PHY 712 Electrodynamics
10-10:50 AM in Olin 103

Notes for Lecture 21:
Propagating EM waves in optical waveguides and cavities

Chap. 8 in Jackson –

1. Review of rectangular waveguides
2. Optical cavitites
3. Cylindrical waveguides
4. Coaxial cables

17	Fri: 02/21/2025	Chap. 7	Electromagnetic plane waves	#16	02/24/2025
18	Mon: 02/24/2025	Chap. 7	Electromagnetic response functions	#17	02/26/2025
19	Wed: 02/26/2025	Chap. 7	Optical effects of refractive indices	#18	02/28/2025
20	Fri: 02/28/2025	Chap. 8	Waveguides		
21	Mon: 03/03/2025	Chap. 8	Waveguides	#19	03/05/2025
22	Wed: 03/05/2025	Chap. 9	Radiation from localized sources	Topics choice	03/07/2025
23	Fri: 03/07/2025		Review		
	Mon: 03/10/2025	No class	*Spring Break*		
	Wed: 03/12/2025	No class	*Spring Break*		
	Fri: 03/14/2025	No class	*Spring Break*		
	Mon: 03/17/2025	No class	*Take-home exam*		
	Wed: 03/19/2025	No class	*Take-home exam*		
	Fri: 03/21/2025	No class	*Take-home exam*		
24	Mon: 03/24/2025	Chap. 9	Radiation from localized sources		

PHY 712 – Problem Set #19

Assigned: 03/03/2025 Due: 03/05/2025

Complete reading Chapter 8 in **Jackson**.

1. Consider the ideal retangular waveguide discussed in Sec. 8.4 of **JDJ** and in class which includes a uniform dielectric material with uniform real response parameters ε and μ within an ideal metallic coating. The device is extended along the **z** direction and the dielectric material has lengths a and b in the **x** and **y** directions. The textbook and class notes analyze the TE modes of this system. For the TM modes, $B_z = H_z = 0$ and for a real amplitude E_0 it can be assumed that the **z** component of the electric field has the form

$$E_z(x,y) = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{b}\right),$$

for integers m and n. For this TM case, find the x and y field components for lowest frequency propagating mode and the corresponding magnitude of the Poynting vector.

Review of waveguides

Note that, JDJ and these lecture notes focus on waveguides and cavities based on devices with ideal metal boundaries and real dielectric interiors.   There are many other possibilities.

Boundary values of EM fields near the surface of an ideal conductor

Inside the conductor :

$$\mathbf{H}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\Re\left(\mathbf{H}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

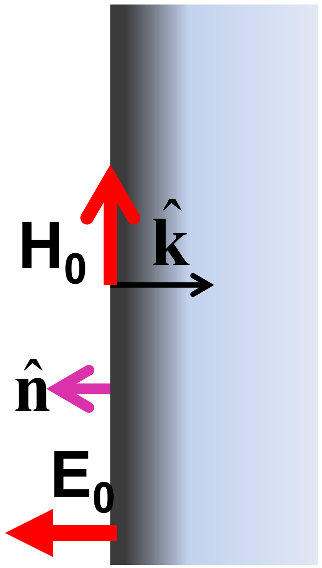
$$\mathbf{E}(\mathbf{r},t) = \delta\mu\omega\frac{1-i}{2}\hat{\mathbf{k}}\times\mathbf{H}(\mathbf{r},t)$$

At the boundary of an ideal conductor, the **E** and **H** fields decay in the direction normal to the interface.

➔These properties control the forms of EM waves that can exist close to these boundaries

Ideal conductor boundary conditions:

$$\hat{\mathbf{n}}\times\mathbf{E}\Big|_S = 0 \qquad \hat{\mathbf{n}}\cdot\mathbf{H}\Big|_S = 0$$

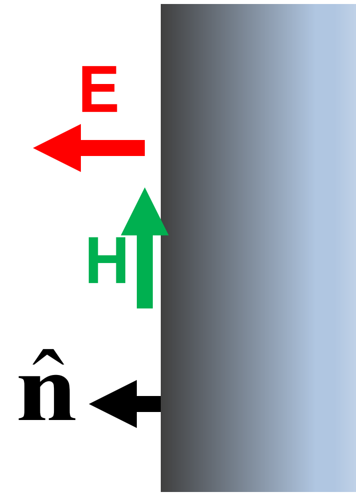


Wave guides – dielectric media with one or more metal boundary

Continuity conditions for fields near metal boundaries --

Ideal conductor boundary conditions:

$$\hat{\mathbf{n}} \times \mathbf{E}\Big|_S = 0 \qquad \hat{\mathbf{n}} \cdot \mathbf{H}\Big|_S = 0$$



Waveguide terminology

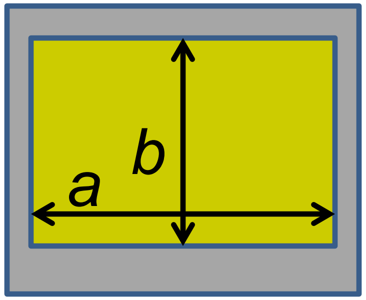
- TEM: transverse electric and magnetic (both E and H fields are perpendicular to wave propagation direction)
- TM: transverse magnetic (H field is perpendicular to wave propagation direction)
- TE: transverse electric (E field is perpendicular to wave propagation direction)

Analysis of rectangular waveguide

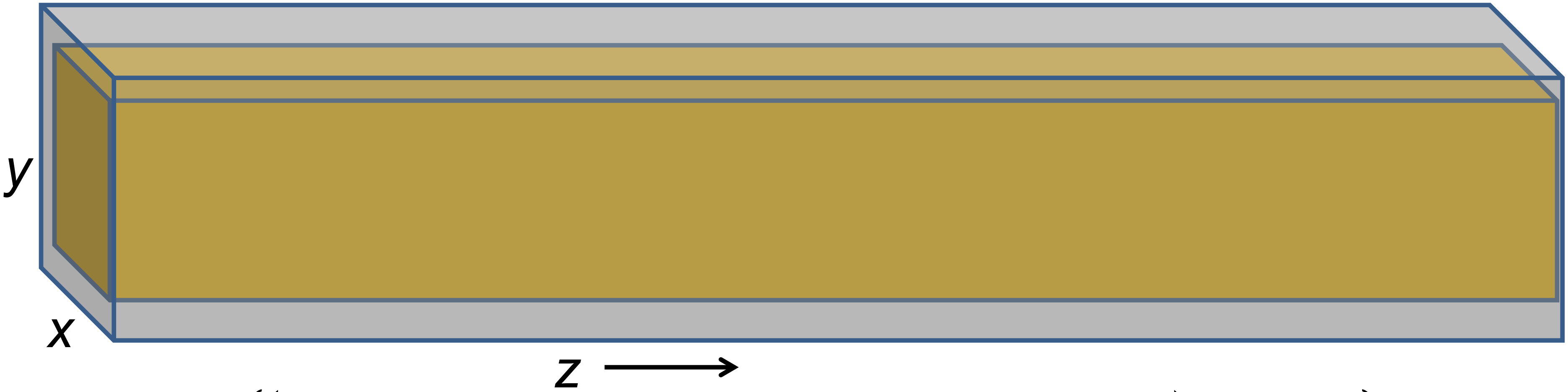
Boundary conditions at surface of waveguide:

$\mathbf{E}_{\text{tangential}}=0, \quad \mathbf{B}_{\text{normal}}=0$

Cross section view



Analysis of rectangular waveguide



$$\mathbf{B} = \Re\left\{\left(B_x(x,y)\hat{\mathbf{x}} + B_y(x,y)\hat{\mathbf{y}} + B_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

$$\mathbf{E} = \Re\left\{\left(E_x(x,y)\hat{\mathbf{x}} + E_y(x,y)\hat{\mathbf{y}} + E_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

Inside the dielectric medium: (assume ε,μ to be real)

$$\nabla\cdot\mathbf{E} = 0 \qquad \nabla\cdot\mathbf{B} = 0$$

$$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \qquad \nabla\times\mathbf{B} - \varepsilon\mu\frac{\partial\mathbf{E}}{\partial t} = 0$$

Solution of Maxwell's equations within the pipe:

Combining Faraday's Law and Ampere's Law, we find that each field component must satisfy a two-dimensional Helmholz equation:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-k^2+\mu\varepsilon\omega^2\right)\mathbf{F}(x,y) \;=\; 0.$$

F = **E** or **B**
propagation along *z*.

For the rectangular wave guide discussed in Section 8.4 of your text a solution for a TE mode can have:

$$E_z(x,y)\equiv 0 \quad \text{and} \quad B_z(x,y)=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$

$$\text{with } k^2\equiv k_{mn}^2=\mu\varepsilon\omega^2-\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]$$

Maxwell's equations within the pipe in terms of all 6 components:

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+ikB_z=0.$$

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+ikE_z=0.$$

For TE mode with $E_z \equiv 0$

$$B_x=-\frac{k}{\omega}E_y$$

$$B_y=\frac{k}{\omega}E_x$$

$$\frac{\partial E_z}{\partial y}-ikE_y=i\omega B_x.$$

$$\frac{\partial B_z}{\partial y}-ikB_y=-i\mu\varepsilon\omega E_x.$$

$$ikE_x-\frac{\partial E_z}{\partial x}=i\omega B_y.$$

$$ikB_x-\frac{\partial B_z}{\partial x}=-i\mu\varepsilon\omega E_y.$$

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=i\omega B_z.$$

$$\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=-i\mu\varepsilon\omega E_z.$$

TE modes for rectangular wave guide continued:

$$E_z(x,y) \equiv 0 \quad \text{and} \quad B_z(x,y) = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$

$$E_x = \frac{\omega}{k} B_y = \frac{-i\omega}{k^2 - \mu\varepsilon\omega^2}\frac{\partial B_z}{\partial y} = \frac{-i\omega}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}\frac{n\pi}{b} B_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),$$

$$E_y = -\frac{\omega}{k} B_x = \frac{i\omega}{k^2 - \mu\varepsilon\omega^2}\frac{\partial B_z}{\partial x} = \frac{i\omega}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}\frac{m\pi}{a} B_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right).$$

Check boundary conditions:

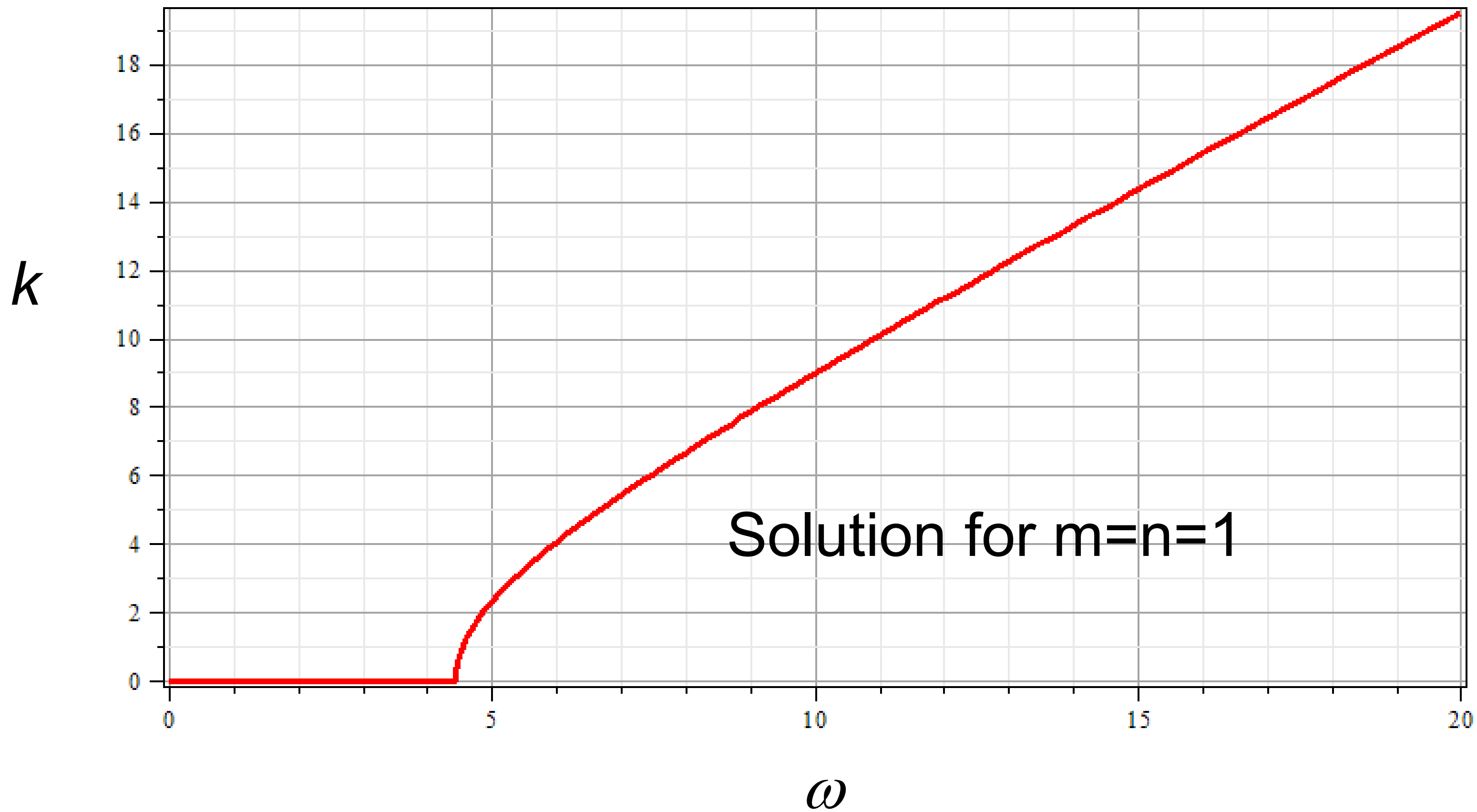
$\mathbf{E}_{\text{tangential}} = 0$ because: $E_z(x,y) \equiv 0, \;\; E_x(x,0) = E_x(x,b) = 0$

and $E_y(0,y) = E_y(a,y) = 0.$

$\mathbf{B}_{\text{normal}} = 0$ because: $B_y(x,0) = B_y(x,b) = 0$

and $B_x(0,y) = B_x(a,y) = 0.$

$$k^2 \equiv k_{mn}^2 = \mu\varepsilon\omega^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$



Summary --

$$\mathbf{B} = \Re\left\{\left(B_x(x,y)\hat{\mathbf{x}} + B_y(x,y)\hat{\mathbf{y}} + B_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

$$\mathbf{E} = \Re\left\{\left(E_x(x,y)\hat{\mathbf{x}} + E_y(x,y)\hat{\mathbf{y}} + E_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

For $E_z(x,y) = 0$ and $B_z(x,y) = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$

$$iE_x(x,y) = B_0\left(\frac{\omega n\pi / b}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) = i\frac{\omega}{k}B_y(x,y)$$

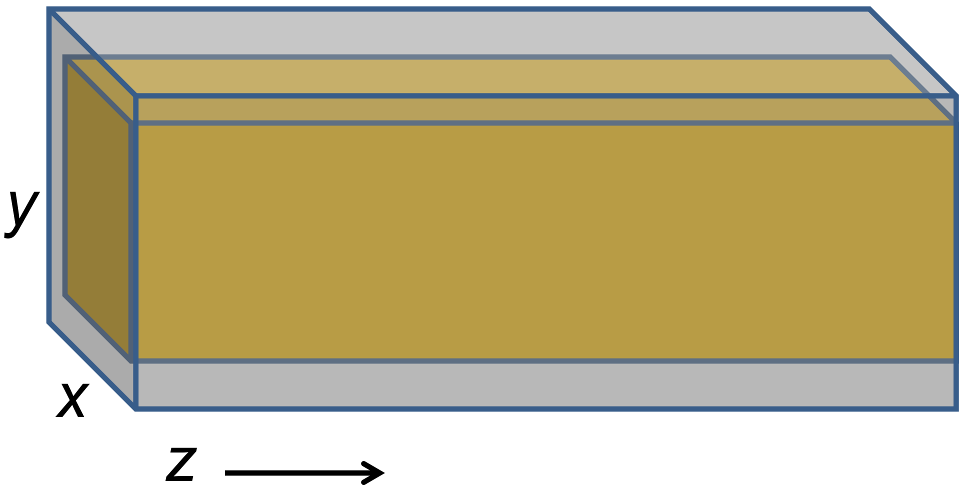
$$iE_y(x,y) = B_0\left(\frac{-\omega m\pi / a}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) = -i\frac{\omega}{k}B_x(x,y)$$

Poynting vector: $\langle \mathbf{S}\rangle_{avg} = \frac{1}{2}\Re\left(\mathbf{E}\times\mathbf{H}^*\right) = \frac{1}{2}\Re\left(\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ E_x & E_y & 0 \\ H_x^* & H_y^* & H_z^* \end{vmatrix}\right)$

$$= \frac{\hat{\mathbf{z}}}{2\mu}\Re\left(E_x B_y^* - E_y B_x^*\right)$$

Now consider the case of a rectangular box bounded by an ideal conductor --

Resonant cavity

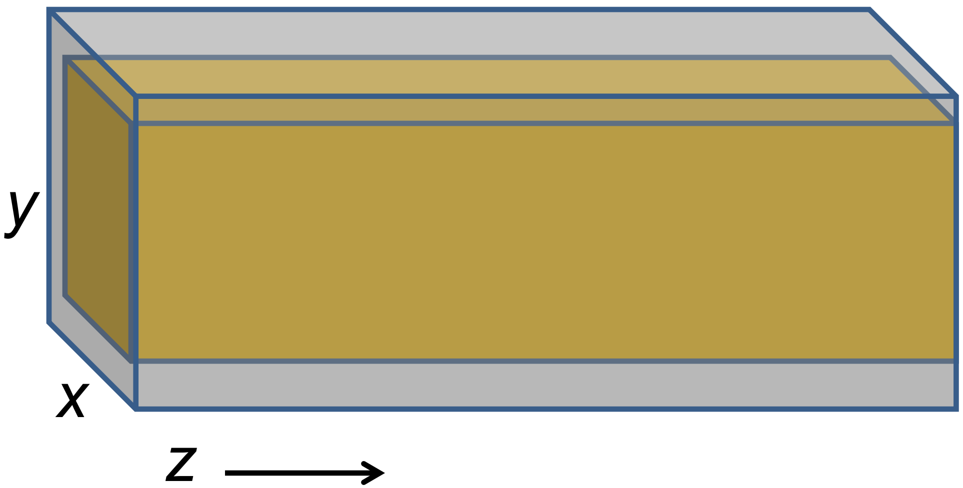


$$0 \le x \le a$$

$$0 \le y \le b$$

$$0 \le z \le d$$

Propagating wave form becomes a standing wave along *z*.



$$0 \le x \le a$$

$$0 \le y \le b$$

$$0 \le z \le d$$

$$\mathbf{B} = \Re\left\{\left(B_x(x,y,z)\hat{\mathbf{x}} + B_y(x,y,z)\hat{\mathbf{y}} + B_z(x,y,z)\hat{\mathbf{z}}\right)e^{-i\omega t}\right\}$$

$$\mathbf{E} = \Re\left\{\left(E_x(x,y,z)\hat{\mathbf{x}} + E_y(x,y,z)\hat{\mathbf{y}} + E_z(x,y,z)\hat{\mathbf{z}}\right)e^{-i\omega t}\right\}$$

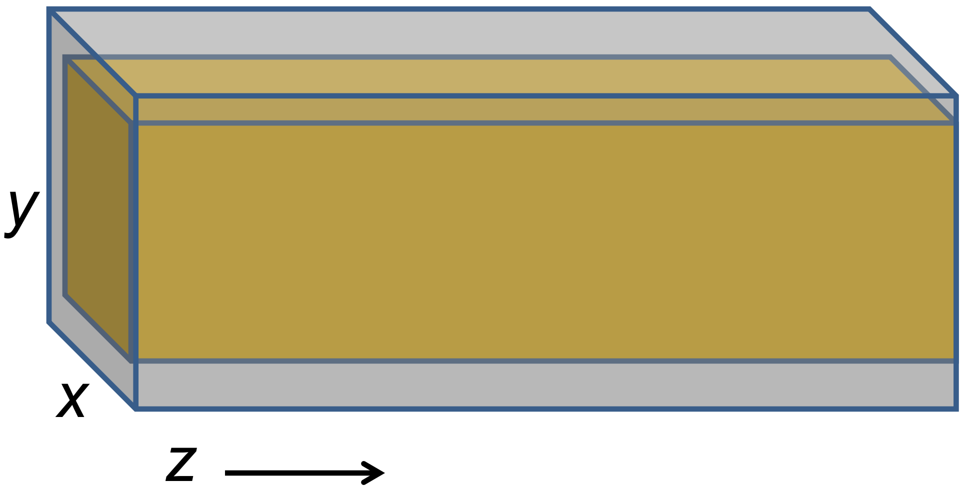
In general: $E_i(x,y,z) = E_i(x,y)\sin(kz)$ or $E_i(x,y)\cos(kz)$

$$B_i(x,y,z) = B_i(x,y)\sin(kz) \text{ or } B_i(x,y)\cos(kz)$$

$$\Rightarrow k = \frac{p\pi}{d}$$

Now consider the case of a rectangular box bounded by an ideal conductor --

Resonant cavity



$$0 \leq x \leq a$$

$$0 \leq y \leq b$$

$$0 \leq z \leq d$$

$$k^2 = \left(\frac{p\pi}{d}\right)^2 = \mu\varepsilon\omega^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

$$\Rightarrow \omega^2 = \frac{1}{\mu\varepsilon}\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right)$$

For example --

$$B_z(x,y,z) = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right)$$

$$\Rightarrow \omega^2 = \frac{1}{\mu\varepsilon}\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right)$$

Typically, microwave oven use frequencies of 2.45 GHz and the wavelength is ~12 cm

Resources online:
https://www.sfu.ca/phys/346/121/resources/physics_of_microwave_ovens.pdf
https://impi.org/wp-content/uploads/2019/09/History-MW-ovens.pdf

Other info on waveguides for fiber optics –
https://www.thefoa.org/tech/wavelength.htm

Other geometries -- a simple cylindrical waveguide

For the rectangular geometry, Maxwell's equations within the pipe:

$$\mathbf{B} = \Re\left\{\left(B_x(x,y)\hat{\mathbf{x}} + B_y(x,y)\hat{\mathbf{y}} + B_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

$$\mathbf{E} = \Re\left\{\left(E_x(x,y)\hat{\mathbf{x}} + E_y(x,y)\hat{\mathbf{y}} + E_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

Inside the dielectric medium: (assume ε, μ to be real)

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \times \mathbf{B} - \varepsilon\mu\frac{\partial \mathbf{E}}{\partial t} = 0$$

Simple cylindrical waveguide -- continued

For the cylindrical geometry, Maxwell's equations within the pipe:

$$\mathbf{B} = \Re\left\{\left(\mathbf{B}_T(r,\varphi) + B_z\left(r,\varphi\right)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

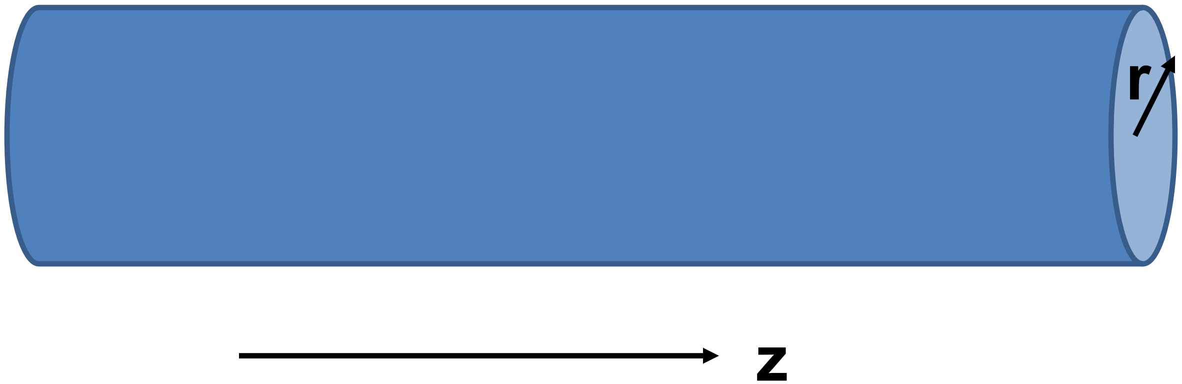
$$\mathbf{E} = \Re\left\{\left(\mathbf{E}_T(r,\varphi) + E_z\left(r,\varphi\right)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

$$\mathbf{E_T} = E_r(r,\varphi)\hat{\mathbf{r}} + E_\varphi(r,\varphi)\hat{\boldsymbol{\varphi}} \qquad \mathbf{B_T} = B_r(r,\varphi)\hat{\mathbf{r}} + B_\varphi(r,\varphi)\hat{\boldsymbol{\varphi}}$$

Using Maxwell's equations within medium:

$$\nabla\cdot\mathbf{E} = 0 \quad \nabla\cdot\mathbf{B} = 0 \quad \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \qquad \nabla\times\mathbf{B} - \varepsilon\mu\frac{\partial\mathbf{E}}{\partial t} = 0$$

Simple cylindrical waveguide



$$\mathbf{B} = \Re\left\{\left(\mathbf{B}_T(\mathbf{r}_T) + B_z\left(\mathbf{r}_T\right)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

$$\mathbf{E} = \Re\left\{\left(\mathbf{E}_T(\mathbf{r}_T) + E_z\left(\mathbf{r}_T\right)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

Decoupling the equations for $F(\mathbf{r}_T) = B_z(\mathbf{r}_T)$ or $F(\mathbf{r}_T) = E_z(\mathbf{r}_T)$:

Combining equations: $\left(\nabla_T^2 - k^2 + \mu\varepsilon\omega^2\right) F(\mathbf{r}_T) \;=\; 0.$

For the cylindrical case, $F(\mathbf{r}_T) \rightarrow F(r,\varphi)$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} - k^2 + \mu\varepsilon\omega^2\right)\mathbf{F}(r,\varphi) \;=\; 0.$$

Simple cylindrical waveguide -- continued

Evaluating $F(r,\varphi) = B_z(\mathbf{r}_T)$ for TE mode or $F(r,\varphi) = E_z(\mathbf{r}_T)$ for TM mode:

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} - k^2 + \mu\varepsilon\omega^2 \right) F(r,\varphi) = 0$$

The components take form $F(r,\varphi) = f_m(r) e^{im\varphi}$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - k^2 + \mu\varepsilon\omega^2 \right) f_m(r) = 0$$

For $\kappa^2 \equiv k^2 - \mu\varepsilon\omega^2$ $\quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - \kappa^2 \right) f_m(r) = 0$

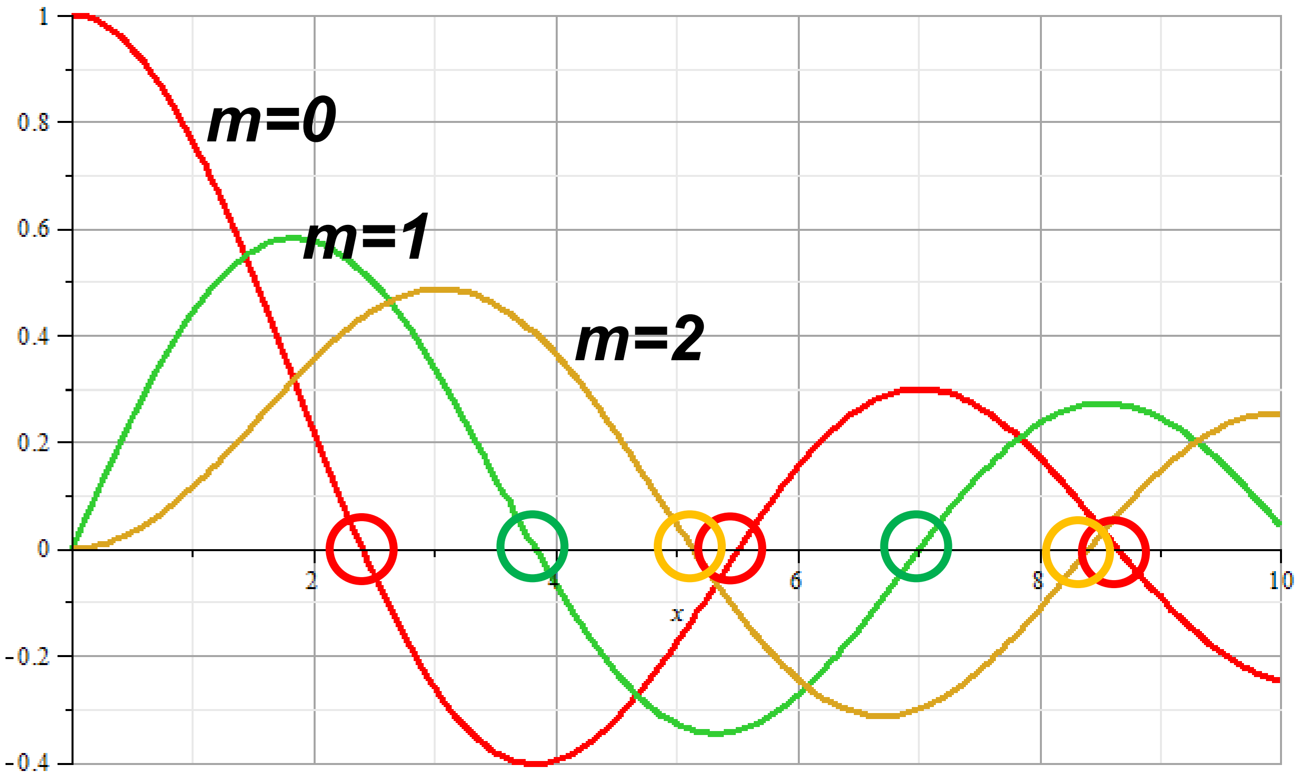
$\Rightarrow f_m(r) = J_m(\kappa r)$ Bessel functions

Boundary values will be applied for $r = a$

In some cases, for zeroes of Bessel function $J_m(\kappa a) = 0$

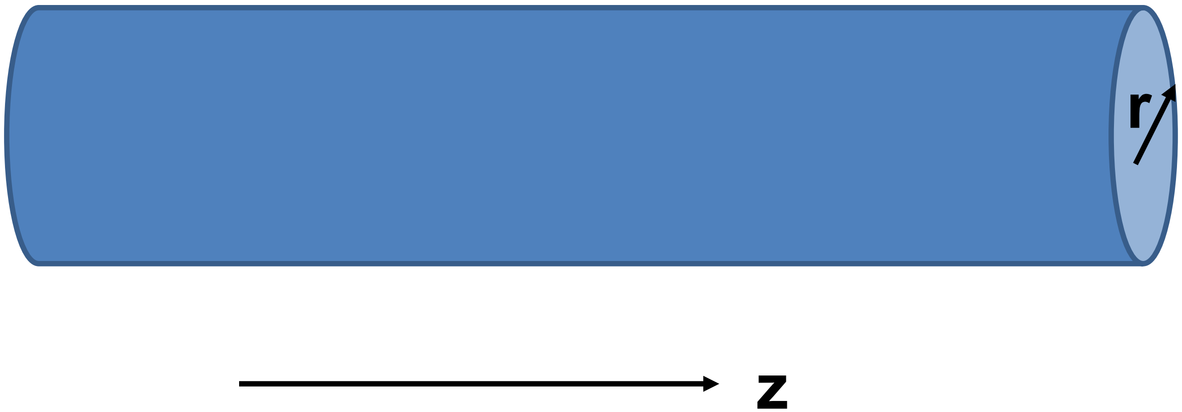
$$\Rightarrow \kappa = \frac{x_{mn}}{a} \quad \text{for } J_m(x_{mn}) = 0$$

Bessel functions : $J_m(x)$



Locations of zeroes of Bessel functions x_{nm}

After finding $F(r,\varphi) = B_z(\mathbf{r}_T)$ for TE mode or $F(r,\varphi) = E_z(\mathbf{r}_T)$ for TM mode, the analysis continues, finding $\mathbf{B_T}(\mathbf{r}_T)$ and $\mathbf{E_T}(\mathbf{r}_T)$ $--$



$$\mathbf{B} = \Re\left\{\left(\mathbf{B}_T(\mathbf{r}_T) + B_z\left(\mathbf{r}_T\right)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

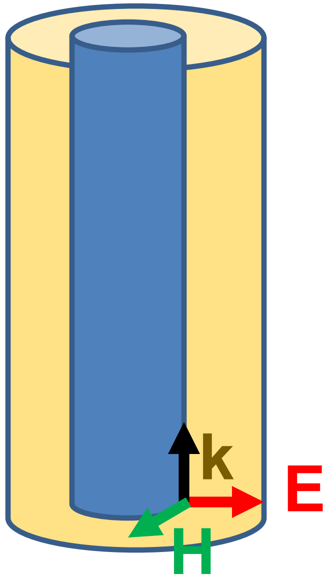
$$\mathbf{E} = \Re\left\{\left(\mathbf{E}_T(\mathbf{r}_T) + E_z\left(\mathbf{r}_T\right)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

Note that in the examples with the “simple” waveguide, having a single outer metallic boundary, TE or TM modes can be produced.

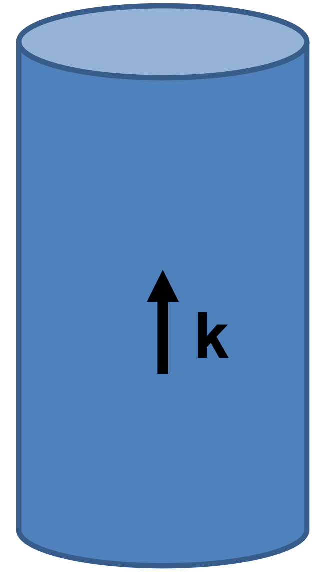
For a more complicated design, it is possible to have TEM modes --

Wave guides – dielectric media with one or more metal boundary

Coaxial cable
TEM modes



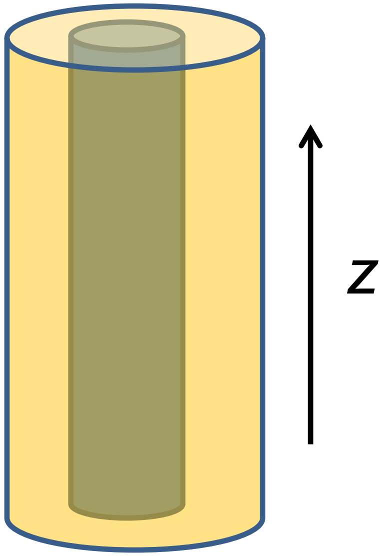
Simple optical pipe
TE or TM modes



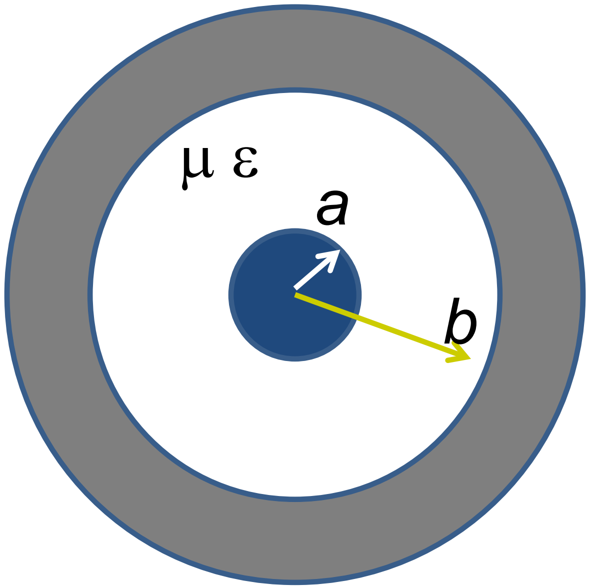
Waveguide terminology

- TEM: transverse electric and magnetic (both E and H fields are perpendicular to wave propagation direction)
- TM: transverse magnetic (H field is perpendicular to wave propagation direction)
- TE: transverse electric (E field is perpendicular to wave propagation direction)

Wave guides



Top view:



Inside medium, μ ε assumed to be real

Coaxial cable
TEM modes

(following problem 8.2 in Jackson's text)

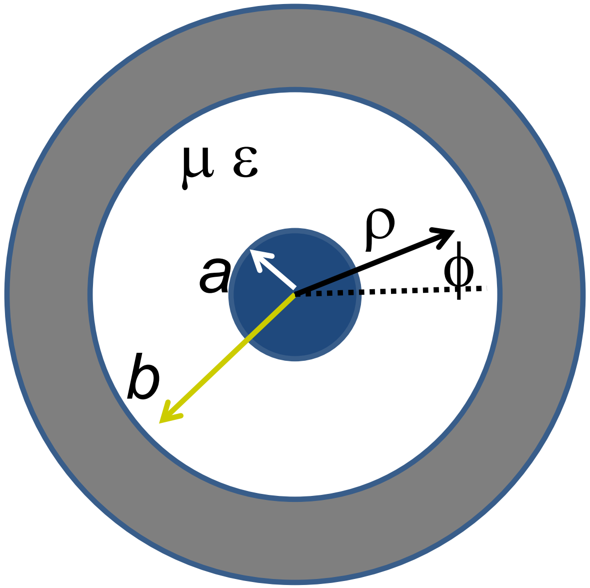
Maxwell's equations inside medium: for $a \leq \rho \leq b$

$$\nabla \times \mathbf{E} = i\omega \mathbf{B} \qquad \nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{B} = -i\omega\mu\varepsilon \mathbf{E} \qquad \nabla \cdot \mathbf{B} = 0$$

Electromagnetic waves in a coaxial cable -- continued

Top view:



Example solution for $a \leq \rho \leq b$

$$\mathbf{E} = \hat{\boldsymbol{\rho}}\Re\left(\frac{E_0 a}{\rho} e^{ikz - i\omega t}\right)$$

$$\mathbf{B} = \hat{\boldsymbol{\varphi}}\Re\left(\frac{B_0 a}{\rho} e^{ikz - i\omega t}\right)$$

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\varphi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

Find:

$$k = \omega\sqrt{\mu\varepsilon}$$

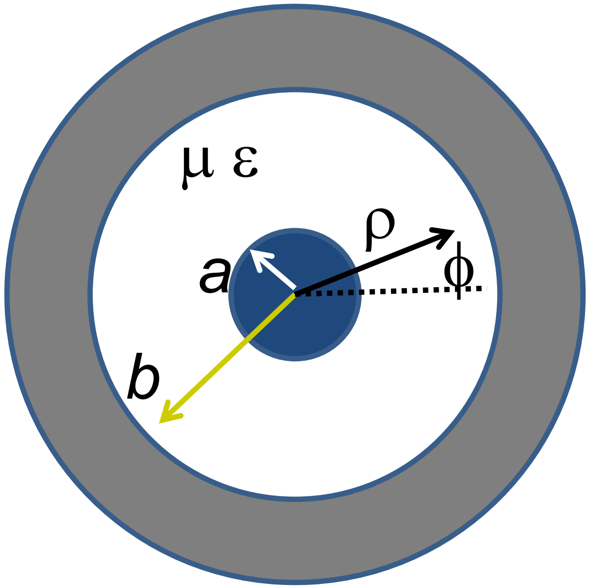
$$E_0 = \frac{B_0}{\sqrt{\mu\varepsilon}}$$

Poynting vector within cable medium (with μ, ε):

$$\langle \mathbf{S} \rangle_{avg} = \frac{1}{2\mu}\Re\left(\mathbf{E} \times \mathbf{B}^*\right) = \frac{|B_0|^2}{2\mu\sqrt{\mu\varepsilon}}\left(\frac{a}{\rho}\right)^2 \hat{\mathbf{z}}$$

Electromagnetic waves in a coaxial cable -- continued

Top view:



Time averaged power in cable material:

$$\int_0^{2\pi} d\phi \int_a^b \rho d\rho \left(\langle \mathbf{S} \rangle_{avg} \cdot \hat{\mathbf{z}} \right) = \frac{|B_0|^2 \pi a^2}{\mu\sqrt{\mu\varepsilon}} \ln\left(\frac{b}{a}\right)$$