

PHY 712 Electrodynamics

10-10:50 AM in Olin 103

Notes for Lecture 20:

Wave Guides - Chap. 8 (Sec. 8.1-8.4 in JDJ)

- 1. TEM, TE, and TM modes of electromagnetic waves**
- 2. Justification for boundary conditions; behavior of waves near conducting surfaces**
- 3. Rectangular waveguide**

13	Wed: 02/12/2025	Chap. 5	Magnetic dipoles and hyperfine interactions	#12	02/14/2025
14	Fri: 02/14/2025	Chap. 5	Magnetic materials and boundary value problems	#13	02/17/2025
15	Mon: 02/17/2025	Chap. 6	Maxwell's Equations	#14	02/19/2025
16	Wed: 02/19/2025	Chap. 6	Electromagnetic energy and forces	#15	02/21/2025
17	Fri: 02/21/2025	Chap. 7	Electromagnetic plane waves	#16	02/24/2025
18	Mon: 02/24/2025	Chap. 7	Electromagnetic response functions	#17	02/26/2025
19	Wed: 02/26/2025	Chap. 7	Optical effects of refractive indices	#18	02/28/2025
20	Fri: 02/28/2025	Chap. 8	Waveguides		
21	Mon: 03/03/2025	Chap. 8	Waveguides		
22	Wed: 03/05/2025				
23	Fri: 03/07/2025		Review		
	Mon: 03/10/2025	No class	<i>Spring Break</i>		
	Wed: 03/12/2025	No class	<i>Spring Break</i>		
	Fri: 03/14/2025	No class	<i>Spring Break</i>		
	Mon: 03/17/2025	No class	<i>Take-home exam</i>		
	Wed: 03/19/2025	No class	<i>Take-home exam</i>		
	Fri: 03/21/2025	No class	<i>Take-home exam</i>		

No HW assignment for Monday, but it would be good to start thinking about your presentation topics.

Some Ideas for Computational Project

The purpose of the "Computational Project" is to provide an opportunity for you to study a topic of your choice in greater depth. The general guideline for your choice of project is that it should have something to do with electrodynamics, and there should be some degree of computation or analysis with the project. The completed project will include a short write-up and a ~15 min presentation to the class. You may design your own project or use one of the following list (which will be updated throughout the term).

- Evaluate the Ewald sum of various ionic crystals using Maple or a programming language. (Template available in Fortran code.)
- Work out the details of the finite difference or finite element methods.
- Work out the details of the hyperfine Hamiltonian as discussed in Chapter 5 of Jackson.
- Work out the details of Jackson problem 7.2 and related problems.
- Work out the details of reflection and refraction from birefringent materials.
- Analyze the Kramers-Kronig transform of some optical data or calculations.
- Determine the classical electrodynamics associated with an infrared or optical laser.
- Analyze the radiation intensity and spectrum from an interesting source such as an atomic or molecular transition, a free electron laser, etc.
- Work out the details of Jackson problem 14.15.

Maxwell's equations

For linear isotropic media and no sources: $\mathbf{D} = \epsilon\mathbf{E}$; $\mathbf{B} = \mu\mathbf{H}$

Coulomb's law: $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

Analysis of Maxwell's equations without sources -- continued:

Coulomb's law : $\nabla \cdot \mathbf{E} = 0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

$$\begin{aligned}\nabla \times \left(\nabla \times \mathbf{B} - \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} \right) &= -\nabla^2 \mathbf{B} - \mu\epsilon \frac{\partial (\nabla \times \mathbf{E})}{\partial t} \\ &= -\nabla^2 \mathbf{B} + \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0\end{aligned}$$

$$\begin{aligned}\nabla \times \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) &= -\nabla^2 \mathbf{E} + \frac{\partial (\nabla \times \mathbf{B})}{\partial t} \\ &= -\nabla^2 \mathbf{E} + \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0\end{aligned}$$

Analysis of Maxwell's equations without sources -- continued:

Both \mathbf{E} and \mathbf{B} fields are solutions to a wave equation:

$$\nabla^2 \mathbf{B} - \frac{1}{v^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{E} - \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\text{where } v^2 \equiv c^2 \frac{\mu_0 \epsilon_0}{\mu \epsilon} \equiv \frac{c^2}{n^2}$$

Plane wave solutions to wave equation :

$$\mathbf{B}(\mathbf{r}, t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \quad \mathbf{E}(\mathbf{r}, t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

Analysis of Maxwell's equations without sources -- continued:

Plane wave solutions to wave equation :

$$\mathbf{B}(\mathbf{r}, t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \quad \mathbf{E}(\mathbf{r}, t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$$

Note: ϵ, μ, n, k can all be complex; for the moment we will assume that they are all real (no dissipation).

Note that \mathbf{E}_0 and \mathbf{B}_0 are not independent;

from Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

$$\Rightarrow \mathbf{B}_0 = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega} = \frac{n \hat{\mathbf{k}} \times \mathbf{E}_0}{c}$$

For real
 ϵ, μ, n, k

also note : $\hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$ and $\hat{\mathbf{k}} \cdot \mathbf{B}_0 = 0$

Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

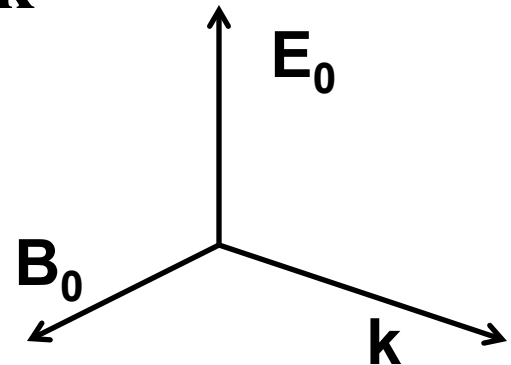
$$\mathbf{B}(\mathbf{r}, t) = \Re\left(\frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \quad \mathbf{E}(\mathbf{r}, t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \quad \text{and } \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$$

Poynting vector and energy density:

$$\langle \mathbf{S} \rangle_{avg} = \frac{n|\mathbf{E}_0|^2}{2\mu c} \hat{\mathbf{k}} = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} |\mathbf{E}_0|^2 \hat{\mathbf{k}}$$

$$\langle u \rangle_{avg} = \frac{1}{2} \epsilon |\mathbf{E}_0|^2$$



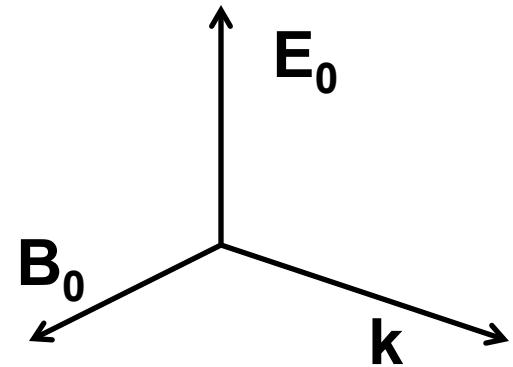
Transverse electric and magnetic waves (TEM)

$$\mathbf{B}(\mathbf{r}, t) = \Re \left(\frac{n \hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \right) \quad \mathbf{E}(\mathbf{r}, t) = \Re \left(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v} \right)^2 = \left(\frac{n\omega}{c} \right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$$

TEM modes describe
electromagnetic waves in lossless
media and vacuum

For real
 ε, μ, n, k



Note that linear combinations of TEM modes are also TEM.

Effects of complex dielectric; fields near the surface on an ideal conductor

Suppose for an isotropic medium: $\mathbf{D} = \epsilon_b \mathbf{E}$ $\mathbf{J} = \sigma \mathbf{E}$

Maxwell's equations in terms of \mathbf{H} and \mathbf{E} :

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon_b \frac{\partial \mathbf{E}}{\partial t}$$

$$\left(\nabla^2 - \mu\sigma \frac{\partial}{\partial t} - \mu\epsilon_b \frac{\partial^2}{\partial t^2} \right) \mathbf{F} = 0 \qquad \mathbf{F} = \mathbf{E}, \mathbf{H}$$

Plane wave form for \mathbf{E} :

$$\mathbf{E}(\mathbf{r}, t) = \Re \left(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \right) \qquad \text{where } \mathbf{k} = \left(n_R + in_I \right) \frac{\omega}{c} \hat{\mathbf{k}}$$

$$\Rightarrow \mathbf{E}(\mathbf{r}, t) = e^{-\hat{\mathbf{k}} \cdot \mathbf{r} / \delta} \Re \left(\mathbf{E}_0 e^{in_R(\omega/c)\hat{\mathbf{k}} \cdot \mathbf{r} - i\omega t} \right)$$

Some details:

Plane wave form for \mathbf{E} :

$$\mathbf{E}(\mathbf{r}, t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \quad \text{where } \mathbf{k} = (n_R + in_I) \frac{\omega}{c} \hat{\mathbf{k}}$$

$$\left(\nabla^2 - \mu\sigma \frac{\partial}{\partial t} - \mu\varepsilon_b \frac{\partial^2}{\partial t^2} \right) \mathbf{E} = 0$$

$$-(n_R + in_I)^2 + i \frac{\mu\sigma c^2}{\omega} + \mu\varepsilon_b c^2 = 0$$

Note that in this formulation, we are assuming ε_b and σ are real numbers. Also note that the full permittivity is

$$\varepsilon = \varepsilon_b + i \frac{\sigma}{\omega} \quad \text{and all response parameters are functions of } \omega.$$

Fields near the surface on an ideal conductor -- continued

For our system :

$$\frac{\omega}{c} n_R = \omega \sqrt{\frac{\mu \epsilon_b}{2} \left(\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon_b} \right)^2} + 1 \right)^{1/2}}$$

$$\frac{\omega}{c} n_I = \omega \sqrt{\frac{\mu \epsilon_b}{2} \left(\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon_b} \right)^2} - 1 \right)^{1/2}}$$

For $\frac{\sigma}{\omega} \gg 1$ $\frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu \sigma \omega}{2}} \equiv \frac{1}{\delta}$ ← “skin depth”

$$\Rightarrow \mathbf{E}(\mathbf{r}, t) = e^{-\hat{\mathbf{k}} \cdot \mathbf{r} / \delta} \Re \left(\mathbf{E}_0 e^{i \hat{\mathbf{k}} \cdot \mathbf{r} / \delta - i \omega t} \right)$$

$$\Rightarrow \mathbf{H}(\mathbf{r}, t) = \frac{n}{c \mu} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t) = \frac{1+i}{\delta \mu \omega} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t)$$

Some representative values of skin depth

Ref: Lorrain² and Corson

$$\frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$$

Note that frequency given in
units of Hz $\Rightarrow \frac{\omega}{2\pi}$

	σ (10^7 S/m)	μ/μ_0	δ (0.001m) at 60 Hz	δ (0.001m) at 1 MHz
Al	3.54	1	10.9	84.6
Cu	5.80	1	8.5	66.1
Fe	1.00	100	1.0	10.0
Mumetal	0.16	2000	0.4	3.0
Zn	1.86	1	15.1	117

Relative energies associated with field

Electric energy density: $\epsilon_b |\mathbf{E}|^2$

Magnetic energy density: $\mu |\mathbf{H}|^2$

Ratio inside conducting media: $\frac{\epsilon_b |\mathbf{E}|^2}{\mu |\mathbf{H}|^2} = \frac{\epsilon_b}{\mu \left| \frac{1+i}{\delta \mu \omega} \right|^2} = \frac{\epsilon_b \mu \omega^2 \delta^2}{2}$

$$\lambda = \frac{2\pi c}{\omega} = \frac{c}{f}$$

$$= 2\pi^2 \frac{\epsilon_b}{\epsilon_0} \frac{\mu}{\mu_0} \frac{\delta^2}{\lambda^2}$$

For $\frac{\epsilon_b |\mathbf{E}|^2}{\mu |\mathbf{H}|^2} \ll 1 \Rightarrow$ magnetic energy dominates

Note that in free space, $\frac{\epsilon_0 |\mathbf{E}|^2}{\mu_0 |\mathbf{H}|^2} = 1$

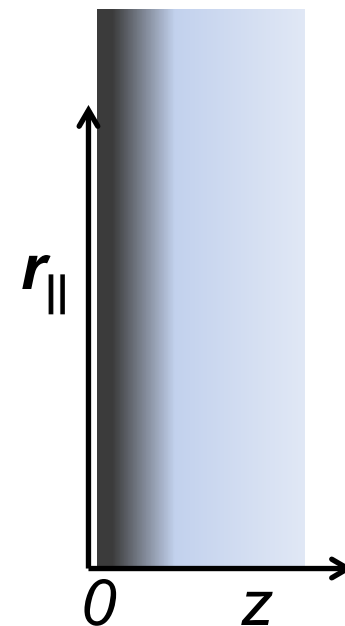
Fields near the surface on an ideal conductor -- continued

$$\text{For } \frac{\sigma}{\omega} \gg 1 \quad \frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$$

$$\text{In this limit, } \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = c\sqrt{\mu\varepsilon} = n_R + in_I = \frac{c}{\omega} \frac{1}{\delta} (1+i)$$

$$\mathbf{E}(\mathbf{r}, t) = e^{-\hat{\mathbf{k}} \cdot \mathbf{r} / \delta} \Re \left(\mathbf{E}_0 e^{i\hat{\mathbf{k}} \cdot \mathbf{r} / \delta - i\omega t} \right)$$

$$\mathbf{H}(\mathbf{r}, t) = \frac{n}{c\mu} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t) = \frac{1+i}{\delta\mu\omega} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t)$$

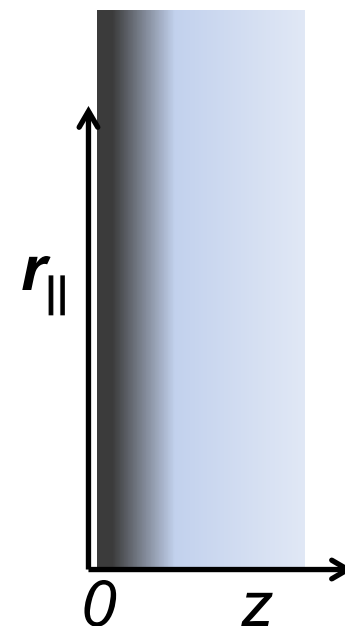


Fields near the surface on an ideal conductor -- continued

(Note that we are assuming no extra charges or currents beyond material responses via ϵ, σ, μ .)

$$\mathbf{E}(\mathbf{r}, t) = e^{-\hat{\mathbf{k}} \cdot \mathbf{r} / \delta} \Re \left(\mathbf{E}_0 e^{i\hat{\mathbf{k}} \cdot \mathbf{r} / \delta - i\omega t} \right)$$

$$\mathbf{H}(\mathbf{r}, t) = \frac{n}{c\mu} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t) = \frac{1+i}{\delta\mu\omega} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t)$$



Note that it is convenient to express the EM fields in terms of the \mathbf{H} amplitude:

$$\mathbf{H}(\mathbf{r}, t) = e^{-\hat{\mathbf{k}} \cdot \mathbf{r} / \delta} \Re \left(\mathbf{H}_0 e^{i\hat{\mathbf{k}} \cdot \mathbf{r} / \delta - i\omega t} \right)$$

$$\mathbf{E}(\mathbf{r}, t) = \delta\mu\omega \frac{1-i}{2} \hat{\mathbf{k}} \times \mathbf{H}(\mathbf{r}, t)$$

Boundary values for ideal conductor

Inside the conductor :

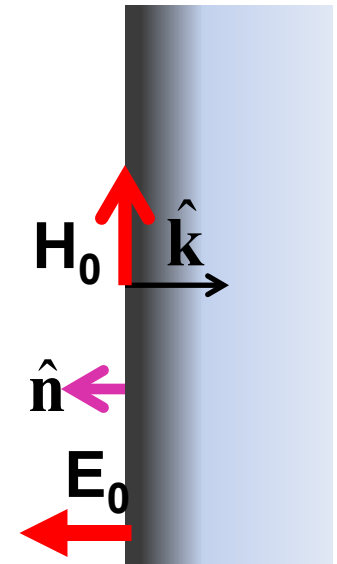
$$\mathbf{H}(\mathbf{r}, t) = e^{-\hat{\mathbf{k}} \cdot \mathbf{r} / \delta} \Re \left(\mathbf{H}_0 e^{i\hat{\mathbf{k}} \cdot \mathbf{r} / \delta - i\omega t} \right)$$

$$\mathbf{E}(\mathbf{r}, t) = \delta \mu \omega \frac{1-i}{2} \hat{\mathbf{k}} \times \mathbf{H}(\mathbf{r}, t)$$

At the boundary of an ideal conductor, the \mathbf{E} and \mathbf{H} fields decay in the direction normal to the interface.

Ideal conductor boundary conditions:

$$\hat{\mathbf{n}} \times \mathbf{E} \Big|_S = 0 \qquad \hat{\mathbf{n}} \cdot \mathbf{H} \Big|_S = 0$$

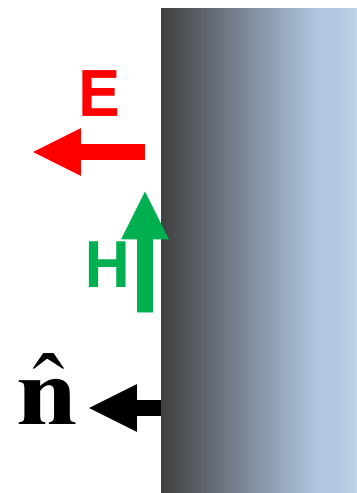


Wave guides – dielectric media with one or more metal boundary

Continuity conditions for fields near metal boundaries --

Ideal conductor boundary conditions:

$$\hat{\mathbf{n}} \times \mathbf{E} \Big|_S = 0 \qquad \hat{\mathbf{n}} \cdot \mathbf{H} \Big|_S = 0$$



Waveguide terminology

- TEM: transverse electric and magnetic (both E and H fields are perpendicular to wave propagation direction)
- TM: transverse magnetic (H field is perpendicular to wave propagation direction)
- TE: transverse electric (E field is perpendicular to wave propagation direction)

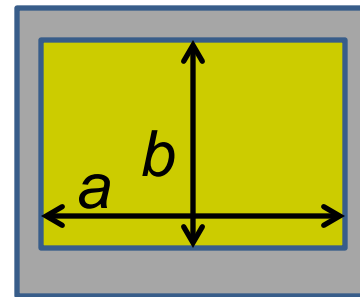
Analysis of rectangular waveguide

Boundary conditions at surface of waveguide:

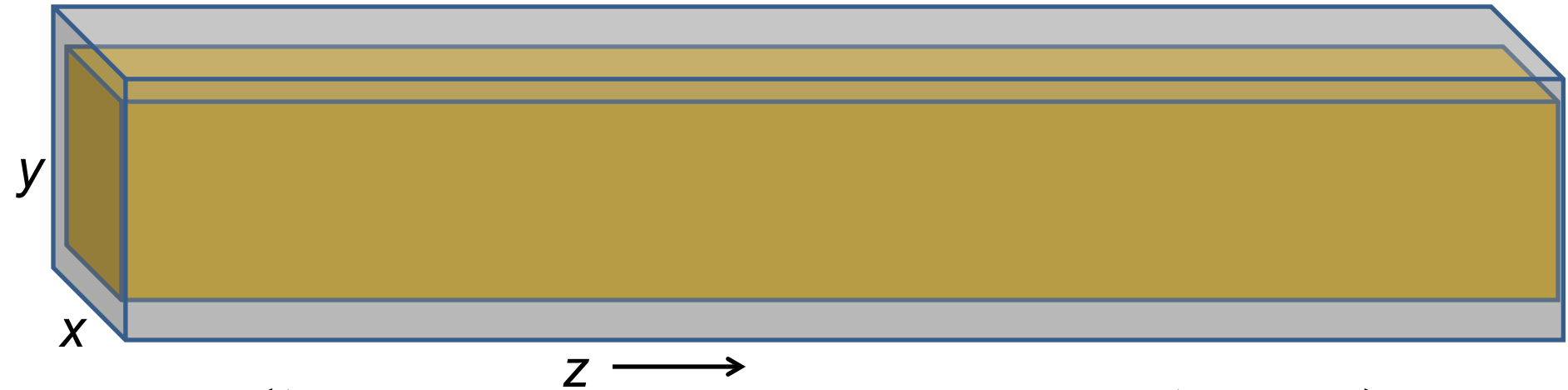
$$\mathbf{E}_{\text{tangential}}=0, \quad \mathbf{B}_{\text{normal}}=0$$



Cross section view



Analysis of rectangular waveguide



$$\mathbf{B} = \Re \left\{ \left(B_x(x, y) \hat{\mathbf{x}} + B_y(x, y) \hat{\mathbf{y}} + B_z(x, y) \hat{\mathbf{z}} \right) e^{ikz - i\omega t} \right\}$$
$$\mathbf{E} = \Re \left\{ \left(E_x(x, y) \hat{\mathbf{x}} + E_y(x, y) \hat{\mathbf{y}} + E_z(x, y) \hat{\mathbf{z}} \right) e^{ikz - i\omega t} \right\}$$

Inside the dielectric medium: (assume ε to be real)

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \times \mathbf{B} - \varepsilon \mu \frac{\partial \mathbf{E}}{\partial t} = 0$$

Solution of Maxwell's equations within the pipe:

Combining Faraday's Law and Ampere's Law, we find that each field component must satisfy a two-dimensional Helmholtz equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2 + \mu\epsilon\omega^2 \right) \mathbf{F}(x, y) = 0. \quad \mathbf{F} = \mathbf{E} \text{ or } \mathbf{H}$$

propagation along z.

For the rectangular wave guide discussed in Section 8.4 of your text a solution for a TE mode can have:

$$E_z(x, y) \equiv 0 \quad \text{and} \quad B_z(x, y) = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right),$$

$$\text{with } k^2 \equiv k_{mn}^2 = \mu\epsilon\omega^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]$$

Maxwell's equations within the pipe in terms of all 6 components:

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + ikB_z = 0.$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z = 0.$$

$$\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x.$$

$$ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y.$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z.$$

For TE mode with $E_z \equiv 0$

$$B_x = -\frac{k}{\omega} E_y$$

$$B_y = \frac{k}{\omega} E_x$$

$$\frac{\partial B_z}{\partial y} - ikB_y = -i\mu\epsilon\omega E_x.$$

$$ikB_x - \frac{\partial B_z}{\partial x} = -i\mu\epsilon\omega E_y.$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -i\mu\epsilon\omega E_z.$$

TE modes for rectangular wave guide continued:

$$E_z(x, y) \equiv 0 \quad \text{and} \quad B_z(x, y) = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right),$$

$$E_x = \frac{\omega}{k} B_y = \frac{-i\omega}{k^2 - \mu\epsilon\omega^2} \frac{\partial B_z}{\partial y} = \frac{-i\omega}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} \frac{n\pi}{b} B_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right),$$

$$E_y = -\frac{\omega}{k} B_x = \frac{i\omega}{k^2 - \mu\epsilon\omega^2} \frac{\partial B_z}{\partial x} = \frac{i\omega}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} \frac{m\pi}{a} B_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right).$$

Check boundary conditions:

$$\mathbf{E}_{\text{tangential}} = 0 \quad \text{because:} \quad E_z(x, y) \equiv 0, \quad E_x(x, 0) = E_x(x, b) = 0$$

$$\text{and} \quad E_y(0, y) = E_y(a, y) = 0.$$

$$\mathbf{B}_{\text{normal}} = 0 \quad \text{because:} \quad B_y(x, 0) = B_y(x, b) = 0$$

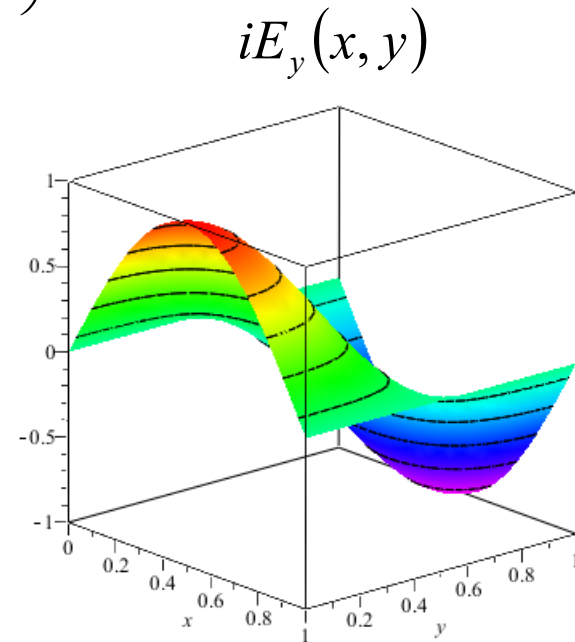
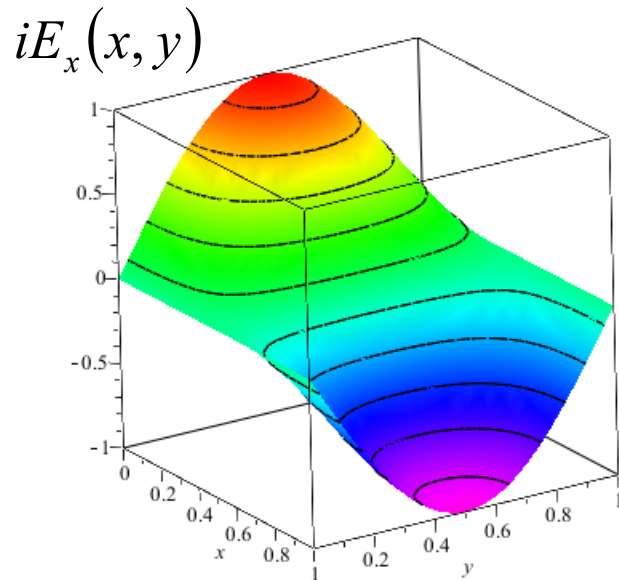
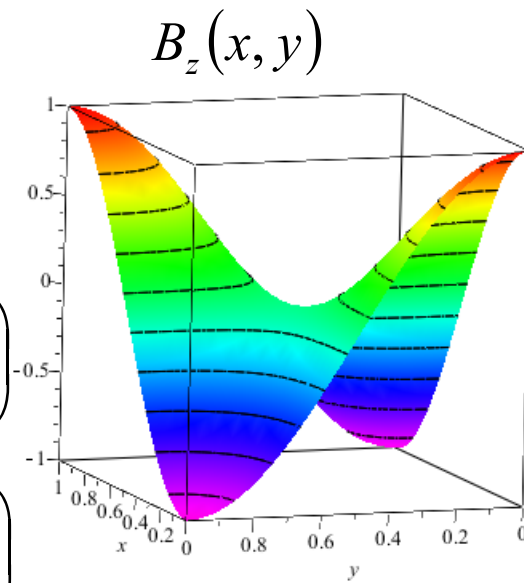
$$\text{and} \quad B_x(0, y) = B_x(a, y) = 0.$$

Solution for $m=n=1$

$$B_z(x, y) = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

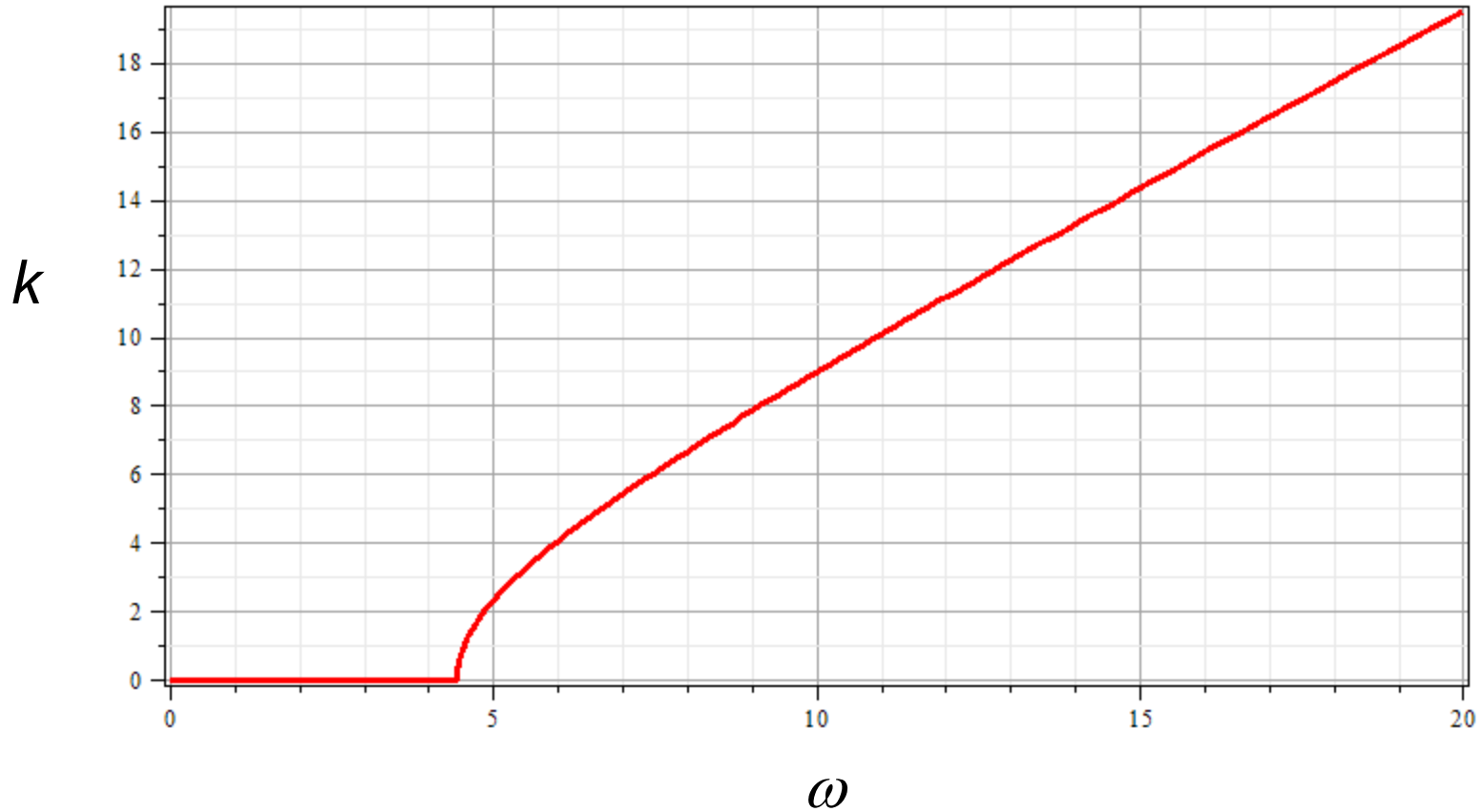
$$iE_x(x, y) = B_0 \left(\frac{\omega n \pi / b}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \right) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$iE_y(x, y) = B_0 \left(\frac{-\omega m \pi / a}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \right) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$



Solution for m=n=1

$$k^2 \equiv k_{mn}^2 = \mu\epsilon\omega^2 - \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]$$



Solution for $m=n=1$ -- more details

$$B_z(x, y) = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$iE_x(x, y) = B_0 \left(\frac{\omega n \pi / b}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \right) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) = i \frac{\omega}{k} B_y(x, y)$$

$$iE_y(x, y) = B_0 \left(\frac{-\omega m \pi / a}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \right) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) = -i \frac{\omega}{k} B_x(x, y)$$

Poynting vector for this case:

$$\langle \mathbf{S} \rangle_{avg} = \frac{1}{2} \Re(\mathbf{E} \times \mathbf{H}^*) = \frac{1}{2} \Re \left(\begin{array}{ccc} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ E_x & E_y & 0 \\ H_x^* & H_y^* & H_z^* \end{array} \right) \quad \text{Note: } \mathbf{H} = \frac{1}{\mu} \mathbf{B}$$
$$= \frac{\hat{\mathbf{z}}}{2} \Re(E_x H_y^* - E_y H_x^*) \quad (\text{direction along the wave guide})$$