PHY 712 Electrodynamics
10-10:50 AM in Olin 103

Notes for Lecture 18:

Continue reading Chapter 7 (Sec. 7.5,7.10 in JDJ)

1. Real and imaginary contributions to electromagnetic response
2. Frequency dependence of dielectric materials; Drude model
3. Kramers-Kronig relationships

17	Fri: 02/21/2025	Chap. 7	Electromagnetic plane waves	#16	02/24/2025
18	Mon: 02/24/2025	Chap. 7	Electromagnetic response functions	#17	02/26/2025
19	Wed: 02/26/2025	Chap. 7	Optical effects of refractive indices		
20	Fri: 02/28/2025				
21	Mon: 03/03/2025				
22	Wed: 03/05/2025				
23	Fri: 03/07/2025		Review		
	Mon: 03/10/2025	No class	*Spring Break*		
	Wed: 03/12/2025	No class	*Spring Break*		
	Fri: 03/14/2025	No class	*Spring Break*		
	Mon: 03/17/2025	No class	*Take-home exam*		
	Wed: 03/19/2025	No class	*Take-home exam*		
	Fri: 03/21/2025	No class	*Take-home exam*		

PHY 712 -- Assignment #17

Assigned: 2/24/2025 Due: 2/26/2025

Continue reading Chapter 7, particularly Sec. 7.10 in **Jackson** .

1. Work problem 7.22 (a) in **Jackson**. In addition to the analytic results, plot the real and imaginary parts of the permittivity as a function of ω for your favorite values of the constants.

BIOGRAPHY OF PAUL DRUDE (1863 - 1906)

born July. 12, 1863, Braunschweig , Germany.
died July 5, 1906, Berlin , Germany.

Paul Karl Ludwig Drude (July 12, 1863 - July 5, 1906) was a German physicist specializing in optics. Known for the Drude model. He wrote a fundamental textbook integrating optics with Maxwell's theories of electromagnetism. He was born in Braunschweig, the son of a physician.

Drude began his studies in mathematics at the University of Go"ttingen, but later changed his major to physics. His dissertation covering the reflection and diffraction of light in crystals was completed in 1887, under Woldemar Voigt.

In 1894 Drude became an extraordinarius professor at the University of Leipzig; in

Drude graduated the year Heinrich Hertz began publishing his findings from his experiments on the electromagnetic theories of James Clerk Maxwell. Thus Drude began his professional career at the time Maxwell's theories were being introduced into Germany. His first experiments were the determination of the optical constants of various solids, measured to unprecedented levels of accuracy. He then worked to derive relationships between the optical and electrical constants and the physical structure of substances. In 1894 he was responsible for introducing the symbol "c" for the speed of light in a perfect vacuum.

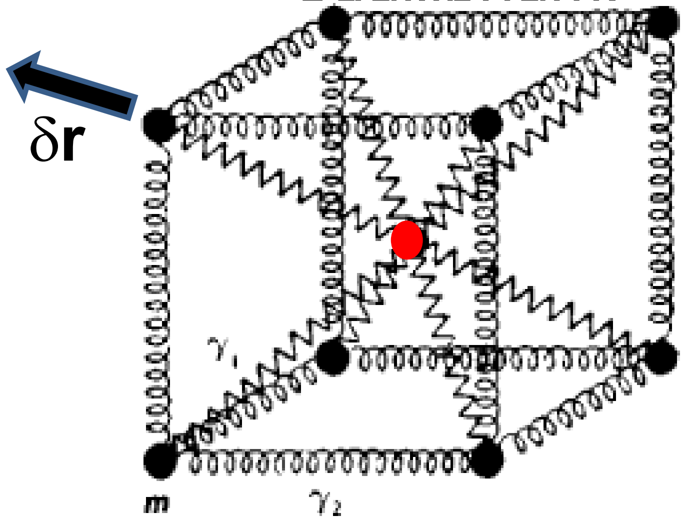
Toward the end of his tenure at Leipzig, Drude was invited to write a textbook on optics, which he accepted. The book, *Lehrbuch der Optik*, published in 1900, brought together the formerly distinct subjects of electricity and optics, which was cited by Drude as an "epoch-making advance in natural science."

In 1900 he developed a powerful model to explain the thermal, electrical, and optical properties of matter. The Drude model would be further advanced in 1933 by Arnold Sommerfeld and Hans Bethe, becoming the *Drude-Sommerfeld-Model.*

http://theor.jinr.ru/~kuzemsky/drudbio.html

Drude model:

Vibration of particle of charge q and mass m near equilibrium:



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$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2 \delta\mathbf{r} - m\gamma\delta\dot{\mathbf{r}}$$

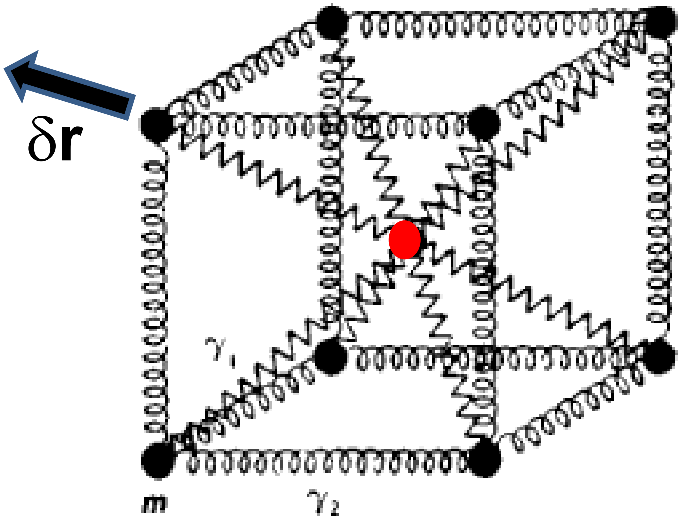
Note that:

- ❑ $\gamma > 0$ represents dissipation of energy.
- ❑ ω_0 represents the natural frequency of the vibration; $\omega_0=0$ would represent a free (unbound) particle

Note that this version of the model does not include consideration of any spatial variation.

Drude model:

Vibration of particle of charge *q* and mass *m* near equilibrium:



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$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2 \delta\mathbf{r} - m\gamma\delta\dot{\mathbf{r}}$$

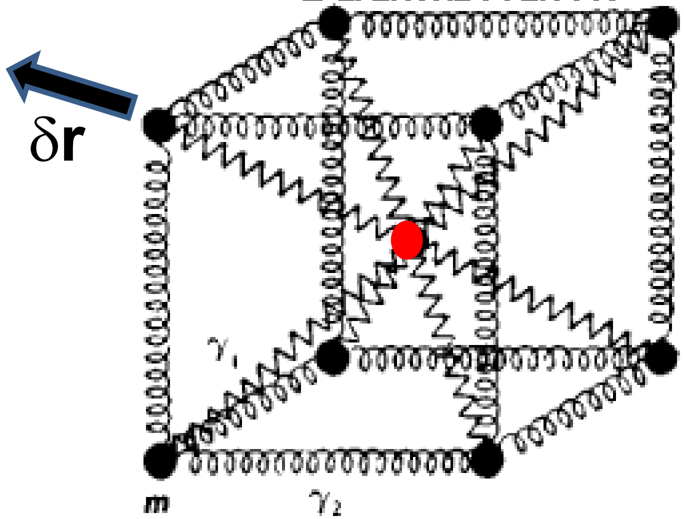
$$\text{For} \quad \delta\mathbf{r} \equiv \delta\mathbf{r}_0 e^{-i\omega t}, \qquad \delta\mathbf{r}_0 = \frac{q\mathbf{E}_0}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

Induced dipole:

$$\mathbf{p} = q\,\delta\mathbf{r} = \frac{q^2\mathbf{E}_0}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} e^{-i\omega t}$$

Drude model:

Vibration of particle of charge *q* and mass *m* near equilibrium:



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$$m\delta\,\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\,\mathbf{r} - m\gamma\delta\,\dot{\mathbf{r}}$$

Displacement field:

$$\mathbf{D} = \varepsilon\mathbf{E} \quad = \varepsilon_0\mathbf{E} + \mathbf{P}$$

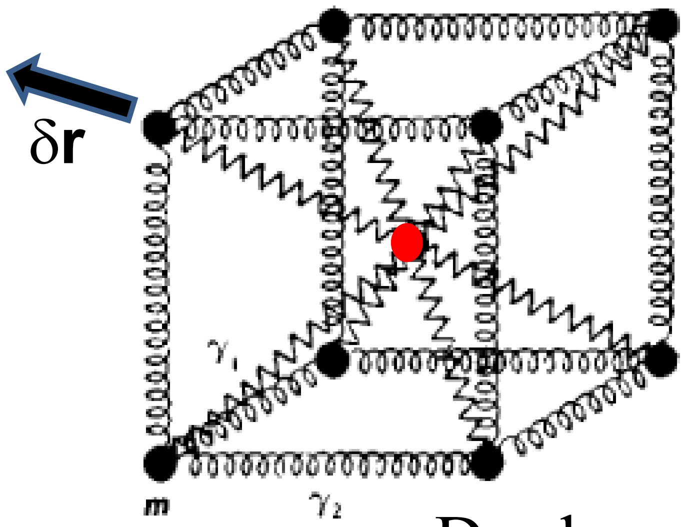
$$\mathbf{P} = \sum_i \mathbf{p}_i \delta^3\left(\mathbf{r} - \mathbf{r}_i\right) \approx N\sum_i f_i\ \mathbf{p}_i$$

$N \equiv$ number of dipoles/volume

$f_i \equiv$ fraction of type i dipoles

Drude model:

Vibration of particle of charge q and mass m near equilibrium:



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Drude model expression for permittivity:

$$\mathbf{D} = \varepsilon \mathbf{E} \;\; = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \mathbf{E} + N \sum_i f_i \; \mathbf{p}_i$$

$$\mathbf{p}_i = q_i \, \delta \mathbf{r} = \frac{q_i^{\,2} \mathbf{E}_0}{m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i} e^{-i\omega t}$$

$$\varepsilon \mathbf{E} \;=\; \varepsilon_0 \mathbf{E}_0 e^{-i\omega t} \left(1 + N \sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i} \right)$$

Drude model dielectric function:

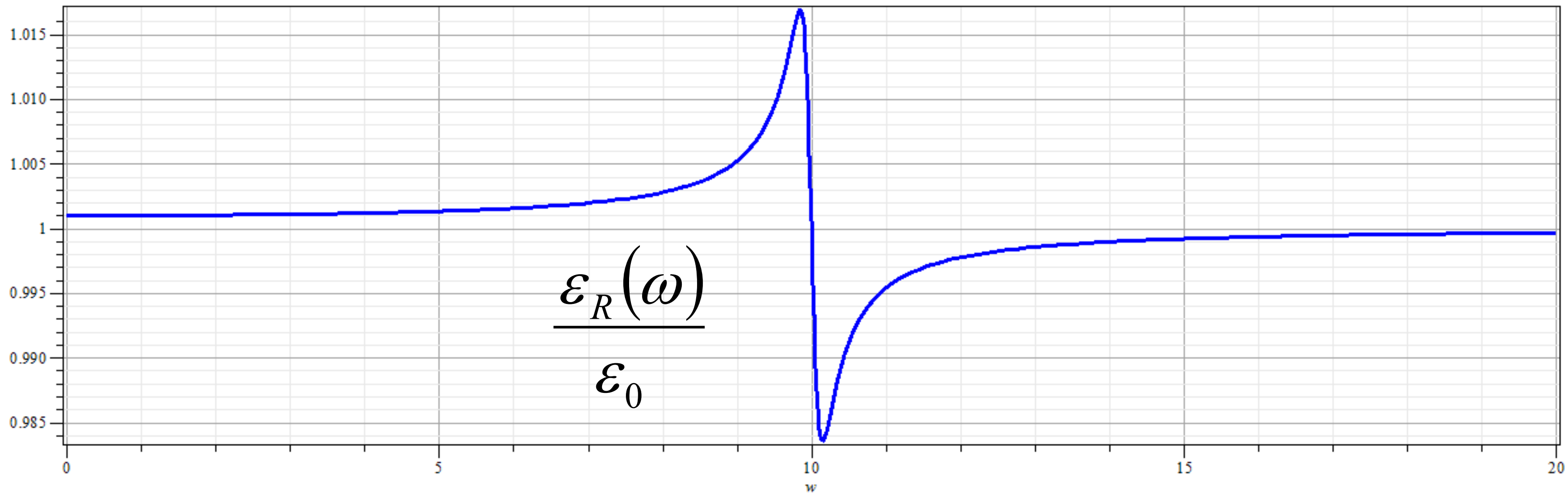
$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\ 2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

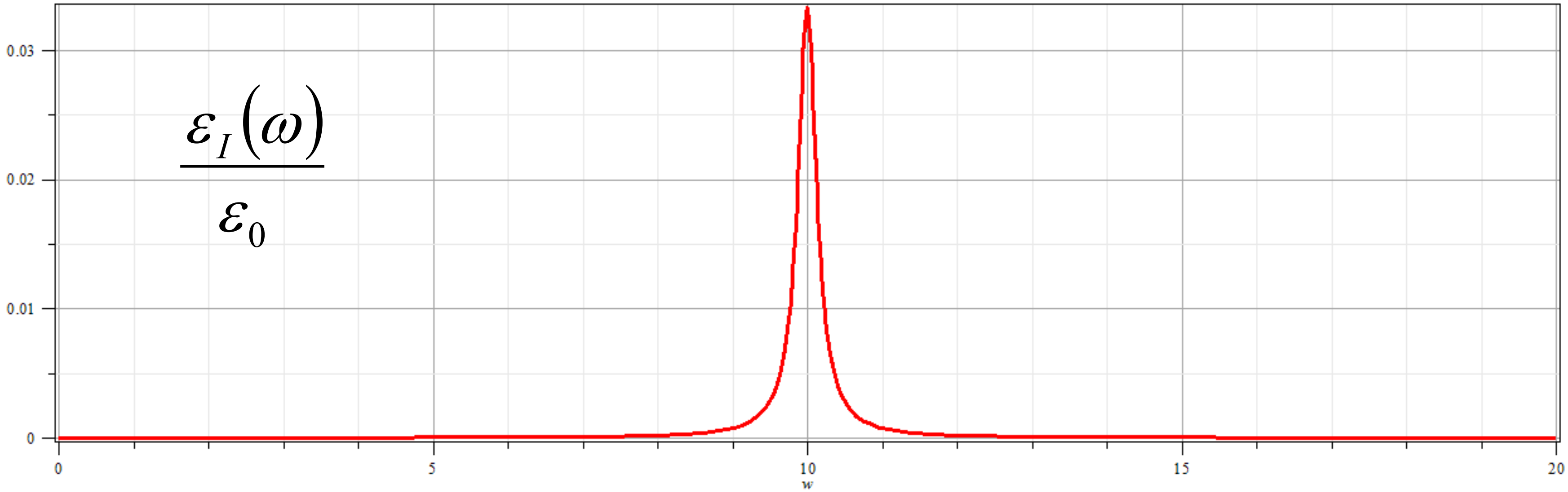
$$= \frac{\varepsilon_R(\omega)}{\varepsilon_0} + i\frac{\varepsilon_I(\omega)}{\varepsilon_0}$$

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\ 2}}{\varepsilon_0 m_i} \frac{\omega_i^2 - \omega^2}{\left(\omega_i^2 - \omega^2\right)^2 + \omega^2 \gamma_i^{\ 2}}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = N\sum_i f_i \frac{q_i^{\ 2}}{\varepsilon_0 m_i} \frac{\omega\gamma_i}{\left(\omega_i^2 - \omega^2\right)^2 + \omega^2 \gamma_i^{\ 2}}$$

Drude model dielectric function:





Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

For $\omega >> \omega_i$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} \approx 1 - \frac{1}{\omega^2}\underbrace{\left(N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\right)}_{\equiv\,\omega_P^2}$$

$$\equiv 1 - \frac{\omega_P^2}{\omega^2}$$

Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i}\frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

For $\omega_0 = 0$ (representing a free particle of charge q_0, mass m_0

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = \underbrace{1 + N\sum_{i>0} f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i}\frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}}_{\equiv \frac{\varepsilon_b(\omega)}{\varepsilon_0}} + \underbrace{iNf_0 \frac{q_0^{\,2}}{\varepsilon_0 m_0}\frac{1}{\omega(\gamma_0 - i\omega)}}_{i\frac{\sigma(\omega)}{\varepsilon_0\omega}}$$

Some details:

$$\mathbf{D} = \varepsilon_b \mathbf{E} \qquad \mathbf{J} = \sigma\mathbf{E}$$

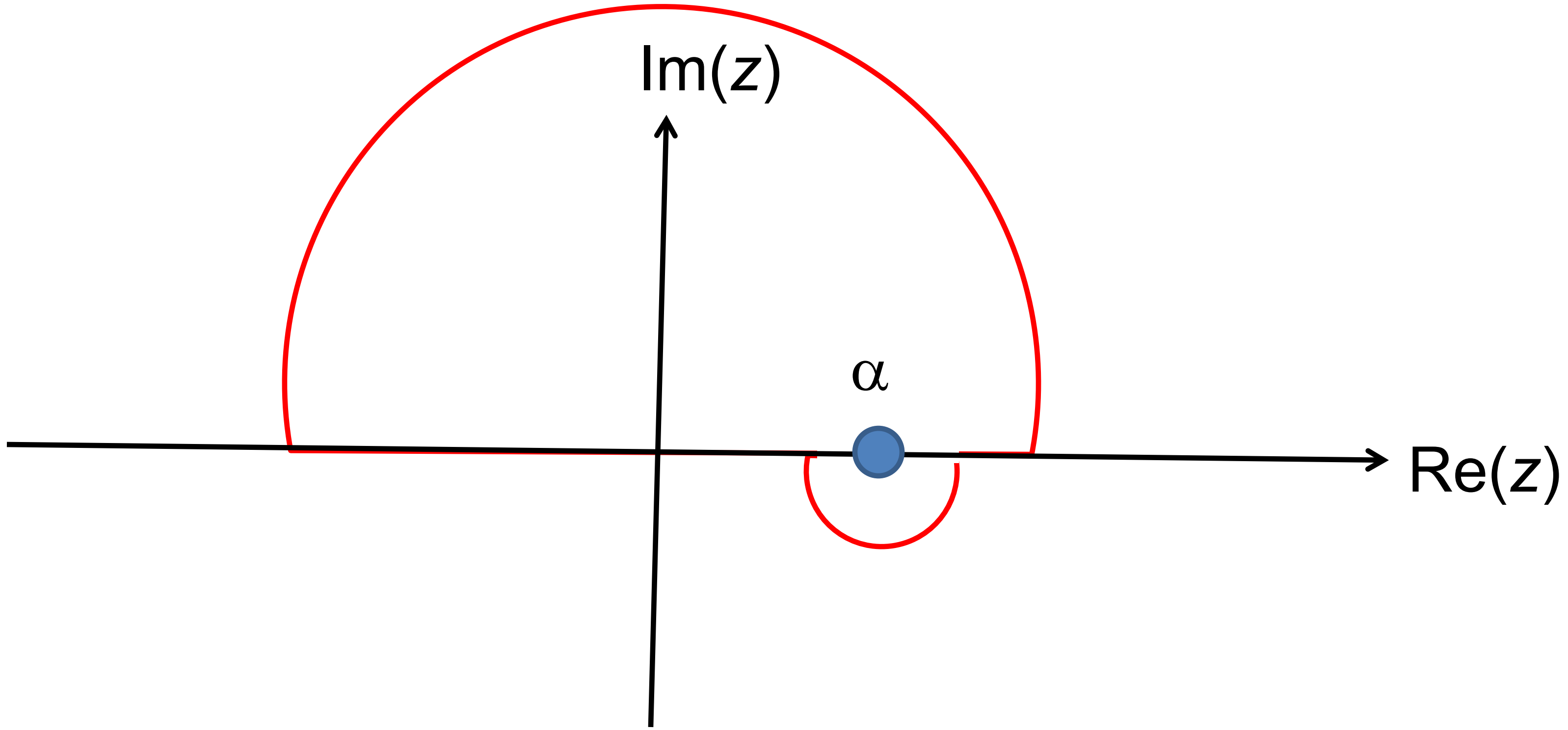
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = (\sigma - i\omega\varepsilon_b)\mathbf{E} = \varepsilon\frac{\partial \mathbf{E}}{\partial t} = -i\omega\left(\varepsilon_b + \frac{i\sigma}{\omega}\right)\mathbf{E}$$

$$\Rightarrow \sigma(\omega) = Nf_0 \frac{q_0^{\,2}}{m_0}\frac{1}{(\gamma_0 - i\omega)}$$

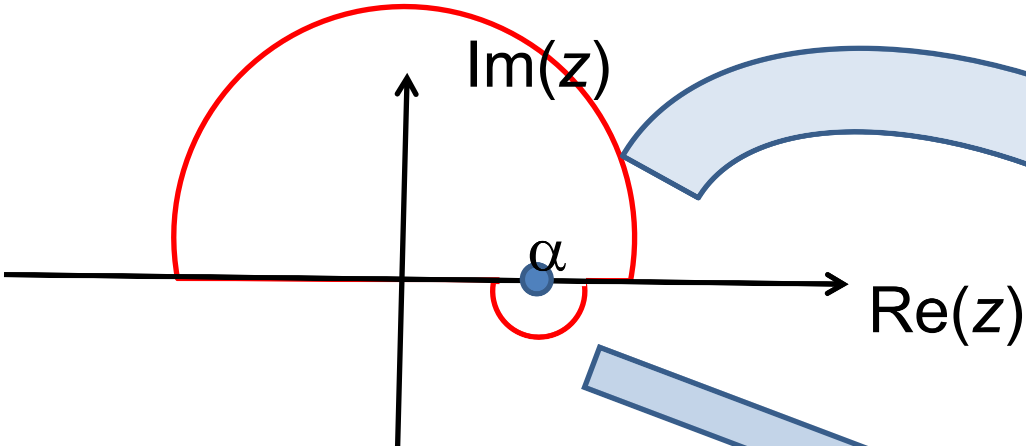
Analytic properties of the dielectric function (in the Drude model or from “first principles” -- Kramers-Kronig transform

Consider Cauchy's integral formula for an analytic function $f(z)$:

$$\oint dz\, f(z) = 0 \qquad\qquad f(\alpha) = \frac{1}{2\pi i} \oint_{includes\ \alpha} dz\, \frac{f(z)}{z - \alpha}$$



Kramers-Kronig transform -- continued



=0

$$f(\alpha) = \frac{1}{2\pi i} \oint_{includes\ \alpha} dz \ \frac{f(z)}{z - \alpha} = \frac{1}{2\pi i}\left(\int_{-\infty}^{\infty} dz_R \ \frac{f(z_R)}{z_R - \alpha} + \int_{rest} dz \ \frac{f(z)}{z - \alpha} \right)$$

$$f(\alpha) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dz_R \frac{f(z_R)}{z_R - \alpha} = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \frac{f(z_R)}{z_R - \alpha} + \frac{1}{2} f(\alpha)$$

Kramers-Kronig transform -- continued

$$f(\alpha) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \frac{f(z_R)}{z_R - \alpha} + \frac{1}{2} f(\alpha)$$

Suppose $f(z_R) = f_R(z_R) + i f_I(z_R)$:

$$\Rightarrow \frac{1}{2}\left(f_R(\alpha) + i f_I(\alpha)\right) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \frac{f_R(z_R) + i f_I(z_R)}{z_R - \alpha}$$

$$\Rightarrow f_R(\alpha) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \frac{f_I(z_R)}{z_R - \alpha}$$

$$f_I(\alpha) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \frac{f_R(z_R)}{z_R - \alpha}$$

Kramers-Kronig transform -- continued

$$f_R(\alpha) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \, \frac{f_I(z_R)}{z_R - \alpha}$$

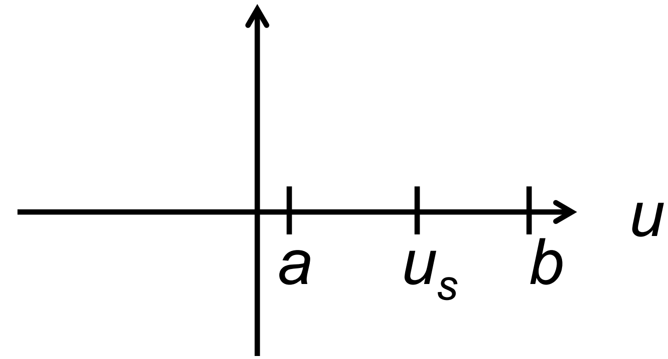
$$f_I(\alpha) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \, \frac{f_R(z_R)}{z_R - \alpha}$$

This Kramers-Kronig transform is useful for the dielectric function

when $f(z_R) \Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} - 1$

Must show that: 1. $f(z)$ is analytic for $z_I > 0$

2. $f(z)$ vanishes for $z \rightarrow \infty$



Some practical considerations

Principal parts integration :

$$P\int_a^b du\; g(u) = \lim_{\upsilon \to 0}\left(\int_a^{u_s-\upsilon} du\; g(u) + \int_{u_s+\upsilon}^{b} du\; g(u) \right)$$

Example :

$$P\int_a^b du\; \frac{1}{u\text{-}u_s} = \lim_{\upsilon \to 0}\left(\int_a^{u_s-\upsilon} du\, \frac{1}{u\text{-}u_s} + \int_{u_s+\upsilon}^{b} du\, \frac{1}{u\text{-}u_s} \right)$$

$$= \lim_{\upsilon \to 0}\left(\ln\left(\frac{\upsilon}{u_s - a}\right) + \ln\left(\frac{b-u_s}{\upsilon}\right) \right) = \ln\left(\frac{b-u_s}{u_s-a}\right)$$

More practical considerations

For dielectric function $\varepsilon(\omega)$:

$$\varepsilon(-\omega) = \varepsilon^*(\omega)$$

$$\Rightarrow \varepsilon_R(-\omega) = \varepsilon_R(\omega)$$

$$\Rightarrow \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

Analytic properties the dielectric function which justify

the treatment of $f(z) \Rightarrow \frac{\varepsilon(z)}{\varepsilon_0} - 1$

Must show that: 1. $f(z)$ is analytic for $z_I > 0$

2. $f(z)$ vanishes for $z \to \infty$ (for $z_I > 0$)

Analysis for Drude model dielectric function:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

Let $f(z) = \dfrac{\varepsilon(z)}{\varepsilon_0} - 1 = N\sum_i f_i \dfrac{q_i^{\,2}}{\varepsilon_0 m_i} \dfrac{1}{\omega_i^2 - z^2 - iz\gamma_i}$

For $|z| >> \omega_i$

$$f(z) \approx -\frac{1}{z^2}\left(N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i}\right) \quad \Rightarrow \text{vanishes at large } z$$

Analysis for Drude model dielectric function – continued -- Analytic properties:

$$f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^{\ 2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2 - z_P^{\ 2} - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2}$$

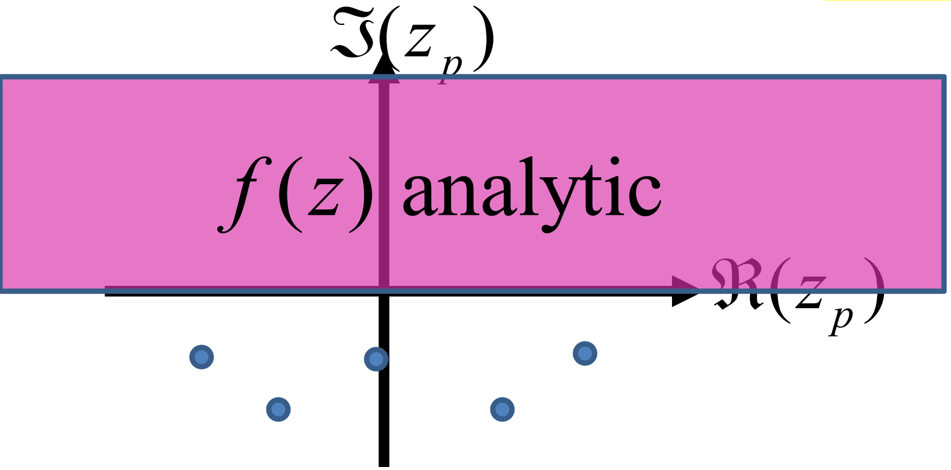
Note that $\Im(z_P) \leq 0 \quad \Rightarrow f(z)$ is analytic for $\Im(z_P) > 0$

$$f(z)=\frac{\varepsilon(z)}{\varepsilon_0}-1=N\sum_i f_i \frac{q_i^{\;2}}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-z^2-iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2-z_P^{\;2}-iz_P\gamma_i=0$

$$z_P=-i\frac{\gamma_i}{2}\pm\sqrt{\omega_i^2-\left(\frac{\gamma_i}{2}\right)^2}$$

Note that $\Im(z_P)\leq 0 \;\Rightarrow f(z)$ is analytic for $\Im(z_P)>0$



Because of these analytic properties, Cauchy's integral theorem results in:

Kramers-Kronig transform – for dielectric function:

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\varepsilon_I(\omega')}{\varepsilon_0} \frac{1}{\omega' - \omega}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \left(\frac{\varepsilon_R(\omega')}{\varepsilon_0} - 1 \right) \frac{1}{\omega' - \omega}$$

$$\text{with} \quad \varepsilon_R(-\omega) = \varepsilon_R(\omega); \quad \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

Is this useful???

Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0 \left\{ \mathbf{E}(\mathbf{r},t) + \int_0^\infty d\tau \; G(\tau)\mathbf{E}(\mathbf{r},t-\tau) \right\}$$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \int_0^\infty d\tau \; G(\tau) e^{i\omega\tau}$$

Some details: Consider a convolution integral such as

$$f(t) = \int_{-\infty}^{\infty} g(t')h(t-t')dt'$$ where the functions $f(t), g(t),$ and $h(t)$

are all well-defined functions with Fourier transforms such as

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t')e^{i\omega t'}dt' \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{-i\omega t}d\omega$$

It follows that: $\tilde{f}(\omega) = \tilde{g}(\omega)\tilde{h}(\omega)$

Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r},t)=\varepsilon_0\left\{\mathbf{E}(\mathbf{r},t)+\int_0^{\infty}d\tau\; G(\tau)\mathbf{E}(\mathbf{r},t-\tau)\right\}$$

$$G(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}-1\right)e^{-i\omega\tau}d\omega \qquad \tilde{G}(\omega)=\frac{\varepsilon(\omega)}{\varepsilon_0}-1=\int_0^{\infty}d\tau\; G(\tau)e^{i\omega\tau}$$

For $\displaystyle \frac{\varepsilon(\omega)}{\varepsilon_0}-1=\frac{N}{\varepsilon_0}\sum_i f_i\frac{q_i^2}{m_i}\frac{1}{\omega_i^2-\omega^2-i\omega\gamma_i}$

$$G(\tau)=\frac{N}{\varepsilon_0}\sum_i f_i\frac{q_i^2}{m_i}e^{-\gamma_i\tau/2}\frac{\sin(\nu_i\tau)}{\nu_i}\Theta(\tau)$$

where $\nu_i\equiv\sqrt{\omega_i^2-\gamma_i^2/4}$

Some details

$$G(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\varepsilon(\omega)}{\varepsilon_0} - 1\right)e^{-i\omega\tau}d\omega = \frac{1}{2\pi}\oint f(z)e^{-iz\tau}dz$$

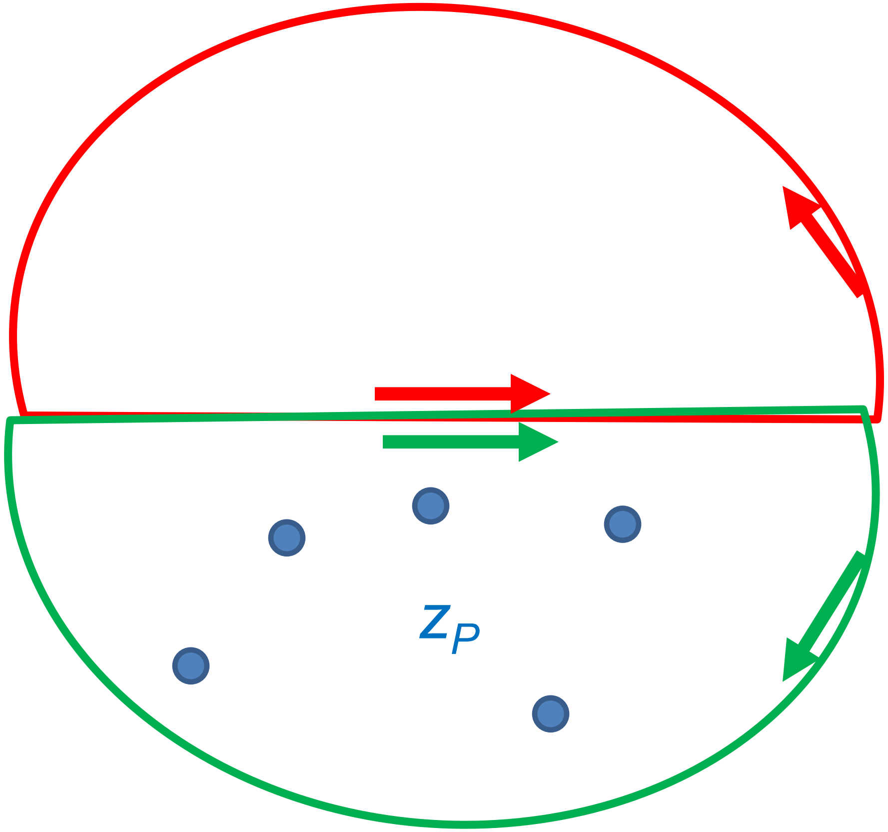
Let $f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N\sum_i f_i \frac{q_i^{\ 2}}{\varepsilon_0 m_i}\frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$

$f(z)$ has poles z_P at $\omega_i^2 - z_P^{\ 2} - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2} \quad \text{or} \quad z_P = -i\left(\frac{\gamma_i}{2} \pm \sqrt{\left(\frac{\gamma_i}{2}\right)^2 - \omega_i^2}\right)$$

$$G(\tau) = \frac{1}{2\pi}\oint f(z) e^{-iz\tau} dz = i \sum_P \text{Res}(z_P)$$

Note that: $e^{-iz\tau} = e^{-iz_R\tau} e^{z_I\tau}$



Valid contour for $\tau < 0$

$G(\tau) = 0$ for $\tau < 0$

Valid contour for $\tau > 0$

$$G(\tau) = \frac{N}{\varepsilon_0} \sum_i f_i \frac{q_i^2}{m_i} e^{-\gamma_i \tau/2} \frac{\sin(\nu_i \tau)}{\nu_i}$$

$$G(\tau) = \frac{1}{2\pi} \oint f(z) e^{-iz\tau} dz = i \sum_P \mathrm{Res}(z_P)$$

Let $f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$

$f(z)$ has poles z_P at $\omega_i^2 - z_P^{\,2} - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2} \quad \text{or} \quad z_P = -i\left(\frac{\gamma_i}{2} \pm \sqrt{\left(\frac{\gamma_i}{2}\right)^2 - \omega_i^2}\right)$$

$$G(\tau) = \frac{N}{\varepsilon_0} \sum_i f_i \frac{q_i^{\,2}}{m_i} e^{-\gamma_i \tau/2} \frac{\sin(\nu_i \tau)}{\nu_i} \Theta(\tau)$$

where $\nu_i \equiv \sqrt{\omega_i^2 - \gamma_i^2 / 4}$ assuming $\omega_i^2 - \gamma_i^2 / 4 \geq 0$