PHY 712 Electrodynamics
10-10:50 AM MWF Olin 103

Notes on Lecture 22:

Radiation from localized sources Chap. 9 (Sec. 9.1-9.3)

A. Electromagnetic waves due to specific sources

B. Dipole radiation patterns

Physics Colloquium

- Thursday -

March 6, 2025

Short Bowel Syndrome
– using enough physics (but no more) to aid in surgical intervention

Mark Roberson, PhD, PMP
President
Goldfinch Sensor Technologies and Analytics, LLC

Reception 3:30
Olin Lobby

Colloquium 4:00
Olin 101

Tell me why I care in 50 words: Small infants with SBS are more likely to die. In SBS the small intestines don't absorb enough nutrients. Surgery planning to cut and rearrange the guts is hard because there are not many infants affected. Using live animals is expensive, takes time, and doesn't mimic SBS very well.

Hey – this is the physics department, so what about the physics? At its root, intestinal malabsorption is about fluid mechanics. I know you don't cover it in undergrad physics, but not to worry. The nutrients get absorbed by villi and micro-villi, there's Newtonian and non-Newtonian fluids, computational microfluidics, and a lot of really cutting-edge physics involved. BUT – and this is really important – solving a problem like SBS is about listening to what needs to be done. Physics is the tool. Our job is to use the tool correctly. There is a time for GPU acceleration, and there is a time for "pretty good" approximations.

Will I be qualified to do surgery after this talk? Seriously – you just asked that? No!

Why should I bother going? I'm not a physics major and/or I'm not a pre-med.
It's fascinating and it matters. I mean, I didn't know anything about this until about two years ago when Meagan told me about the problem. I'm a physics guy but never took biology after 9th grade. Dr. Rosenberg is a surgeon in residency but only took first year physics. This problem is about checking egos at the door and solving a problem to save infant lives. I'll scare everyone with some fancy physics talk (and yes, some equations) early on and also talk anatomy. More importantly, I'll talk about how, to help a surgeon plan surgery, you need to keep your eye on what's important.

Anything you want to warn me about? Well, we're talking about intestines, so there will be diagrams and a picture or two of the intestines, and a short video of intestines squishing around (peristalsis). You can close your eyes on those slides. No gut pieces will be passed around.

Fancy buzzwords you can use to impress your friends:
bioinformatics, computational and systems biology, computational

17	Fri: 02/21/2025	Chap. 7	Electromagnetic plane waves	#16	02/24/2025
18	Mon: 02/24/2025	Chap. 7	Electromagnetic response functions	#17	02/26/2025
19	Wed: 02/26/2025	Chap. 7	Optical effects of refractive indices	#18	02/28/2025
20	Fri: 02/28/2025	Chap. 8	Waveguides		
21	Mon: 03/03/2025	Chap. 8	Waveguides	#19	03/05/2025
22	Wed: 03/05/2025	Chap. 9	Radiation from localized sources	Topics choice	03/07/2025
23	Fri: 03/07/2025		Review		
	Mon: 03/10/2025	No class	*Spring Break*		
	Wed: 03/12/2025	No class	*Spring Break*		
	Fri: 03/14/2025	No class	*Spring Break*		
	Mon: 03/17/2025	No class	*Take-home exam*		
	Wed: 03/19/2025	No class	*Take-home exam*		
	Fri: 03/21/2025	No class	*Take-home exam*		
24	Mon: 03/24/2025	Chap. 9	Radiation from localized sources		

Maxwell's equations

Microscopic or vacuum form $(\mathbf{P}=0;\ \mathbf{M}=0)$:

Coulomb's law: $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\varepsilon_0 \mu_0}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials

$$\nabla \cdot \mathbf{B} = 0 \qquad \Rightarrow \boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

$$\text{or} \quad \boxed{\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 :$$

$$-\nabla^2 \Phi - \frac{\partial (\nabla \cdot \mathbf{A})}{\partial t} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \left(\frac{\partial (\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

Complicated coupled mess!

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Lorenz gauge form -- require: $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2}\frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2}\frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

This choice decouples the equations for the scalar and vector potentials.

General equation form:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi = -4\pi f$$

$$\Psi(\mathbf{r},t) = \begin{cases} \Phi(\mathbf{r},t) \\ A_x(\mathbf{r},t) \\ A_y(\mathbf{r},t) \\ A_z(\mathbf{r},t) \end{cases} \qquad f(\mathbf{r},t) = \begin{cases} \rho(\mathbf{r},t)/(4\pi\varepsilon_0) \\ \mu_0 J_x(\mathbf{r},t)/(4\pi) \\ \mu_0 J_y(\mathbf{r},t)/(4\pi) \\ \mu_0 J_z(\mathbf{r},t)/(4\pi) \end{cases}$$

Solution of Maxwell's equations in the Lorenz gauge -- continued

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-\left(t-|\mathbf{r}-\mathbf{r}'|/c\right)\right)$$

Solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-\left(t-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

Electromagnetic waves from time harmonic sources

Charge density: $\rho(\mathbf{r},t) = \Re\left(\tilde{\rho}(\mathbf{r},\omega)e^{-i\omega t}\right)$

Current density: $\mathbf{J}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{J}}(\mathbf{r},\omega)e^{-i\omega t}\right)$

Note that the continuity condition applies:

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t} + \nabla\cdot\mathbf{J}(\mathbf{r},t) = 0 \quad \Rightarrow -i\omega\tilde{\rho}(\mathbf{r},\omega) + \nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega) = 0$$

General source: $f(\mathbf{r},t) = \Re\left(\tilde{f}(\mathbf{r},\omega)e^{-i\omega t}\right)$

For $\tilde{f}(\mathbf{r},\omega) = \frac{1}{4\pi\varepsilon_0}\tilde{\rho}(\mathbf{r},\omega)$

or $\tilde{f}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\tilde{J}_i(\mathbf{r},\omega)$

Note that this is a very different situation from that considered in for Liénard-Wiechert radiation.

Electromagnetic waves from time harmonic sources – continued:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

$$\tilde{\Psi}(\mathbf{r},\omega)e^{-i\omega t} = \tilde{\Psi}_{f=0}(\mathbf{r},\omega)e^{-i\omega t} +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) \tilde{f}(\mathbf{r}',\omega)e^{-i\omega t'}$$

$$= \tilde{\Psi}_{f=0}(\mathbf{r},\omega)e^{-i\omega t} + \int d^3r' \frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{f}(\mathbf{r}',\omega)e^{-i\omega t}$$

After evaluating time integral.

Electromagnetic waves from time harmonic sources – continued – using usual notation --

For scalar potential (Lorenz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega),$$

where $\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\Phi}_0(\mathbf{r},\omega) = 0$

For vector potential (Lorenz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega),$$

where $\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\mathbf{A}}_0(\mathbf{r},\omega) = 0$

Electromagnetic waves from time harmonic sources – continued:

Useful identity:

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

Spherical Bessel function: $j_l(kr)$

Spherical Hankel function: $h_l(kr) = j_l(kr) + in_l(kr)$

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm}\tilde{\phi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

Electromagnetic waves from time harmonic sources – continued:

Note that:

$$\tilde{\varphi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\, j_l(kr_<)\, h_l(kr_>)\, Y^*_{lm}(\hat{\mathbf{r}}')$$

$$= \frac{ik}{\varepsilon_0}\int d\Omega' Y^*_{lm}(\hat{\mathbf{r}}')\int r'^2\, dr'\,\tilde{\rho}(\mathbf{r}',\omega)\, j_l(kr_<)\, h_l(kr_>)$$

$$\int r'^2\, dr'\,\tilde{\rho}(\mathbf{r}',\omega)\, j_l(kr_<)\, h_l(kr_>) = h_l(kr)\int_0^r r'^2\, dr'\,\tilde{\rho}(\mathbf{r}',\omega)\, j_l(kr') +$$

$$j_l(kr)\int_r^\infty r'^2\, dr'\,\tilde{\rho}(\mathbf{r}',\omega)\, h_l(kr')$$

Electromagnetic waves from time harmonic sources – continued:

Useful identity:

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

Spherical Bessel function: $j_l(kr)$

Spherical Hankel function: $h_l(kr) = j_l(kr) + in_l(kr)$

$$\widetilde{\mathbf{A}}(\mathbf{r},\omega) = \widetilde{\mathbf{A}}_0(\mathbf{r},\omega) + \sum_{lm}\widetilde{\mathbf{a}}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\widetilde{\mathbf{a}}_{lm}(r,\omega) = ik\mu_0\int d^3r'\,\widetilde{\mathbf{J}}(\mathbf{r}',\omega)j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

Forms of spherical Bessel and Hankel functions:

$$j_0(x) = \frac{\sin(x)}{x} \qquad h_0(x) = \frac{e^{ix}}{ix}$$

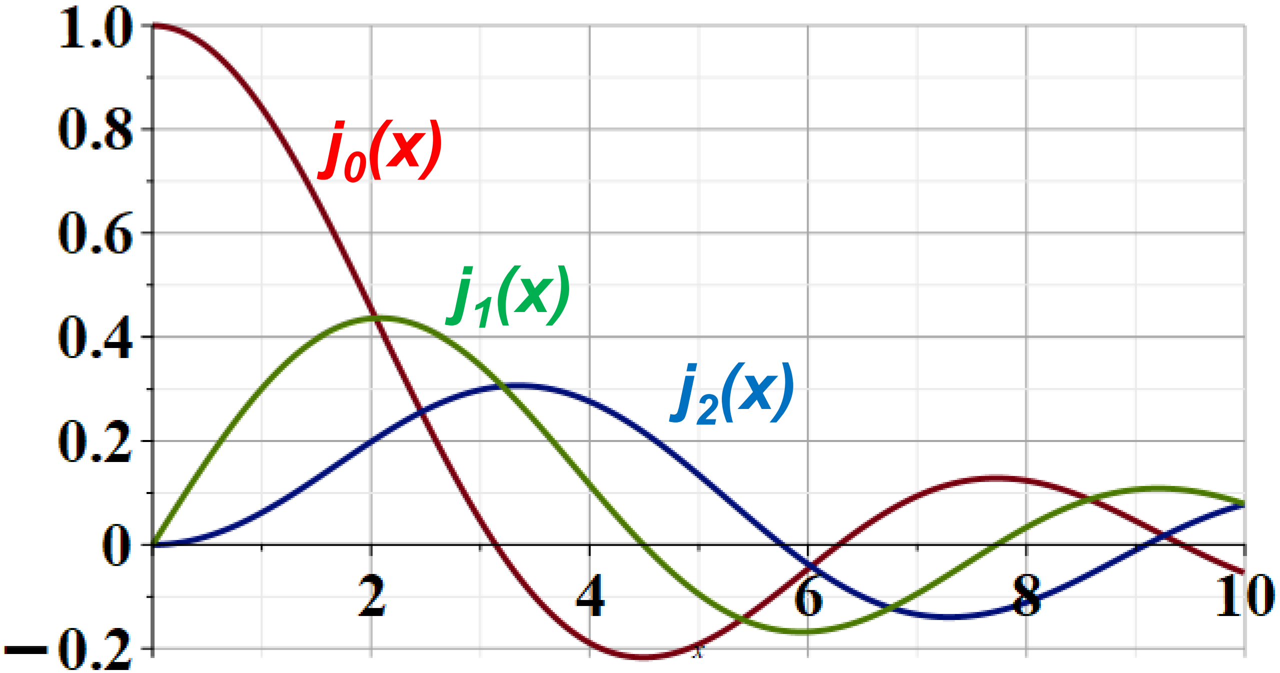
$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \qquad h_1(x) = -\left(1 + \frac{i}{x}\right)\frac{e^{ix}}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin(x) - \frac{3\cos(x)}{x^2} \qquad h_2(x) = i\left(1 + \frac{3i}{x} - \frac{3}{x^2}\right)\frac{e^{ix}}{x}$$

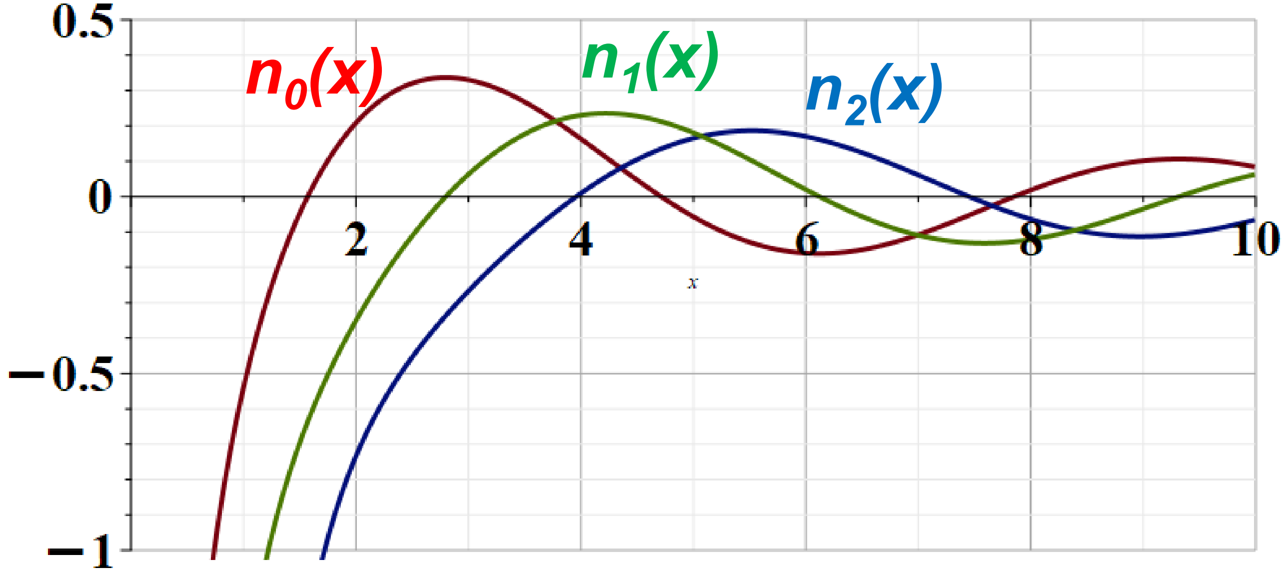
Asymptotic behavior:

$$x << 1 \qquad \Rightarrow j_l(x) \approx \frac{(x)^l}{(2l+1)!!}$$

$$x >> 1 \qquad \Rightarrow h_l(x) \approx (-i)^{l+1}\frac{e^{ix}}{x}$$



1.0
0.8
0.6
0.4
0.2
0
−0.2
2
4
6
8
10
x
$j_0(x)$
$j_1(x)$
$j_2(x)$



0.5
0
−0.5
−1
2
4
6
8
10
x
$n_0(x)$
$n_1(x)$
$n_2(x)$

Digression on spherical Bessel functions --

Consider the homogeneous wave equation

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\Phi}_0(\mathbf{r},\omega) = 0$$

Suppose $\tilde{\Phi}_0(\mathbf{r},\omega) = \psi_{lm}(r)Y_{lm}(\hat{\mathbf{r}})$

$\Rightarrow \psi_{lm}(r)$ must satisfy the following for $k = \omega / c$:

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2\right)\psi_{lm}(r) = 0$$

General spherical Bessel function equation:

$$\left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{l(l+1)}{x^2} + 1\right)w_l(x) = 0 \qquad \Rightarrow \psi_{lm}(r) = w_l(kr)$$

Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm} \tilde{\phi}_{lm}(r,\omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0} \int d^3r' \, \tilde{\rho}(\mathbf{r}',\omega) \, j_l(kr_<) h_l(kr_>) Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \sum_{lm} \tilde{\mathbf{a}}_{lm}(r,\omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega) = ik\mu_0 \int d^3r' \, \tilde{\mathbf{J}}(\mathbf{r}',\omega) \, j_l(kr_<) h_l(kr_>) Y^*_{lm}(\hat{\mathbf{r}}')$$

For $r >>$ (extent of source)

$$\tilde{\phi}_{lm}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_l(kr) \int d^3r' \, \tilde{\rho}(\mathbf{r}',\omega) \, j_l(kr') Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega) \approx ik\mu_0 h_l(kr) \int d^3r' \, \tilde{\mathbf{J}}(\mathbf{r}',\omega) \, j_l(kr') Y^*_{lm}(\hat{\mathbf{r}}')$$

Some details:

$$\tilde{\Phi}(\mathbf{r},\omega)=\tilde{\Phi}_0(\mathbf{r},\omega)+\sum_{lm}\tilde{\varphi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\varphi}_{lm}(r,\omega)=\frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\,j_l(kr_<)\,h_l(kr_>)\,Y^*_{lm}(\hat{\mathbf{r}}')$$

$$=\frac{ik}{\varepsilon_0}\int d\Omega' Y^*_{lm}(\hat{\mathbf{r}}')\left(h_l(kr)\int_0^r r'^2\,dr'\,j_l(kr')\,\tilde{\rho}(\mathbf{r}',\omega)+j_l(kr)\int_r^\infty r'^2\,dr'\,h_l(kr')\,\tilde{\rho}(\mathbf{r}',\omega)\right)$$

For $r >>$ (extent of source)

$$\tilde{\phi}_{lm}(r,\omega)\approx\frac{ik}{\varepsilon_0}h_l(kr)\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\,j_l(kr')Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega)\approx ik\mu_0 h_l(kr)\int d^3r'\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)\,j_l(kr')Y^*_{lm}(\hat{\mathbf{r}}')$$

Electromagnetic waves from time harmonic sources – continued -- some details:

$$\tilde{\varphi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\, j_l(kr_<)\, h_l(kr_>)\, Y^*_{lm}(\hat{\mathbf{r}}')$$

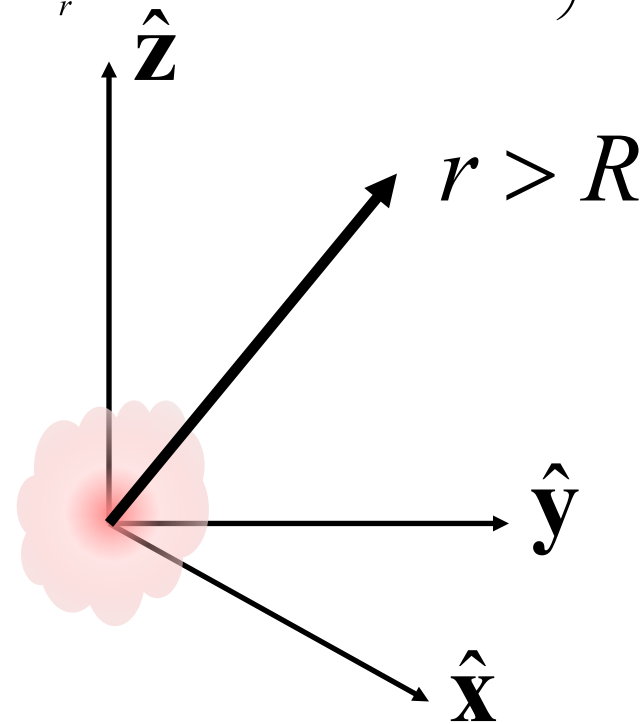
$$= \frac{ik}{\varepsilon_0}\left(h_l(kr)\int_0^r r'^2\, dr'\,\rho_{lm}(\mathbf{r}',\omega) j_l(kr') + j_l(kr)\int_r^\infty r'^2\, dr'\,\rho_{lm}(\mathbf{r}',\omega) h_l(kr') \right)$$

where $\rho_{lm}(\mathbf{r}',\omega) \equiv \int d\Omega'\,\tilde{\rho}(\mathbf{r}',\omega) Y^*_{lm}(\hat{\mathbf{r}}')$

note that for $r > R$, where $\tilde{\rho}(\mathbf{r},\omega) \approx 0$,

$$\tilde{\varphi}_{lm}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_l(kr)\int_0^\infty r'^2\, dr'\,\rho_{lm}(\mathbf{r}',\omega) j_l(kr')$$

Similar relationships can be written for $\tilde{\mathbf{a}}_{lm}(r,\omega)$ and $\tilde{\mathbf{J}}(\mathbf{r}',\omega)$.



Electromagnetic waves from time harmonic sources – continued:

For $r >>$ (extent of source)

$$\tilde{\phi}_{lm}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_l(kr) \int d^3r' \, \tilde{\rho}(\mathbf{r}',\omega) j_l(kr') Y^*{}_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega) \approx ik\mu_0 h_l(kr) \int d^3r' \, \tilde{\mathbf{J}}(\mathbf{r}',\omega) j_l(kr') Y^*{}_{lm}(\hat{\mathbf{r}}')$$

Note that these results are "exact" when r is outside the extent of the charge and current density.

Note that $\widetilde{\rho}(\mathbf{r}',\omega)$ and $\widetilde{\mathbf{J}}(\mathbf{r}',\omega)$ are connected via the continuity condition : $\quad -i\omega\,\widetilde{\rho}(\mathbf{r},\omega)+\nabla\cdot\widetilde{\mathbf{J}}(\mathbf{r},\omega)=0$

$$\widetilde{\phi}_{lm}(r,\omega)\approx\frac{ik}{\varepsilon_0}h_l(kr)\int d^3r'\,\widetilde{\rho}(\mathbf{r}',\omega)\,j_l(kr')Y^*{}_{lm}(\hat{\mathbf{r}}')$$

$$=-\frac{k}{\omega\varepsilon_0}h_l(kr)\int d^3r'\,\widetilde{\mathbf{J}}(\mathbf{r}',\omega)\cdot\nabla'\Big(j_l(kr')Y^*{}_{lm}(\hat{\mathbf{r}}')\Big)$$

Electromagnetic waves from time harmonic sources – continued -- now considering the dipole approximation

Various approximations:

$$kr >> 1 \qquad \Rightarrow h_l(kr) \approx (-i)^{l+1}\frac{e^{ikr}}{kr}$$

$$kr' << 1 \qquad \Rightarrow j_l(kr') \approx \frac{(kr')^l}{(2l+1)!!}$$

Lowest (non-trivial) contributions in l expansions:

$$\tilde{\varphi}_{1m}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_1(kr) \int d^3r' \tilde{\rho}(\mathbf{r}',\omega) \frac{kr'}{3} Y^*_{1m}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{00}(r,\omega) \approx ik\mu_0 h_0(kr) \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}',\omega) Y^*_{00}(\hat{\mathbf{r}}')$$

Some details -- continued: (assuming confined source)

Recall continuity condition: $-i\omega\,\tilde{\rho}(\mathbf{r},\omega)+\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)=0$

$$-i\omega\mathbf{r}\;\tilde{\rho}(\mathbf{r},\omega)+\mathbf{r}\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)$$

$$\int d^3r\;\mathbf{r}\;\tilde{\rho}(\mathbf{r},\omega)=\frac{1}{i\omega}\int d^3r\;\mathbf{r}\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)$$

$$=-\frac{1}{i\omega}\int d^3r\;\tilde{\mathbf{J}}(\mathbf{r},\omega)=\mathbf{p}(\omega)$$

Here we have used the identity:

$$\nabla\cdot(\psi\mathbf{V})=\nabla\psi\cdot\mathbf{V}+\psi(\nabla\cdot\mathbf{V})$$

We have also assumed that

$$\lim_{r\to\infty}\left(x\tilde{\mathbf{J}}(\mathbf{r},\omega)\right)=0$$

Electromagnetic waves from time harmonic sources – in the dipole approximation continued:

Lowest order contribution; dipole radiation:

Define dipole moment at frequency ω:

$$\mathbf{p}(\omega) \equiv \int d^3r\ \mathbf{r}\tilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega}\int d^3r\ \tilde{\mathbf{J}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{ik}{4\pi\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$

Note: in this case we have assumed a restricted extent of the source such that *kr'<<1* for all *r'* with significant charge/current density.

Electromagnetic waves from time harmonic sources – in dipole approximation -- continued:

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = -\nabla\tilde{\Phi}(\mathbf{r},\omega) + i\omega\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left(k^2\left(\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right) + \left(\frac{3\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\mathbf{p}(\omega)\right) - \mathbf{p}(\omega)}{r^2}\right)(1-ikr)\right)$$

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla\times\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0 c^2}\frac{e^{ikr}}{r}k^2\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\left(1-\frac{1}{ikr}\right)$$

Power radiated for $kr >> 1$:

$$\frac{dP}{d\Omega} = r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg} = \frac{r^2}{2\mu_0}\hat{\mathbf{r}}\cdot\Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right)$$

$$= \frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right|^2$$

Example of radiation source -- exact treatment

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0 e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega)=\frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0\,(ik\mu_0)\int_0^\infty r'^2\,dr'\,e^{-r'/R}h_0(kr_>)j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r},\omega)=-\frac{J_0 k}{\varepsilon_0\omega R}\cos\theta\int_0^\infty r'^2\,dr'\,e^{-r'/R}h_1(kr_>)j_1(kr_<)$$

Evaluation for $r >> R$:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0\mu_0\frac{e^{ikr}}{r}\;\frac{2R^3}{(1+k^2R^2)^2}$$

$$\tilde{\Phi}(\mathbf{r},\omega)=\frac{J_0 k}{\varepsilon_0\omega}\cos\theta\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\;\frac{2R^3}{(1+k^2R^2)^2}$$

Example of radiation source – exact treatment continued

Evaluation for $r >> R$:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \; \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = \frac{J_0 k}{\varepsilon_0 \omega} \cos\theta \frac{e^{ikr}}{r} \left(1+\frac{i}{kr}\right) \; \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

Relationship to dipole approximation (exact when $kR \rightarrow 0$)

$$\mathbf{p}(\omega) \equiv \int d^3r \; \mathbf{r}\tilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega}\int d^3r \; \tilde{\mathbf{J}}(\mathbf{r},\omega) = -\frac{8\pi R^3 J_0}{i\omega}\hat{\mathbf{z}}$$

Corresponding dipole fields: $\tilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{ik}{4\pi\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$

Summary of results

Exact -- Evaluation for $r >> R$:

$$\widetilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \; \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

$$\widetilde{\Phi}(\mathbf{r},\omega) = \frac{J_0 k}{\varepsilon_0 \omega} \cos\theta \frac{e^{ikr}}{r} \left(1 + \frac{i}{kr}\right) \; \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

Dipole approximation --

$$\mathbf{p}(\omega) \equiv \int d^3r \; \mathbf{r} \tilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega} \int d^3r \; \tilde{\mathbf{J}}(\mathbf{r},\omega) = -\frac{8\pi R^3 J_0}{i\omega} \hat{\mathbf{z}}$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi} \mathbf{p}(\omega) \frac{e^{ikr}}{r} = 2R^3 J_0 \mu_0 \hat{\mathbf{z}} \frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{ik}{4\pi\varepsilon_0} \mathbf{p}(\omega) \cdot \hat{\mathbf{r}} \left(1 + \frac{i}{kr}\right) \frac{e^{ikr}}{r} = \frac{2R^3 J_0 k}{\varepsilon_0 \omega} \cos\theta \left(1 + \frac{i}{kr}\right) \frac{e^{ikr}}{r}$$