PHY 712 Electrodynamics
10-10:50 AM in Olin 103

Discussion for Lecture 19:

Complete reading of Chapter 7 (Sec. 7.1-7.5, 7.10)

1. **Additional results of Drude model of permittivity in materials**
2. **Summary of complex response functions for electromagnetic fields**
3. **Comment on spectral properties of electromagnetic waves**

Physics Colloquium

- Thursday -

4 PM Olin 101

February 27, 2025

Defects, Devices, and Degradation in Metal Halide Perovskite Solar Cells

Metal halide perovskites are an exciting class of printable semiconductor materials with applications in solar, photo and X-ray detectors, LEDs, lasers, and photoelectrochemical systems. Central to technology adaptation is resolving underlying mechanisms of degradation that arise from near-band-edge defects within the perovskite crystal lattice. This talk will discuss the evolution of patented (spectro) electrochemistry-based measurement science approaches to quantify the distribution and energetics of donor and acceptor defects in prototypical perovskite solar cell materials and at buried charge selective interlayers (i.e., hole transport layers). Connections to device performance, benchmarked with time-resolved photoluminescence measurements, will be shown.

Briefly, the group utilizes a solid-state electrolyte top contact that equilibrates with the perovskite film to create "half-cells" of device-relevant material stacks and study the stacks under solar cell-relevant electric fields. This allows for spectroscopic assessment of valence and conduction bands under operando conditions (i.e., relevant electric fields and demonstration of charge transport), as well as quantify near-band edge defects using redox-active hole or electron capturing molecular probes. The combination of spectroscopy and electrochemistry characterizes the energetic distribution of donor defect states at an energy resolution of <10 meV over a range of 10^{11} to 10^{19} defects/ cm^3 . Such detection limits are better than spectroscopic, electronic and photoemission protocols, with speciation (anion versus cation defects) not available in those other approaches. Advancements towards development of in-line characterization (i.e., roll-to-roll) and connections to stability will also be described and benchmarked with respect to photoluminescence and photoelectron spectroscopies.

Prof. Erin L. Ratcliff

Professor of Materials Science and Engineering

Chemistry and Biochemistry

Georgia Institute of Technology

Reception 3:30

Olin Lobby

Colloquium 4:00

Olin 101

15	Mon: 02/17/2025	Chap. 6	Maxwell's Equations	#14	02/19/2025
16	Wed: 02/19/2025	Chap. 6	Electromagnetic energy and forces	#15	02/21/2025
17	Fri: 02/21/2025	Chap. 7	Electromagnetic plane waves	#16	02/24/2025
18	Mon: 02/24/2025	Chap. 7	Electromagnetic response functions	#17	02/26/2025
19	Wed: 02/26/2025	Chap. 7	Optical effects of refractive indices	#18	02/28/2025
20	Fri: 02/28/2025	Chap. 8	Waveguides		
21	Mon: 03/03/2025				
22	Wed: 03/05/2025				
23	Fri: 03/07/2025		Review		
	Mon: 03/10/2025	No class	*Spring Break*		
	Wed: 03/12/2025	No class	*Spring Break*		
	Fri: 03/14/2025	No class	*Spring Break*		
	Mon: 03/17/2025	No class	*Take-home exam*		
	Wed: 03/19/2025	No class	*Take-home exam*		
	Fri: 03/21/2025	No class	*Take-home exam*		

PHY 712 -- Assignment #18

Assigned: 2/26/2025 Due: 2/28/2025

Continue reading Chapter 7, particularly Sec. 7.4 in **Jackson** .

1. This problem concerns the phenomenon of total internal reflection. Imagine that plane polarized monochromatic light is relected and refracted at an interface between two media which have real refractive indices $n=2$ and $n'=1.1$. Draw a diagram of the incident, reflected, and refracted beams. What is the range of incident angles i for which total internal reflection occurs?

More results for Drude model dielectric function:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\;2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

Let $f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N\sum_i f_i \frac{q_i^{\;2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$

For $|z| >> \omega_i$

$$f(z) \approx -\frac{1}{z^2}\left(N\sum_i f_i \frac{q_i^{\;2}}{\varepsilon_0 m_i} \right) \Rightarrow \text{vanishes at large } z$$

Analysis for Drude model dielectric function – continued -- Analytic properties:

$$f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2 - z_P^{\,2} - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2}$$

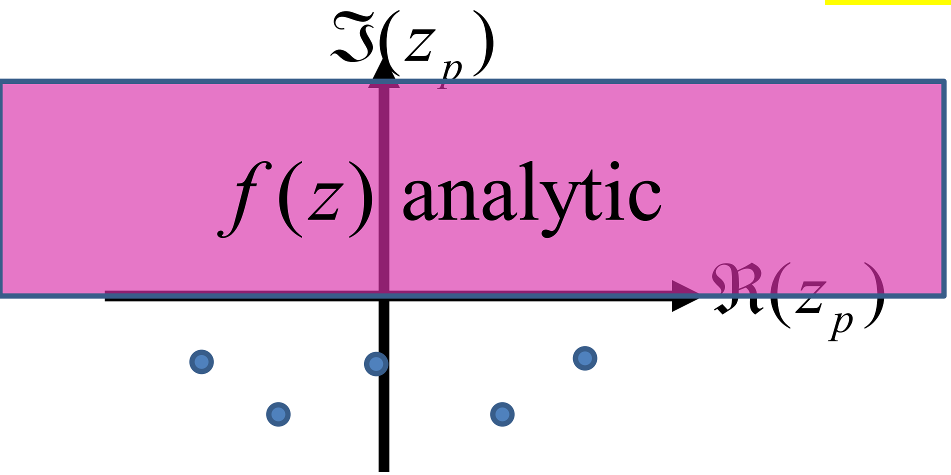
Note that $\Im(z_P) \leq 0 \quad \Rightarrow f(z)$ is analytic for $\Im(z_P) > 0$

$$f(z)=\frac{\varepsilon(z)}{\varepsilon_0}-1=N\sum_i f_i\frac{q_i^{\ 2}}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-z^2-iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2-z_P^{\ 2}-iz_P\gamma_i=0$

$$z_P=-i\frac{\gamma_i}{2}\pm\sqrt{\omega_i^2-\left(\frac{\gamma_i}{2}\right)^2}$$

Note that $\Im(z_P)\leq 0$ $\Rightarrow$ $f(z)$ is analytic for $\Im(z_P)>0$



Because of these analytic properties, Cauchy's integral theorem results in:

Kramers-Kronig transform – for dielectric function:

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\varepsilon_I(\omega')}{\varepsilon_0} \frac{1}{\omega' - \omega}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \left(\frac{\varepsilon_R(\omega')}{\varepsilon_0} - 1 \right) \frac{1}{\omega' - \omega}$$

$$\text{with} \quad \varepsilon_R(-\omega) = \varepsilon_R(\omega); \quad \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0 \left\{ \mathbf{E}(\mathbf{r},t) + \int_0^\infty d\tau \; G(\tau)\mathbf{E}(\mathbf{r},t-\tau) \right\}$$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \int_0^\infty d\tau \; G(\tau) e^{i\omega\tau}$$

Some details: Consider a convolution integral such as

$$f(t) = \int_{-\infty}^{\infty} g(t')h(t-t')dt'$$ where the functions $f(t), g(t),$ and $h(t)$

are all well-defined functions with Fourier transforms such as

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t')e^{i\omega t'}dt' \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{-i\omega t}d\omega$$

It follows that: $\tilde{f}(\omega) = \tilde{g}(\omega)\tilde{h}(\omega)$

Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0 \left\{ \mathbf{E}(\mathbf{r},t) + \int_0^\infty d\tau \; G(\tau)\mathbf{E}(\mathbf{r},t-\tau) \right\}$$

$$G(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \right) e^{-i\omega\tau} d\omega \qquad \tilde{G}(\omega) = \frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \int_0^\infty d\tau \; G(\tau) e^{i\omega\tau}$$

For $\dfrac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \dfrac{N}{\varepsilon_0}\sum_i f_i \dfrac{q_i^2}{m_i} \dfrac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$

$$G(\tau) = \frac{N}{\varepsilon_0}\sum_i f_i \frac{q_i^2}{m_i} e^{-\gamma_i \tau/2} \frac{\sin(\nu_i \tau)}{\nu_i} \Theta(\tau)$$

where $\nu_i \equiv \sqrt{\omega_i^2 - \gamma_i^2 / 4}$

Some details

$$G(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}-1\right)e^{-i\omega\tau}d\omega=\frac{1}{2\pi}\oint f(z)e^{-iz\tau}dz$$

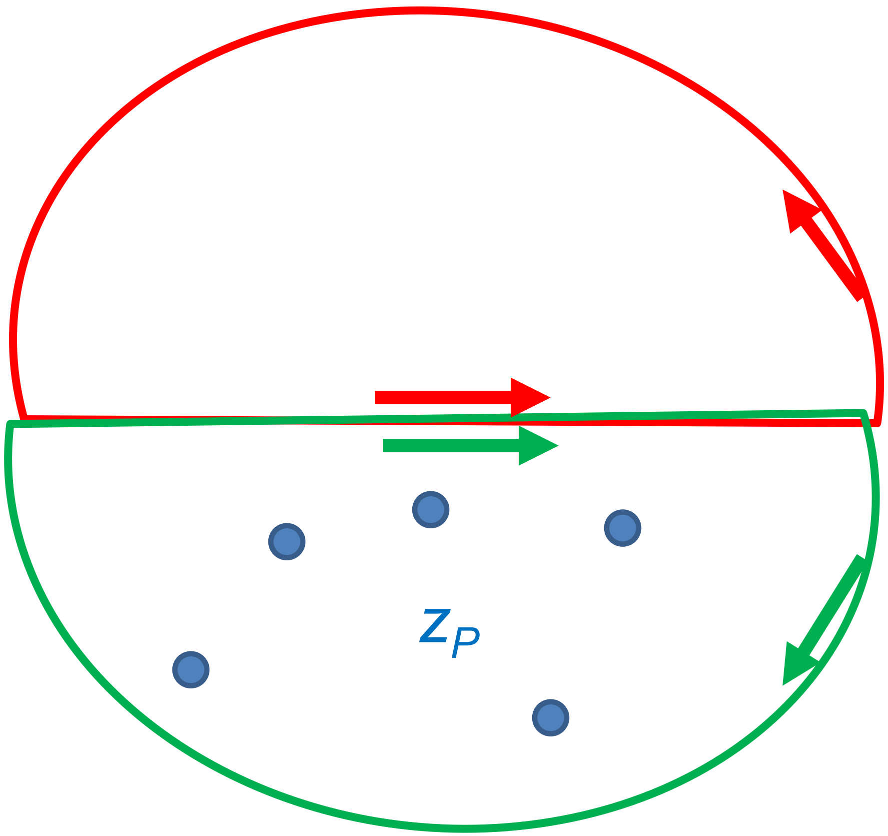
$$\text{Let}\quad f(z)=\frac{\varepsilon(z)}{\varepsilon_0}-\ 1=N\sum_i f_i\frac{q_i^{\ 2}}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-z^2-iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2-z_P^{\ 2}-iz_P\gamma_i=0$

$$z_P=-i\frac{\gamma_i}{2}\pm\sqrt{\omega_i^2-\left(\frac{\gamma_i}{2}\right)^2}\quad\text{or}\quad z_P=-i\left(\frac{\gamma_i}{2}\pm\sqrt{\left(\frac{\gamma_i}{2}\right)^2-\omega_i^2}\right)$$

$$G(\tau) = \frac{1}{2\pi}\oint f(z)e^{-iz\tau}dz = i\sum_P \text{Res}(z_P)$$

Note that: $e^{-iz\tau} = e^{-iz_R\tau}e^{z_I\tau}$



Valid contour for $\tau < 0$

$G(\tau) = 0$ for $\tau < 0$

Valid contour for $\tau > 0$

$$G(\tau) = \frac{N}{\varepsilon_0}\sum_i f_i \frac{q_i^2}{m_i} e^{-\gamma_i\tau/2}\frac{\sin(\nu_i\tau)}{\nu_i}$$

$$G(\tau) = \frac{1}{2\pi}\oint f(z) e^{-iz\tau} dz = i \sum_P \mathrm{Res}(z_P)$$

Let $\quad f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$

$f(z)$ has poles z_P at $\omega_i^2 - z_P^{\,2} - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2} \quad \text{or} \quad z_P = -i\left(\frac{\gamma_i}{2} \pm \sqrt{\left(\frac{\gamma_i}{2}\right)^2 - \omega_i^2}\right)$$

$$G(\tau) = \frac{N}{\varepsilon_0}\sum_i f_i \frac{q_i^{\,2}}{m_i} e^{-\gamma_i \tau/2} \frac{\sin(\nu_i \tau)}{\nu_i} \Theta(\tau)$$

where $\nu_i \equiv \sqrt{\omega_i^2 - \gamma_i^2 / 4}$ $\qquad$ assuming $\omega_i^2 - \gamma_i^2 / 4 \geq 0$

Review of Fresnel equations --
Electromagnetic plane waves in isotropic medium with linear and real permeability and permittivity: μ ε.

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}\cdot\mathbf{r}-ct\right)}\right) \qquad n^2 = c^2\mu\varepsilon$$

$$\mathbf{B}(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

Note that special care is needed when these quantities are complex.

Poynting vector for plane electromagnetic waves:

$$\langle\mathbf{S}\rangle_{avg} = \frac{n\left|\mathbf{E}_0\right|^2}{2\mu c}\hat{\mathbf{k}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left|\mathbf{E}_0\right|^2\hat{\mathbf{k}}$$

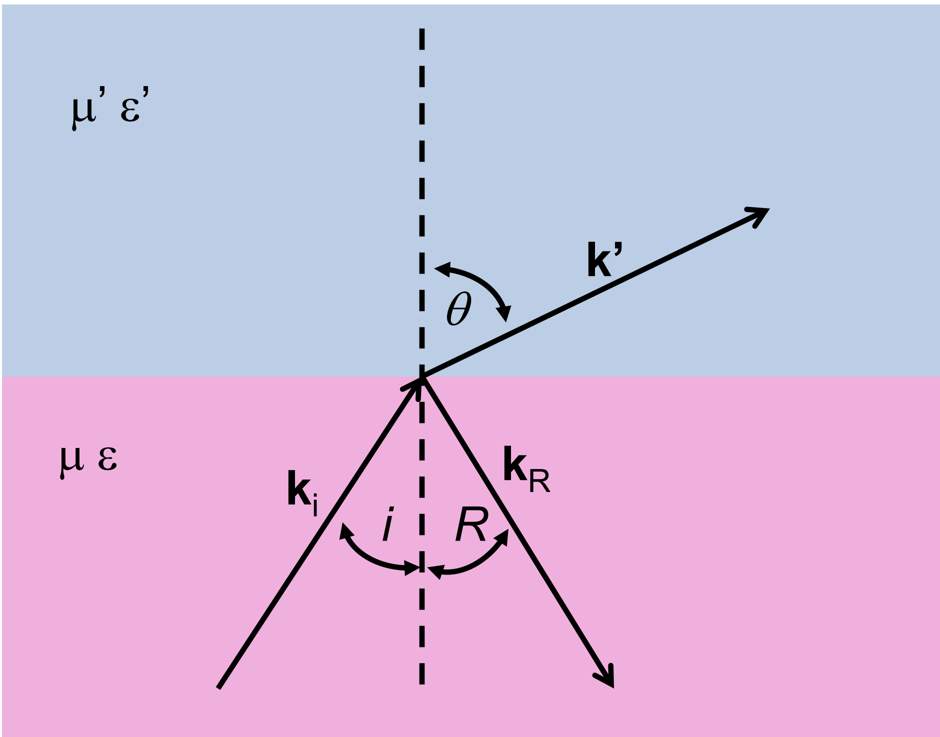
Energy density for plane electromagnetic waves:

$$\langle u\rangle_{avg} = \frac{1}{2}\varepsilon\left|\mathbf{E}_0\right|^2$$

Some comments on the Fresnel Equations

1. Different behaviors of *s* and *p* polarization
2. Brewster's angle
3. Total internal reflection

Review:
Reflection and refraction of plane electromagnetic waves at a plane interface between dielectrics (assumed to be lossless)



$$n' \equiv \sqrt{\frac{\varepsilon' \mu'}{\varepsilon_0 \mu_0}}$$

$$n \equiv \sqrt{\frac{\varepsilon\ \mu}{\varepsilon_0 \mu_0}}$$

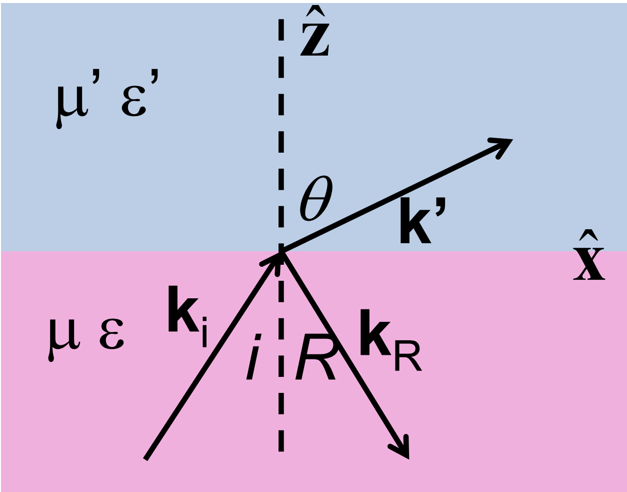
$$i = R$$

$$n \sin i = n' \sin \theta$$

$$|\mathbf{k}_i| = |\mathbf{k}_R| = n\frac{\omega}{c}$$

$$|\mathbf{k}'| = n'\frac{\omega}{c}$$

Review:
Reflection and refraction between two isotropic media



Reflectance, transmittance:

$$\mathrm{R} = \frac{\mathbf{S}_R \cdot \hat{\mathbf{z}}}{\mathbf{S}_i \cdot \hat{\mathbf{z}}} = \left| \frac{E_{0R}}{E_{0i}} \right|^2 \qquad \mathrm{T} = \frac{\mathbf{S}' \cdot \hat{\mathbf{z}}}{\mathbf{S}_i \cdot \hat{\mathbf{z}}} = \left| \frac{E'_0}{E_{0i}} \right|^2 \frac{n'}{n} \frac{\mu}{\mu'} \frac{\cos\theta}{\cos i}$$

Note that $\mathrm{R} + \mathrm{T} = 1$

For s-polarization (E perpendicular to plane of incidence)

$$\frac{E_{0R}}{E_{0i}} = \frac{n\cos i - \frac{\mu}{\mu'} n'\cos\theta}{n\cos i + \frac{\mu}{\mu'} n'\cos\theta} \qquad \frac{E'_0}{E_{0i}} = \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'} n'\cos\theta}$$

Note that: $n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$

For p-polarization (E in plane of incidence)

$$\frac{E_{0R}}{E_{0i}} = \frac{\frac{\mu}{\mu'} n'^2\cos i - nn'\cos\theta}{\frac{\mu}{\mu'} n'^2\cos i + nn'\cos\theta} \qquad \frac{E'_0}{E_{0i}} = \frac{2nn'\cos i}{\frac{\mu}{\mu'} n'^2\cos i + nn'\cos\theta}$$

Note that: $n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$

Special case: normal incidence (i=0, θ=0)

$$\frac{E_{0R}}{E_{0i}} = \frac{\frac{\mu}{\mu'}n' - n}{\frac{\mu}{\mu'}n' + n} \qquad \frac{E'_0}{E_{0i}} = \frac{2n}{\frac{\mu}{\mu'}n' + n}$$

Reflectance, transmittance:

$$\mathrm{R} = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n' - n}{\frac{\mu}{\mu'}n' + n}\right|^2$$

$$\mathrm{T} = \left|\frac{E'_0}{E_{0i}}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'} = \left|\frac{2n}{\frac{\mu}{\mu'}n' + n}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'}$$

Fresnel equations for reflectivity in general --

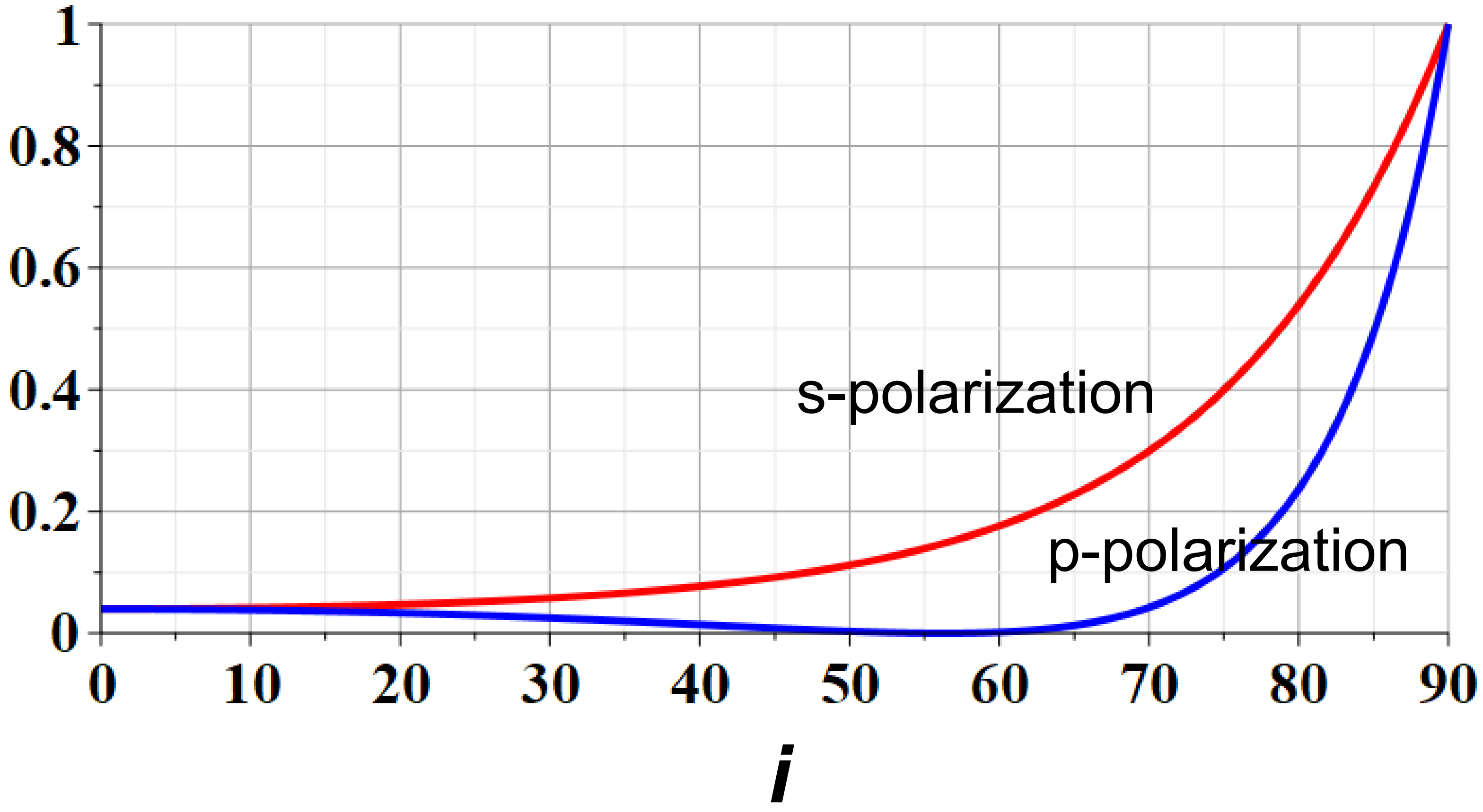
Reflectance for s-polarization

$$R_s = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}\right|^2$$

Reflectance for p-polarization

$$R_p = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n'\cos i - \frac{n}{n'}\sqrt{n'^2 - n^2\sin^2 i}}{\frac{\mu}{\mu'}n'\cos i + \frac{n}{n'}\sqrt{n'^2 - n^2\sin^2 i}}\right|^2$$

Example for $\mu = \mu'$; $n = 1$ and $n' = 1.5$



Analysis --

Reflectance for s-polarization

$$R_s = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}\right|^2 \Rightarrow R_s \neq 0 \quad \text{for typical angles } i$$

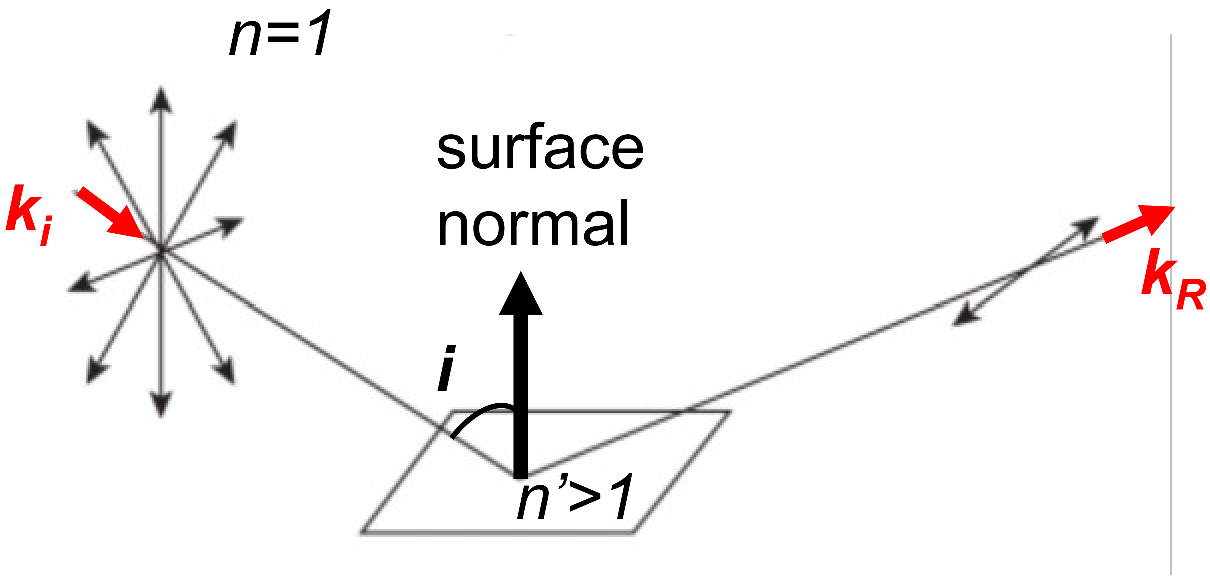
Reflectance for p-polarization

$$R_p = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n'\cos i - \frac{n}{n'}\sqrt{n'^2 - n^2\sin^2 i}}{\frac{\mu}{\mu'}n'\cos i + \frac{n}{n'}\sqrt{n'^2 - n^2\sin^2 i}}\right|^2$$

$$\Rightarrow R_p = 0 \quad \text{when}$$

$$\tan i_B = \frac{n'}{n} \quad \text{for } \mu = \mu'$$

$$i_B \equiv \text{Brewster's angle}$$

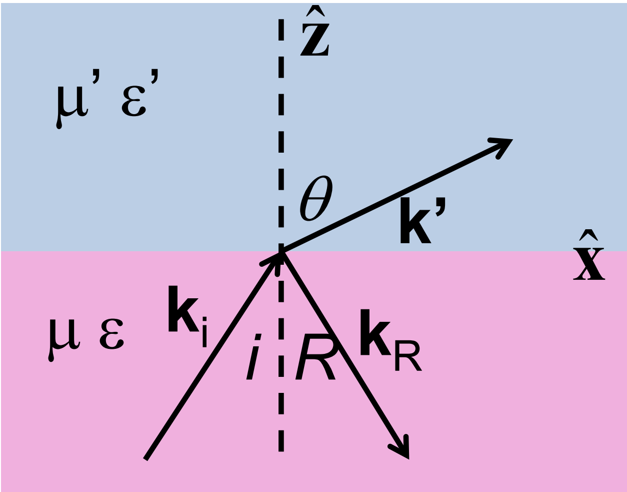


Polarization due to reflection from a refracting surface

Brewster's angle: for $i = i_B$, $\mathrm{R}_p(i_B) = 0$

$$R_p = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n'\cos i - \frac{n}{n'}\sqrt{n'^2 - n^2\sin^2 i}}{\frac{\mu}{\mu'}n'\cos i + \frac{n}{n'}\sqrt{n'^2 - n^2\sin^2 i}}\right|^2 \qquad \text{For } \mu' = \mu, \quad i_B = \tan^{-1}\left(\frac{n'}{n}\right)$$

Reflection and refraction between two isotropic media -- continued



For each wave:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}\cdot\mathbf{r} - ct\right)}\right) \qquad n^2 = c^2\mu\varepsilon$$

$$\mathbf{B}(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

Matching condition at interface:

$$n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$$

Total internal reflection:

If $n > n'$, for $i > i_0 \equiv \sin^{-1}\left(\frac{n'}{n}\right)$,

refracted field no longer propagates in medium $\mu'\ \varepsilon'$

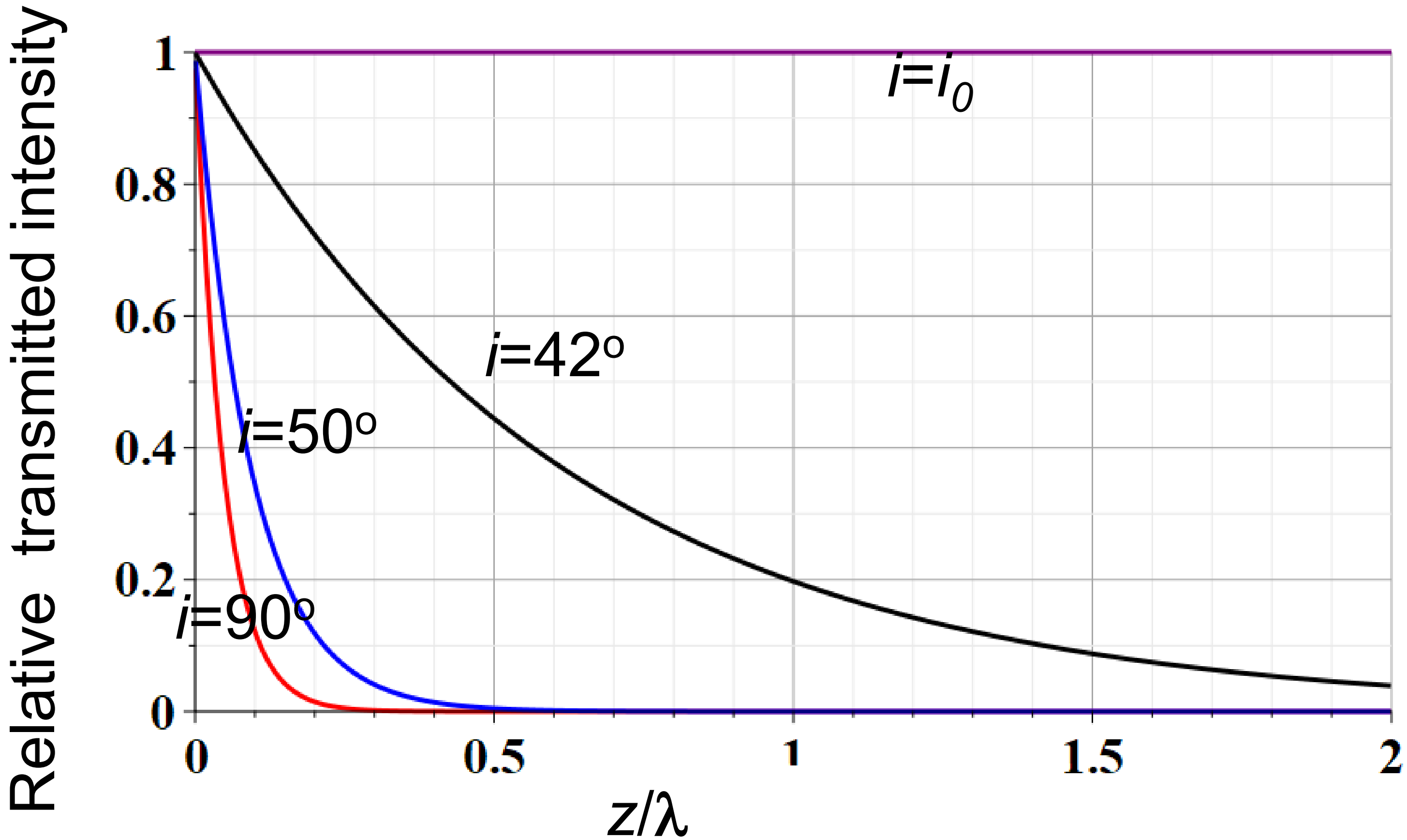
For $i > i_0$

$$n'\cos\theta = i\sqrt{n^2\sin^2 i - n'^2} = i\, n\sqrt{\frac{\sin^2 i}{\sin^2 i_0} - 1}$$

$$\mathbf{E}'(\mathbf{r},t) = e^{-\left(\frac{n\omega}{c}\sqrt{\frac{\sin^2 i}{\sin^2 i_0}-1}\right)z}\,\Re\left(\mathbf{E}'_0\, e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}_{\parallel}\cdot\mathbf{r} - ct\right)}\right)$$

Example of total internal reflection

n'=1 and n=1.5 ➔ $i_0 = \sin^{-1}(1/1.5) = 41.8103148°$



Transmitted illumination confined within a few wavelengths of the surface.

TIRF (total internal reflection fluorescence)

www.nikon.com/products/microscope-solutions/bioscience.../nikon_note_10_lr.pdf

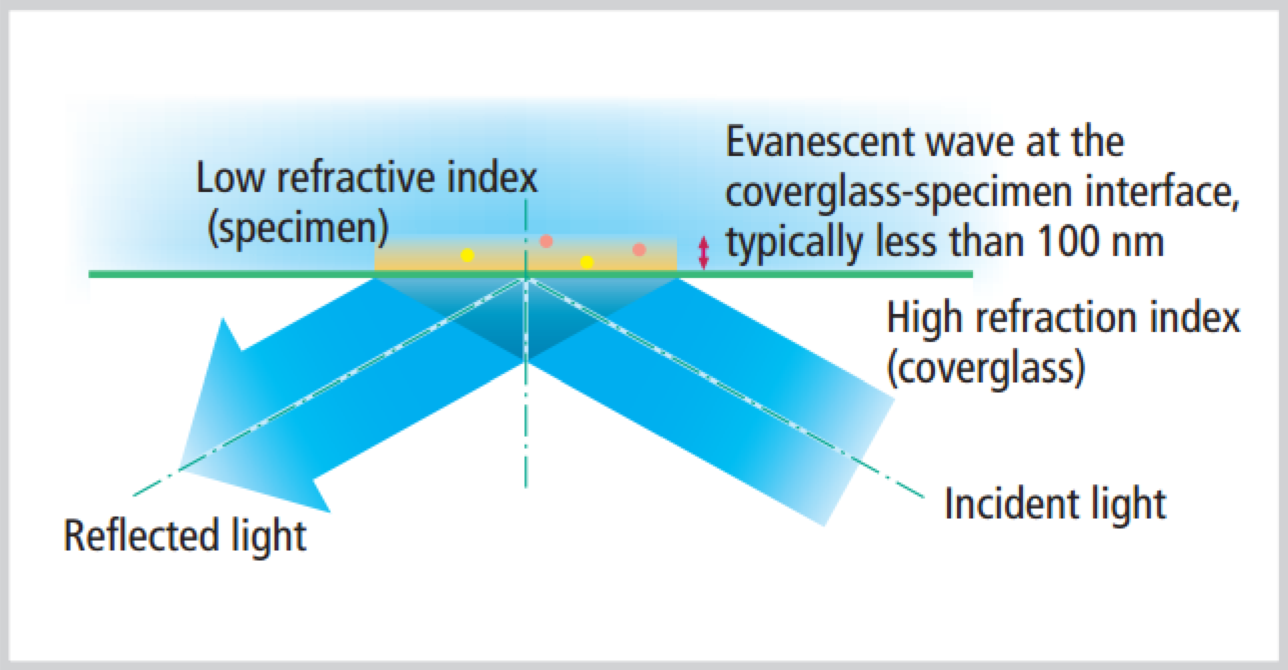


Figure 1: Creation of an evanescent wave at the coverglass-specimen interface

Design of TIRF device using laser and high power lens

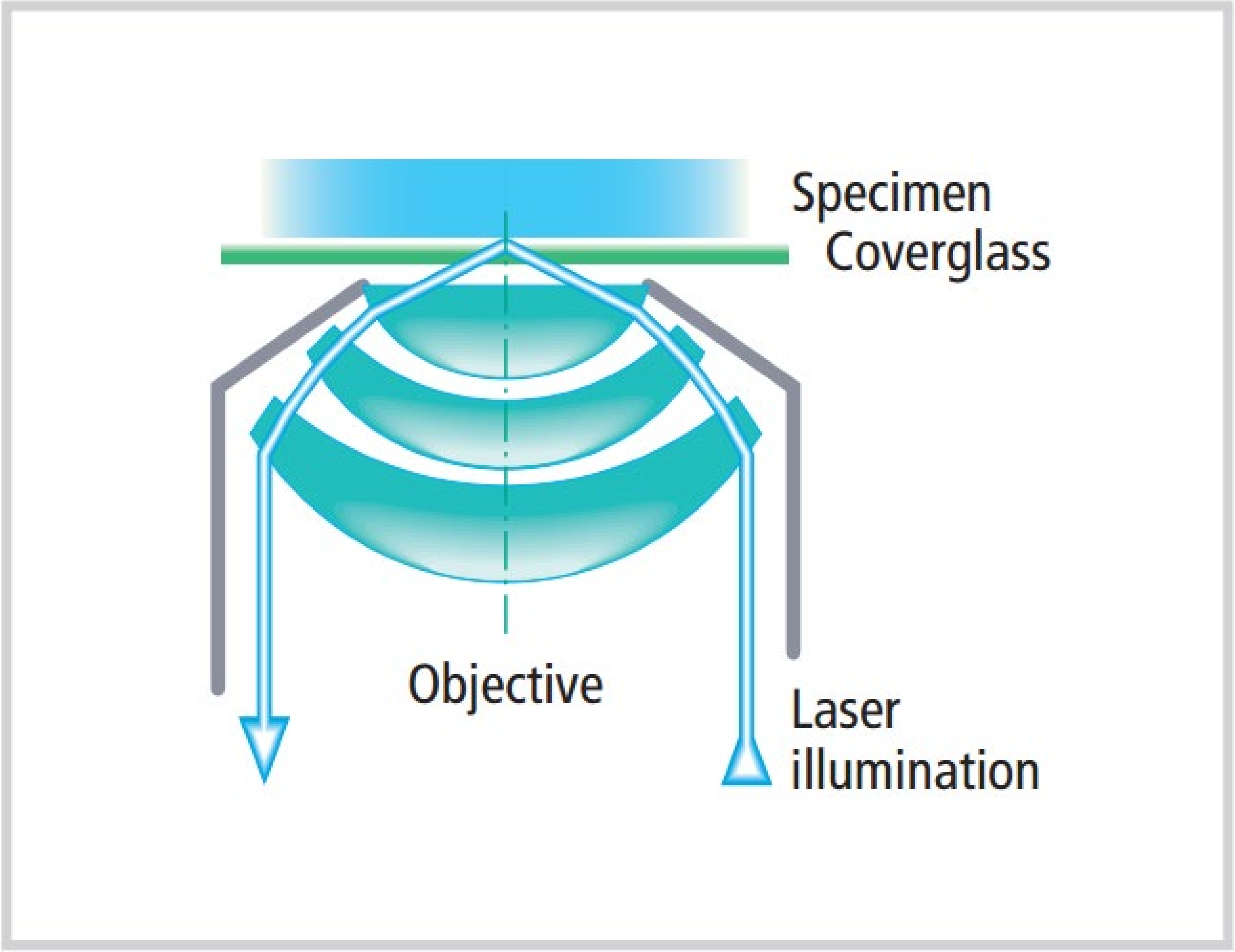


Figure 2: Through-the-lens laser TIRF.

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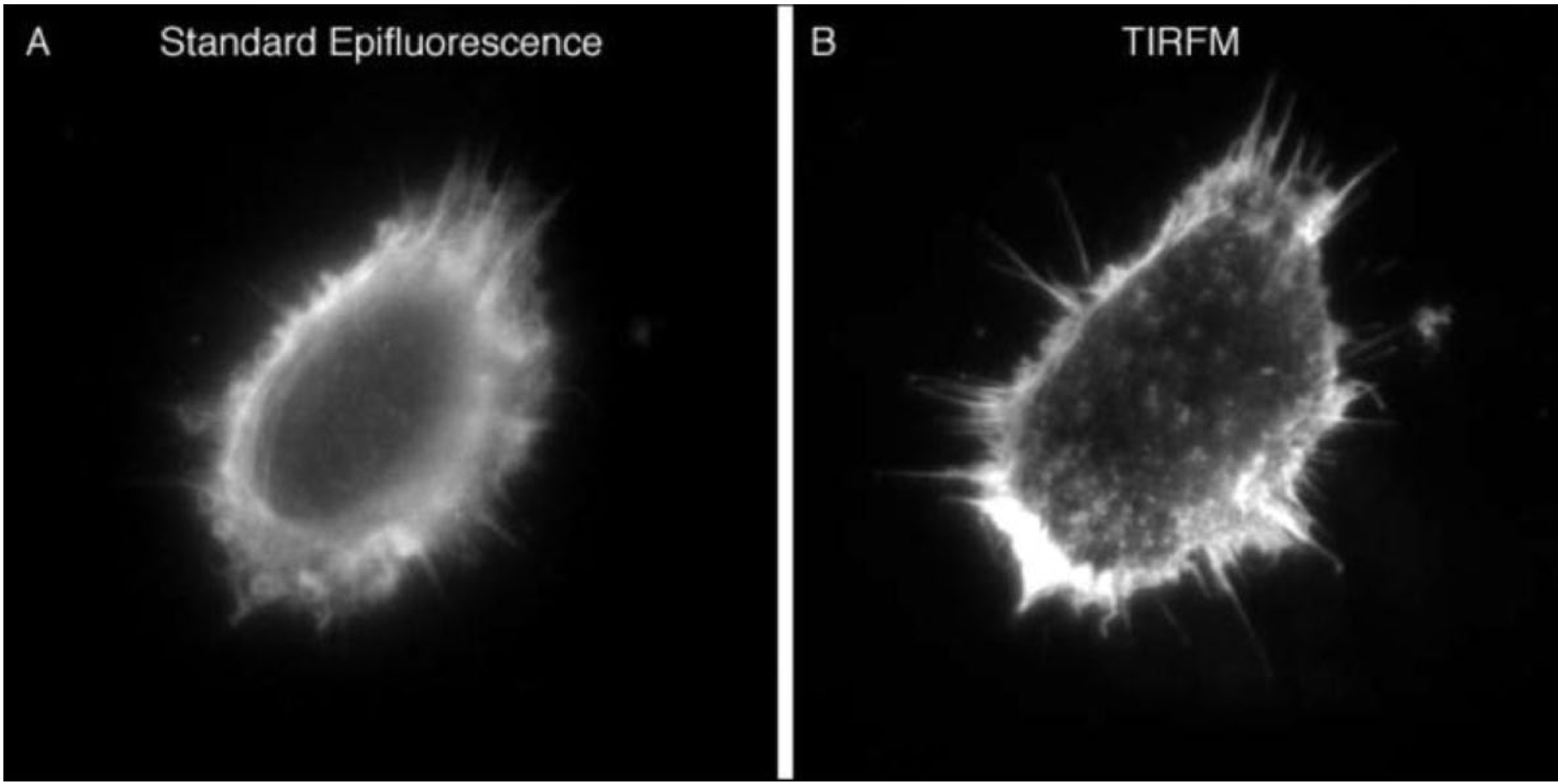
Figure 1



Extension to complex refractive index $n = n_R + i\, n_I$

$$\text{Suppose } \mu = \mu', \;\; n = \text{real}, \;\; n' = n'_R + i n'_I$$

Reflectance at normal incidence:

$$\mathrm{R} = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n' - n}{\frac{\mu}{\mu'}n' + n}\right|^2 = \frac{\left(n'_R - n\right)^2 + \left(n'_I\right)^2}{\left(n'_R + n\right)^2 + \left(n'_I\right)^2}$$

Note that for $n'_I >> \left|n'_R \pm n\right|$:

$$\mathrm{R} \approx 1$$

The general Fresnel equations can be similarly adapted for complex refractive indices.

Origin of imaginary contributions to permittivity --
Review: Drude model dielectric function:

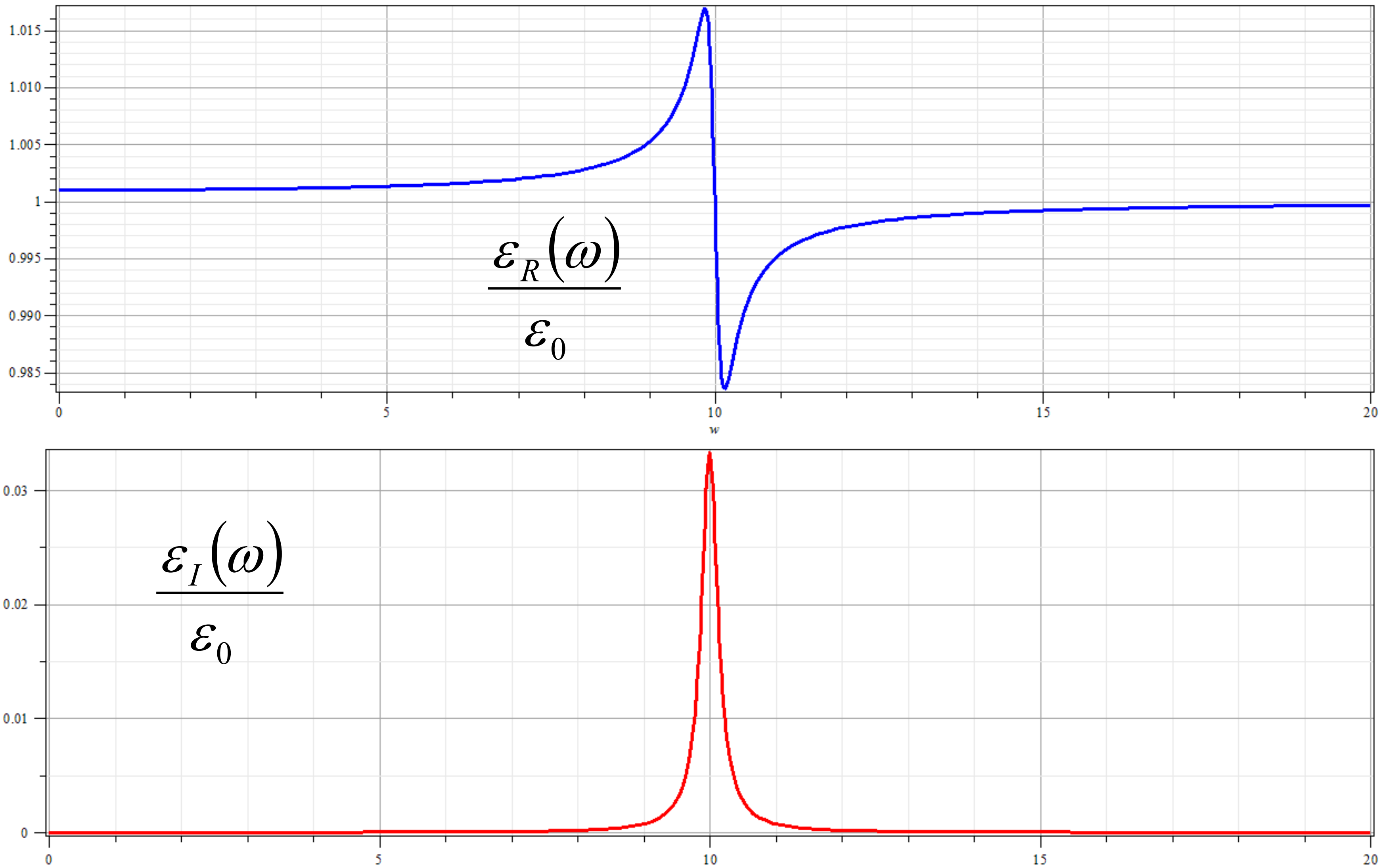
$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i}\frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$= \frac{\varepsilon_R(\omega)}{\varepsilon_0} + i\,\frac{\varepsilon_I(\omega)}{\varepsilon_0}$$

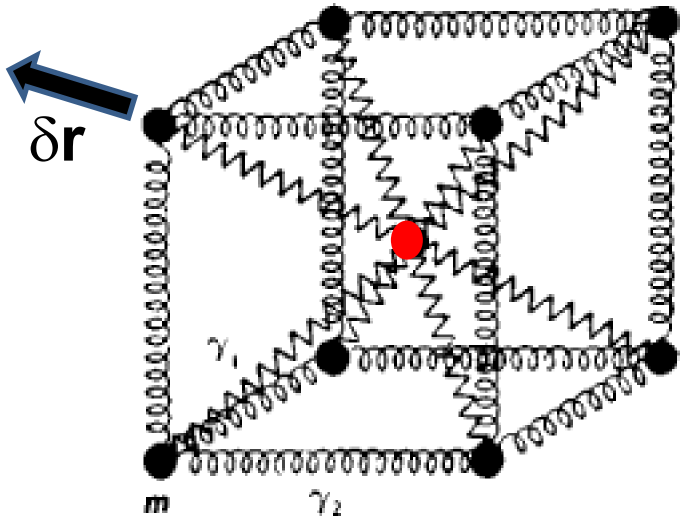
$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i}\frac{\omega_i^2 - \omega^2}{\left(\omega_i^2 - \omega^2\right)^2 + \omega^2\gamma_i^{\,2}}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i}\frac{\omega\gamma_i}{\left(\omega_i^2 - \omega^2\right)^2 + \omega^2\gamma_i^{\,2}}$$

Drude model dielectric function:



Extensions of the Drude model for lattice vibrations



In principle, the ideas of the Drude model apply both to the ionic vibrations which occur at low frequency (~10^{12} Hz) contributing to the so called static permittivity function ε_s and to the electronic vibrations which occur at high frequency (~10^{15} Hz) contributing to the so called high frequency permittivity function ε_∞.

In this model at high frequencies, only the electrons contribute to the polarization: $\epsilon_\infty = \epsilon_0 + \frac{|\mathbf{P}_{electron}|}{|\mathbf{E}|}$

At low frequencies both electrons and ions contribute to the polarization: $\epsilon_s = \epsilon_0 + \frac{|\mathbf{P}_{electron}|}{|\mathbf{E}|} + \frac{|\mathbf{P}_{ion}|}{|\mathbf{E}|}$

$$\Rightarrow \frac{|\mathbf{P}_{ion}|}{|\mathbf{E}|} = \epsilon_s - \epsilon_\infty$$

In terms of the Drude model form:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

This form applies to classical lattice vibration modes and also to quantum treatments of electronic transitions in which case, the prefactor $f_i \frac{q_i^2}{\varepsilon_0 m_i}$ should be reinterpreted as an "oscillator" strength calculated as a transition matrix element.

$$\frac{\varepsilon_\infty(\omega)}{\varepsilon_0} = 1 + N \sum_{i\,\in\,\text{electrons}} f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$\frac{\varepsilon_s(\omega)}{\varepsilon_0} = 1 + N \sum_{i\,\in\,\text{electrons}} f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i} + N \sum_{i\,\in\,\text{vibrations}} f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$\omega_i = 2\pi\nu_i$$

$$\nu_i \sim 10^{15}\,\text{Hz}$$

$$\omega_i = 2\pi\nu_i$$

$$\nu_i \sim 10^{12}\,\text{Hz}$$

Comment: The Drude model allows us to “derive”:

Kramers-Kronig transform – for dielectric function:

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\varepsilon_I(\omega')}{\varepsilon_0} \frac{1}{\omega' - \omega}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \left(\frac{\varepsilon_R(\omega')}{\varepsilon_0} - 1 \right) \frac{1}{\omega' - \omega}$$

$$\text{with} \quad \varepsilon_R(-\omega) = \varepsilon_R(\omega); \quad \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

Practical applications -- It is often possible/more convenient to calculate the imaginary response and use KK to deduce the real response or visa versa.

Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

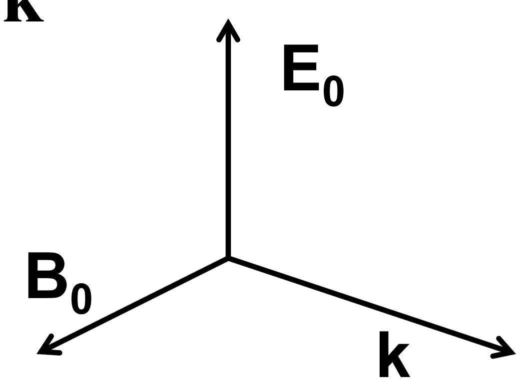
$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

Poynting vector and energy density:

$$\langle \mathbf{S} \rangle_{avg} = \frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\,\hat{\mathbf{k}}$$

$$\langle u \rangle_{avg} = \frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$

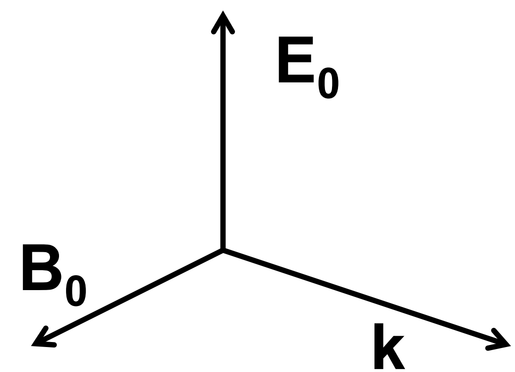


Transverse electric and magnetic waves (TEM)

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$\left|\mathbf{k}\right|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

TEM modes describe electromagnetic waves in lossless media and vacuum



For real
ε, μ, n, k

Effects of complex dielectric; fields near the surface on an ideal conductor

Suppose for an isotropic medium: $\mathbf{D} = \varepsilon_b \mathbf{E} \qquad \mathbf{J} = \sigma \mathbf{E}$

Maxwell's equations in terms of $\mathbf{H}$ and $\mathbf{E}$:

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon_b \frac{\partial \mathbf{E}}{\partial t}$$

$$\left(\nabla^2 - \mu\sigma \frac{\partial}{\partial t} - \mu\varepsilon_b \frac{\partial^2}{\partial t^2} \right) \mathbf{F} = 0 \qquad \mathbf{F} = \mathbf{E}, \mathbf{H}$$

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \right) \qquad \text{where } \mathbf{k} = \left(n_R + i n_I \right) \frac{\omega}{c} \hat{\mathbf{k}}$$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta} \Re\left(\mathbf{E}_0 e^{i n_R (\omega/c) \hat{\mathbf{k}}\cdot\mathbf{r} - i\omega t} \right)$$

Some details:

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \text{where } \mathbf{k} = \left(n_R + i n_I\right)\frac{\omega}{c}\hat{\mathbf{k}}$$

$$\left(\nabla^2 - \mu\sigma\frac{\partial}{\partial t} - \mu\varepsilon_b\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = 0$$

$$-\left(n_R + i n_I\right)^2 + i\frac{\mu\sigma c^2}{\omega} + \mu\varepsilon_b c^2 = 0$$

Fields near the surface on an ideal conductor -- continued

For our system:

$$\frac{\omega}{c}n_R = \omega\sqrt{\frac{\mu\varepsilon_b}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_b}\right)^2}+1\right)^{1/2}$$

$$\frac{\omega}{c}n_I = \omega\sqrt{\frac{\mu\varepsilon_b}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_b}\right)^2}-1\right)^{1/2}$$

$$\text{For } \frac{\sigma}{\omega} >> 1 \qquad \frac{\omega}{c}n_R \approx \frac{\omega}{c}n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\Rightarrow \mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

Some representative values of skin depth

Ref: Lorrain and Corson

$$\frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$$

	σ (10^7 S/m)	μ/μ_0	δ (0.001m) at 60 Hz	δ (0.001m) at 1 MHz
Al	3.54	1	10.9	84.6
Cu	5.80	1	8.5	66.1
Fe	1.00	100	1.0	10.0
Mumetal	0.16	2000	0.4	3.0
Zn	1.86	1	15.1	117

Relative energies associated with field

Electric energy density: $\varepsilon_b |\mathbf{E}|^2$

Magnetic energy density: $\mu |\mathbf{H}|^2$

Ratio inside conducting media: $$\frac{\varepsilon_b |\mathbf{E}|^2}{\mu |\mathbf{H}|^2} = \frac{\varepsilon_b}{\mu \left|\frac{1+i}{\delta\mu\omega}\right|^2} = \frac{\varepsilon_b \mu \omega^2 \delta^2}{2}$$

Here wavelength is defined:

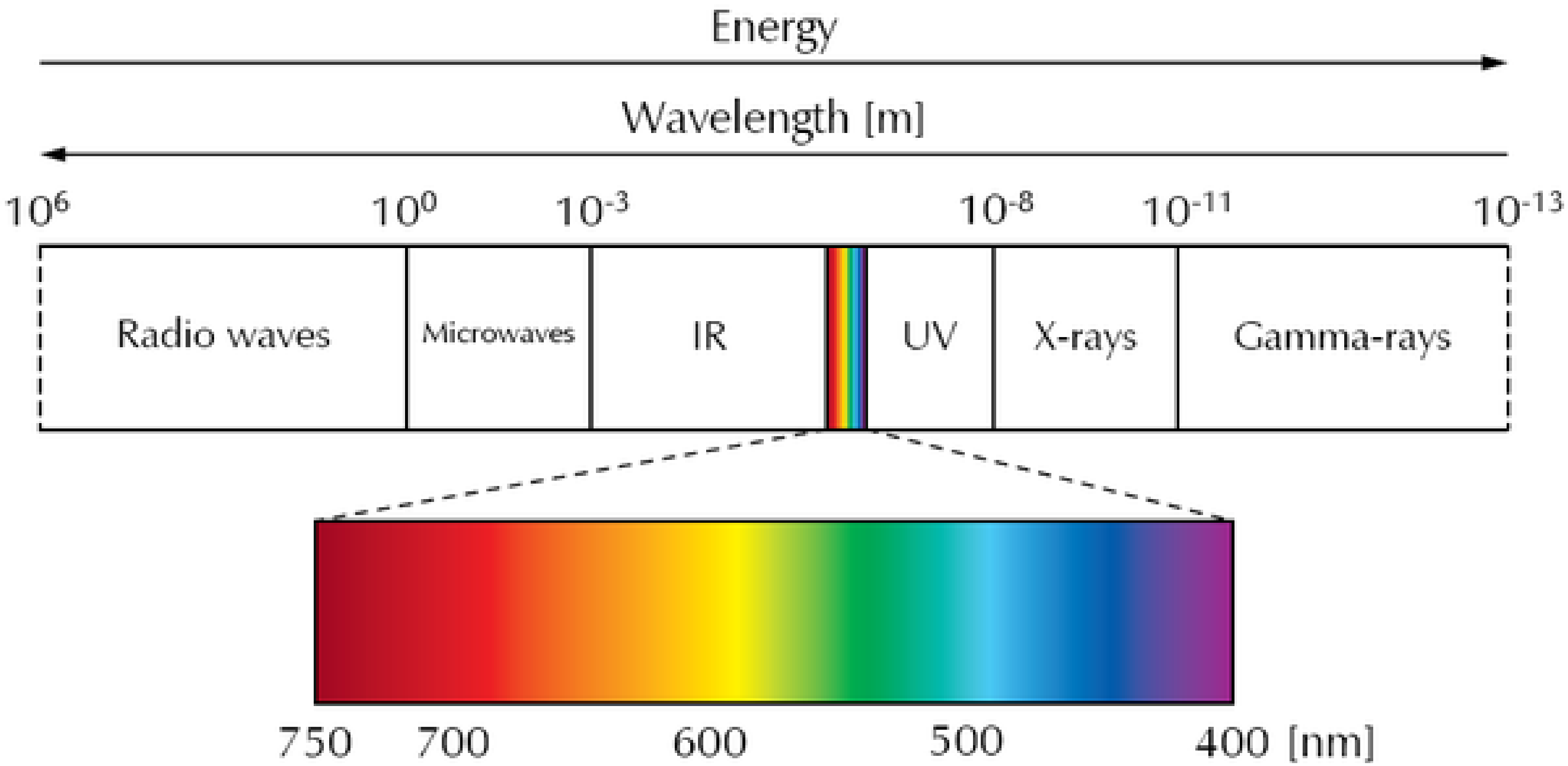
$$\lambda = \frac{2\pi c}{\omega}$$

$$= 2\pi^2 \frac{\varepsilon_b}{\varepsilon_0} \frac{\mu}{\mu_0} \frac{\delta^2}{\lambda^2}$$

For $\frac{\varepsilon_b |\mathbf{E}|^2}{\mu |\mathbf{H}|^2} \ll 1 \quad \Rightarrow$ magnetic energy dominates

Note that in free space, $\frac{\varepsilon_0 |\mathbf{E}|^2}{\mu_0 |\mathbf{H}|^2} = 1$

Various wavelengths λ --



General relationships

Comment on complex dielectric and refractive index functions

For $\mu = \mu_0$:

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\varepsilon_R}{\varepsilon_0} + i\frac{\varepsilon_I}{\varepsilon_0} \equiv a + ib = \left(n_R + in_I\right)^2$$

$$a = n_R^2 - n_I^2$$

$$b = 2n_R n_I$$

$$\Rightarrow n_R^2 = \frac{1}{2}\left(a + \sqrt{a^2 + b^2}\right) \qquad n_I^2 = \frac{1}{2}\left(-a + \sqrt{a^2 + b^2}\right)$$