omega t}\right)$$

Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Equations	in time domain	in frequency domain
Coulomb's law:	$\nabla \cdot \mathbf{D} = \rho_{free}$	$\nabla \cdot \widetilde{\mathbf{D}} = \widetilde{\rho}_{free}$
Ampere-Maxwell's law:	$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$	$\nabla \times \widetilde{\mathbf{H}} + i\omega\widetilde{\mathbf{D}} = \widetilde{\mathbf{J}}_{free}$
Faraday's law:	$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$	$\nabla \times \widetilde{\mathbf{E}} - i\omega\widetilde{\mathbf{B}} = 0$
No magnetic monopoles:	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \widetilde{\mathbf{B}} = 0$

Note -- in all of these, the real part is taken at the end of the calculation.

Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields :

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Poynting vector: $\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)$

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{4}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)\times\left(\widetilde{\mathbf{H}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{H}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

$$= \frac{1}{4}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)\times\widetilde{\mathbf{H}}^*(\mathbf{r},\omega) + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)\times\widetilde{\mathbf{H}}(\mathbf{r},\omega)\right)$$

$$+ \frac{1}{4}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)\times\widetilde{\mathbf{H}}(\mathbf{r},\omega)e^{-2i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)\times\widetilde{\mathbf{H}}^*(\mathbf{r},\omega)e^{2i\omega t}\right)$$

$$\left\langle\mathbf{S}(\mathbf{r},t)\right\rangle_{t\,\text{avg}} = \Re\left(\frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)\times\widetilde{\mathbf{H}}^*(\mathbf{r},\omega)\right)\right)$$

Maxwell's equations

Coulomb's law: $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere-Maxwell's law: $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

Maxwell's equations

For linear isotropic media -- $\mathbf{D} = \varepsilon\mathbf{E}; \quad \mathbf{B} = \mu\mathbf{H}$

and no sources:

Coulomb's law: $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

Analysis of Maxwell's equations without sources -- continued:

Coulomb's law: $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

$$\nabla \times \left(\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) = -\nabla^2 \mathbf{B} - \mu\varepsilon \frac{\partial \left(\nabla \times \mathbf{E} \right)}{\partial t}$$

$$= -\nabla^2 \mathbf{B} + \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla \times \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) = -\nabla^2 \mathbf{E} + \frac{\partial \left(\nabla \times \mathbf{B} \right)}{\partial t}$$

$$= -\nabla^2 \mathbf{E} + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Analysis of Maxwell's equations without sources -- continued:

Both E and B fields are solutions to a wave equation:

$$\nabla^2 \mathbf{B} - \frac{1}{v^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{E} - \frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\text{where} \quad v^2 \equiv c^2 \frac{\mu_0 \varepsilon_0}{\mu \varepsilon} \equiv \frac{c^2}{n^2}$$

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

Analysis of Maxwell's equations without sources -- continued:

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \qquad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$$

Note: ε, μ, n, k can all be complex; for the moment we will assume that they are all real (no dissipation).

Note that $\mathbf{E}_0$ and $\mathbf{B}_0$ are not independent;

from Faraday's law: $\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$

$$\Rightarrow \mathbf{B}_0 = \frac{\mathbf{k}\times\mathbf{E}_0}{\omega} = \frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}$$

also note: $\hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$ and $\hat{\mathbf{k}}\cdot\mathbf{B}_0 = 0$

Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

$$\mathbf{B}(\mathbf{r},t)=\Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2=\left(\frac{\omega}{v}\right)^2=\left(\frac{n\omega}{c}\right)^2 \quad \text{where } n\equiv\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0=0$$

Poynting vector for plane electromagnetic waves:

$$\langle\mathbf{S}\rangle_{avg}=\frac{1}{2}\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\times\frac{1}{\mu}\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right)$$

$$=\frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}}=\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\hat{\mathbf{k}}$$

Note that:

$$\mathbf{E}_0\times\left(\hat{\mathbf{k}}\times\mathbf{E}_0\right)=\hat{\mathbf{k}}\left(\mathbf{E}_0\cdot\mathbf{E}_0\right)-\mathbf{E}_0\left(\hat{\mathbf{k}}\cdot\mathbf{E}_0\right)$$

$$=\hat{\mathbf{k}}|\mathbf{E}_0|^2$$

Analysis of Maxwell's equations without sources -- continued:

Transverse Electric and Magnetic (TEM) waves

Summary of plane electromagnetic waves:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

Energy density for plane electromagnetic waves:

$$\langle u\rangle_{avg} = \frac{1}{4}\Re\left(\varepsilon\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\cdot\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right) +$$

$$\frac{1}{4}\Re\left(\frac{1}{\mu}\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\cdot\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right)$$

$$= \frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$