

PHY 712 Electrodynamics

10-10:50 AM MWF Olin 103

Notes on Lecture 24:

Radiation from time harmonic sources

Chap. 9 (Sec. 9.1-9.4)

- A. Discussion of course schedule**
- B. Review of Lecture 22 – radiation from time harmonic localized microscopic sources**
- C. Radiation from time harmonic macroscopic sources**

Tentative schedule for PHY 712

23	Fri: 03/07/2025		Review		
	Mon: 03/10/2025	No class	<i>Spring Break</i>		
	Wed: 03/12/2025	No class	<i>Spring Break</i>		
	Fri: 03/14/2025	No class	<i>Spring Break</i>		
	Mon: 03/17/2025	No class	<i>Take-home exam</i>		
	Wed: 03/19/2025	No class	<i>Take-home exam</i>		
	Fri: 03/21/2025	No class	<i>Take-home exam</i>		
24	Mon: 03/24/2025	Chap. 9	Radiation from time harmonic sources	#20	03/26/2025
25	Wed: 03/26/2025	Chap. 9 & 10	Radiation from scattering		
26	Fri: 03/28/2025	Chap. 11	Special Theory of Relativity		
27	Mon: 03/31/2025	Chap. 11	Special Theory of Relativity		
28	Wed: 04/2/2025	Chap. 11	Special Theory of Relativity		
29	Fri: 04/4/2024	Chap. 14	Radiation from accelerating charged particles		
30	Mon: 04/07/2025	Chap. 14	Radiation from accelerating charged particles		
31	Wed: 04/09/2025	Chap. 14	Synchrotron radiation and Compton scattering		
32	Fri: 04/11/2025	Chap. 13 & 15	Other radiation -- Cherenkov & bremsstrahlung		
33	Mon: 04/14/2025	Special Topics			
34	Wed: 04/16/2025	Special Topics			
35	Fri: 04/18/2025	Special Topics			
	Mon: 04/21/2025		Presentations I		
	Wed: 04/23/2025		Presentations II		
	Fri: 04/25/2025		Presentations III		
36	Mon: 04/28/2025		Review		

Questions for discussion –

1. Preferences for presentations
 - A. 3 presentations/day for 2 days
 - B. 2 presentations/day for 3 days

2. Timing of presentations
 - A. Last full week of classes
 - B. Earlier?
 - C. Other?

Homework assignment from this lecture

PHY 712 -- Assignment #20

Assigned: 3/24/2025 Due: 3/26/2025

Continue reading Chapter 9 (Sec. 9.1-9.4) in **Jackson** .

1. Problem 9.10 in **Jackson** lists the harmonic frequency dependent charge and current densities of a radiating H atom. Instead of answering **Jackson's** questions, calculate the exact scalar $\Phi(\mathbf{r}, \omega_0)$ field for $r \gg a_0$ and compare your results with the scalar potential field calculated within the dipole approximation.

In this problem, you are given the following harmonically oscillating source:

$$\rho(r, \theta, \phi, t) = \frac{2e}{\sqrt{6}a_0^4} r e^{-3r/(2a_0)} Y_{00}(\theta, \phi) Y_{10}(\theta, \phi) e^{-i\omega_0 t}$$

$$\mathbf{J}(r, \theta, \phi, t) = \frac{-iv_0}{2} \left(\frac{1}{2} \hat{\mathbf{r}} + \frac{a_0}{z} \hat{\mathbf{z}} \right) \rho(r, \theta, \phi, t)$$

Maxwell's equations

Microscopic or vacuum form ($\mathbf{P} = 0$; $\mathbf{M} = 0$):

Coulomb's law : $\nabla \cdot \mathbf{E} = \rho / \epsilon_0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\epsilon_0 \mu_0}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \quad \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

$$\text{or} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 :$$

$$-\nabla^2 \Phi - \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \left(\frac{\partial(\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

Complicated coupled mess!

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Lorenz gauge form -- require: $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2} \frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2} \frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \epsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

This choice decouples the equations for the scalar and vector potentials.

General equation form:

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Psi = -4\pi f$$

$$\Psi(\mathbf{r}, t) = \begin{cases} \Phi(\mathbf{r}, t) \\ A_x(\mathbf{r}, t) \\ A_y(\mathbf{r}, t) \\ A_z(\mathbf{r}, t) \end{cases}$$

$$f(\mathbf{r}, t) = \begin{cases} \rho(\mathbf{r}, t) / (4\pi\epsilon_0) \\ \mu_0 J_x(\mathbf{r}, t) / (4\pi) \\ \mu_0 J_y(\mathbf{r}, t) / (4\pi) \\ \mu_0 J_z(\mathbf{r}, t) / (4\pi) \end{cases}$$

Solution of Maxwell's equations in the Lorenz gauge -- continued

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - |\mathbf{r} - \mathbf{r}'|/c\right)\right)$$

Solution for field $\Psi(\mathbf{r}, t)$:

$$\Psi(\mathbf{r}, t) = \Psi_{f=0}(\mathbf{r}, t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right)\right) f(\mathbf{r}', t')$$

Electromagnetic waves from time harmonic sources

Charge density: $\rho(\mathbf{r}, t) = \Re(\tilde{\rho}(\mathbf{r}, \omega) e^{-i\omega t})$

Current density: $\mathbf{J}(\mathbf{r}, t) = \Re(\tilde{\mathbf{J}}(\mathbf{r}, \omega) e^{-i\omega t})$

Note that the continuity condition applies:

$$\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r}, t) = 0 \quad \Rightarrow \quad -i\omega \tilde{\rho}(\mathbf{r}, \omega) + \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r}, \omega) = 0$$

General source: $f(\mathbf{r}, t) = \Re(\tilde{f}(\mathbf{r}, \omega) e^{-i\omega t})$

For $\tilde{f}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0} \tilde{\rho}(\mathbf{r}, \omega)$

or $\tilde{f}(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \tilde{J}_i(\mathbf{r}, \omega)$

Note that this is a very different situation from that considered in for Liénard-Wiechert radiation.

Electromagnetic waves from time harmonic sources – continued:

$$\Psi(\mathbf{r}, t) = \Psi_{f=0}(\mathbf{r}, t) + \int d^3 r' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right)\right) f(\mathbf{r}', t')$$

$$\begin{aligned} \tilde{\Psi}(\mathbf{r}, \omega) e^{-i\omega t} &= \tilde{\Psi}_{f=0}(\mathbf{r}, \omega) e^{-i\omega t} + \\ &\int d^3 r' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right)\right) \tilde{f}(\mathbf{r}', \omega) e^{-i\omega t'} \\ &= \tilde{\Psi}_{f=0}(\mathbf{r}, \omega) e^{-i\omega t} + \int d^3 r' \frac{e^{i\frac{\omega}{c}|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \tilde{f}(\mathbf{r}', \omega) e^{-i\omega t} \end{aligned}$$

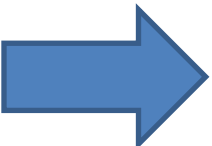
After evaluating time integral.

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik \sum_{lm} j_l(kr_<) h_l(kr_>) Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}')$$

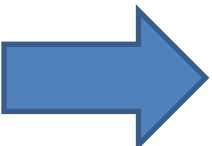
Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$



$$\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\phi}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r, \omega) = \frac{ik}{\epsilon_0} \int d^3r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$$



$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\mathbf{a}}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r, \omega) = ik\mu_0 \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$$

Example radiation source -- exact results for $r \gg R$:

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \quad \tilde{\rho}(\mathbf{r}, \omega) = \frac{J_0}{-i\omega R} \cos \theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 (ik\mu_0) \int_0^\infty r'^2 dr' e^{-r'/R} h_0(kr_>) j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{J_0 k}{\varepsilon_0 \omega R} \cos \theta \int_0^\infty r'^2 dr' e^{-r'/R} h_1(kr_>) j_1(kr_<)$$

Evaluation for $r \gg R$:

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \frac{2R^3}{(1 + k^2 R^2)^2}$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = \frac{J_0 k}{\varepsilon_0 \omega} \cos \theta \frac{e^{ikr}}{r} \left(1 + \frac{i}{kr}\right) \frac{2R^3}{(1 + k^2 R^2)^2}$$

Example radiation source – continued --

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \quad \tilde{\rho}(\mathbf{r}, \omega) = \frac{J_0}{-i\omega R} \cos \theta e^{-r/R}$$

Note that the continuity of charge and current must be satisfied. For the Fourier amplitudes, the relations are as below:

Recall continuity condition: $-i\omega \tilde{\rho}(\mathbf{r}, \omega) + \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r}, \omega) = 0$

Consider the time-harmonic dipole $\mathbf{p}(\omega) \equiv \int d^3r \, \mathbf{r} \tilde{\rho}(\mathbf{r}, \omega)$

Jackson's clever trick*: $-i\omega \mathbf{r} \tilde{\rho}(\mathbf{r}, \omega) + \mathbf{r} \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r}, \omega) = 0$

$$\mathbf{p}(\omega) = \int d^3r \, \mathbf{r} \tilde{\rho}(\mathbf{r}, \omega) = \frac{1}{i\omega} \int d^3r \, \mathbf{r} \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r}, \omega) = -\frac{1}{i\omega} \int d^3r \, \tilde{\mathbf{J}}(\mathbf{r}, \omega)$$

* Provided that $(\mathbf{r} \mathbf{J}(\mathbf{r}, \omega))_{r \rightarrow \infty} = 0$

Example of radiation source

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \quad \tilde{\rho}(\mathbf{r}, \omega) = \frac{J_0}{-i\omega R} \cos \theta e^{-r/R}$$

Dipole moment: $\mathbf{p}(\omega) = \hat{\mathbf{z}} \frac{J_0}{-i\omega R} \int d^3r \, r \cos(\theta) \left(\cos(\theta) e^{-r/R} \right)$

$$= \hat{\mathbf{z}} \frac{J_0}{-i\omega R} \frac{4\pi}{3} \int_0^\infty dr \, r^3 e^{-r/R} = \hat{\mathbf{z}} J_0 \frac{8\pi R^3}{-i\omega}$$
$$= \frac{1}{-i\omega} \int d^3r \, \tilde{\mathbf{J}}(\mathbf{r}, \omega)$$

From the analysis valid for $kr \gg 1$ and $kR \ll 1$:

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = -\frac{i\mu_0\omega}{4\pi} \mathbf{p}(\omega) \frac{e^{ikr}}{r} = \hat{\mathbf{z}} J_0 \mu_0 2R^3 \frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{ik}{4\pi\epsilon_0} \mathbf{p}(\omega) \cdot \hat{\mathbf{r}} \left(1 + \frac{i}{kr} \right) \frac{e^{ikr}}{r} = \frac{J_0 2R^3}{\epsilon_0 c} \left(1 + \frac{i}{kr} \right) \frac{e^{ikr}}{r} \cos \theta$$

More details --

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \quad \tilde{\rho}(\mathbf{r}, \omega) = \frac{J_0}{-i\omega R} \cos \theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 (ik\mu_0) \int_0^\infty r'^2 dr' e^{-r'/R} h_0(kr_>) j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{J_0 k}{\epsilon_0 \omega R} \cos \theta \int_0^\infty r'^2 dr' e^{-r'/R} h_1(kr_>) j_1(kr_<)$$

$$\tilde{\mathbf{A}}(r, \omega) = \hat{\mathbf{z}} J_0 \mu_0 \left(\frac{e^{ikr}}{kr} \int_0^r r' dr' e^{-r'/R} \sin(kr') + \frac{\sin(kr)}{kr} \int_r^\infty r' dr' e^{-r'/R + ikr'} \right)$$

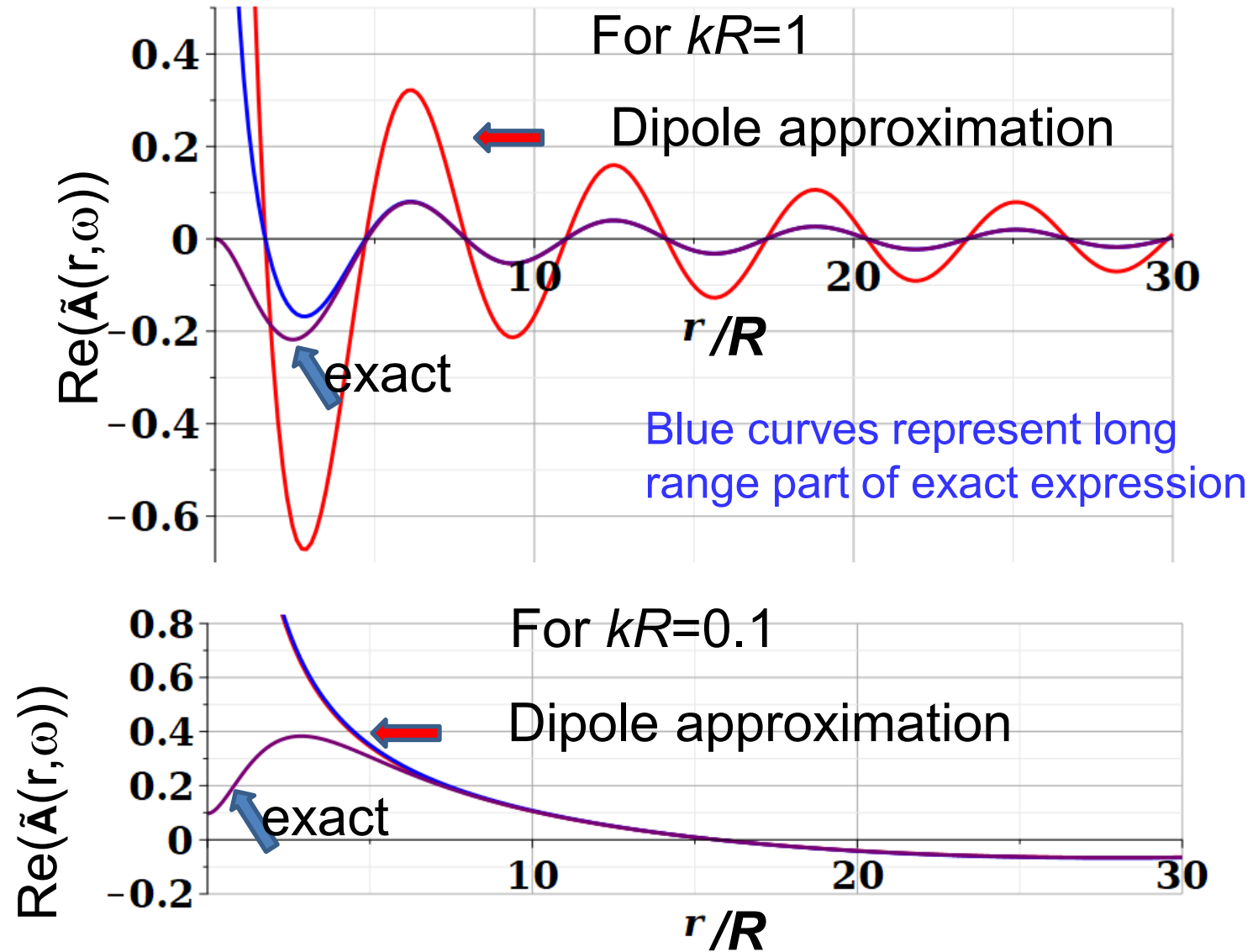
$$\underset{r \gg R}{\approx} \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \frac{2R^3}{(k^2 R^2 + 1)^2}$$

Correct when this term is negligible.

For dipole approximation:

$$\tilde{\mathbf{A}}_{\text{dipole approx}}(r, \omega) = \hat{\mathbf{z}} J_0 \mu_0 \left(\frac{e^{ikr}}{kr} \int_0^\infty r' dr' e^{-r'/R} (kr') \right) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} 2R^3$$

Example continued



Electromagnetic waves from time harmonic sources – continued:

Dipole radiation case:

Define dipole moment at frequency ω :

$$\mathbf{p}(\omega) \equiv \int d^3r \, \mathbf{r} \tilde{\rho}(\mathbf{r}, \omega) = -\frac{1}{i\omega} \int d^3r \, \tilde{\mathbf{J}}(\mathbf{r}, \omega)$$

For fields outside extent of source and $kr' \ll 1$ within the source:

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = -\frac{i\mu_0\omega}{4\pi} \mathbf{p}(\omega) \frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{ik}{4\pi\epsilon_0} \mathbf{p}(\omega) \cdot \hat{\mathbf{r}} \left(1 + \frac{i}{kr} \right) \frac{e^{ikr}}{r}$$

Note: in this case we have assumed a restricted extent of the source such that $kr' \ll 1$ for all r' with significant charge/current density.

Electromagnetic waves from time harmonic sources –
considering the dipole approximation:

$$\begin{aligned}\tilde{\mathbf{E}}(\mathbf{r}, \omega) &= -\nabla \tilde{\Phi}(\mathbf{r}, \omega) + i\omega \tilde{\mathbf{A}}(\mathbf{r}, \omega) \\ &= \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \left(k^2 \left((\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}} \right) + \left(\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}(\omega)) - \mathbf{p}(\omega)}{r^2} \right) (1 - ikr) \right)\end{aligned}$$

$$\begin{aligned}\tilde{\mathbf{B}}(\mathbf{r}, \omega) &= \nabla \times \tilde{\mathbf{A}}(\mathbf{r}, \omega) \\ &= \frac{1}{4\pi\epsilon_0 c} \frac{e^{ikr}}{r} k^2 (\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \left(1 - \frac{1}{ikr} \right)\end{aligned}$$

Power radiated for $kr \gg 1$:

$$\begin{aligned}\frac{dP}{d\Omega} &= r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S} \rangle_{avg} = \frac{r^2}{2\mu_0} \hat{\mathbf{r}} \cdot \Re \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{B}}^*(\mathbf{r}, \omega) \right) \\ &= \frac{c^2 k^4}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} \left| (\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}} \right|^2\end{aligned}$$

Relating dipole radiation results to notation in Jackson --

Power in the dipole approximation; Section 9.2 of Jackson

Here we use our notation with $\mathbf{n} \rightarrow \hat{\mathbf{r}}$ and $Z_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}}$

$$\frac{dP}{d\Omega} = \frac{r^2}{2} \Re \left| \left(\hat{\mathbf{r}} \cdot (\mathbf{E} \times \mathbf{H}^*) \right) \right|^2$$

Using the expressions for the dipole fields far from the source:

$$\mathbf{H} = \frac{ck^2}{4\pi} (\hat{\mathbf{r}} \times \mathbf{p}) \frac{e^{ikr}}{r} \quad \mathbf{E} = Z_0 \mathbf{H} \times \hat{\mathbf{r}}$$

The power can be written
$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \left| ((\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}}) \right|^2$$

Defining the angle θ by $\mathbf{p} \cdot \hat{\mathbf{r}} = |\mathbf{p}| \cos \theta$,

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 |\mathbf{p}|^2 \sin^2 \theta \quad \text{integrating over solid angles} \quad P = \frac{c^2 Z_0}{12\pi} k^4 |\mathbf{p}|^2$$

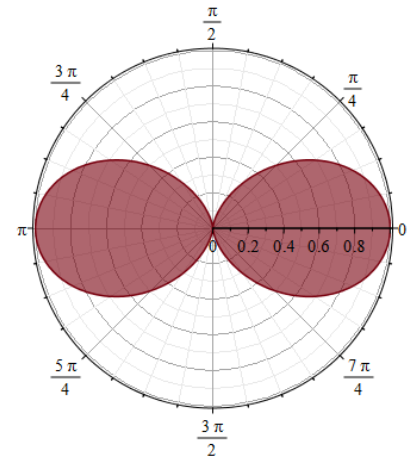
Properties of dipole radiation field for $kr \gg 1$:

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \left(k^2 ((\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}) \right)$$

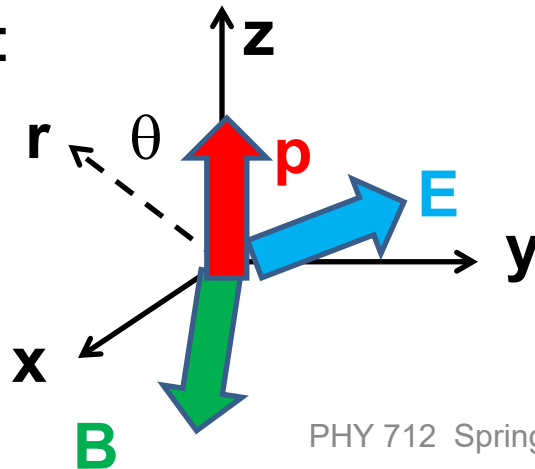
$$\tilde{\mathbf{B}}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0 c^2} \frac{e^{ikr}}{r} k^2 (\hat{\mathbf{r}} \times \mathbf{p}(\omega))$$

Power radiated for $kr \gg 1$:

$$\frac{dP}{d\Omega} = r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S} \rangle_{avg} = \frac{c^2 k^4}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} |(\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}|^2$$



Example:



Note that vectors $\mathbf{r}$, $\mathbf{E}$, $\mathbf{B}$ are mutually orthogonal

Now consider an alternative approach to the spherical Bessel function expansion (neglecting the homogeneous solutions --

Fields from time harmonic source:

$$\tilde{\Phi}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\rho}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

$$\text{For } r \gg r': \quad |\mathbf{r} - \mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}' + \dots$$

$$\tilde{\Phi}(\mathbf{r}, \omega) \approx \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\rho}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

For our example:

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \quad \tilde{\rho}(\mathbf{r}, \omega) = \frac{J_0}{-i\omega R} \cos \theta e^{-r/R}$$

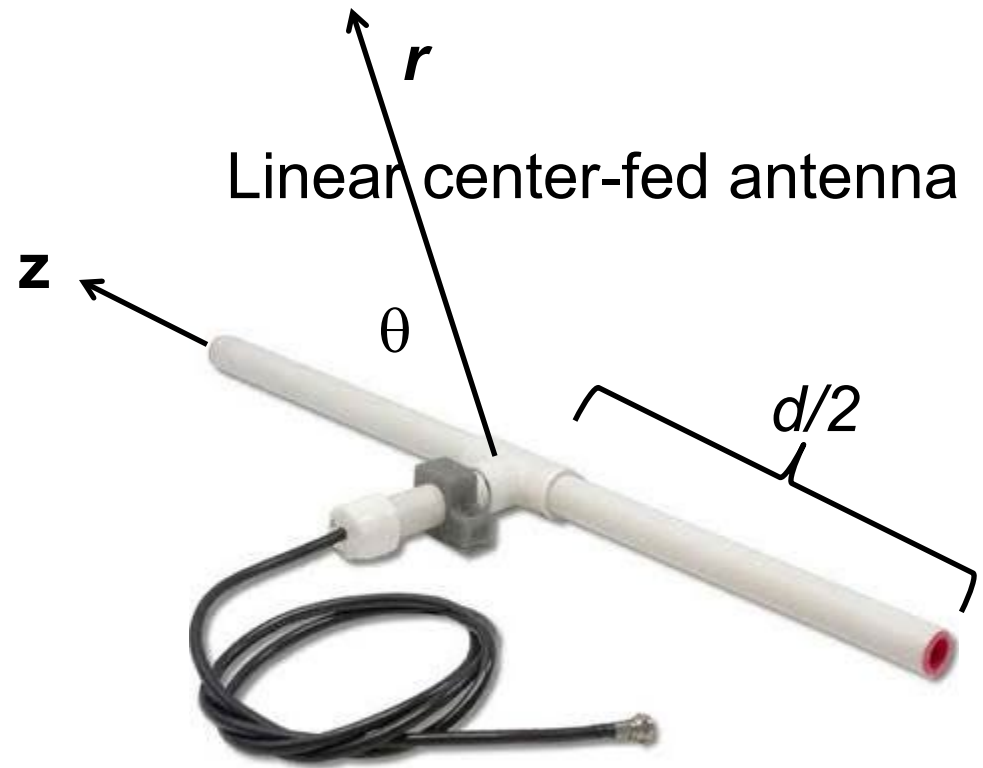
For $r \gg r'$: $|\mathbf{r} - \mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}' + \dots$

$$\tilde{\Phi}(\mathbf{r}, \omega) \approx \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\rho}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

→ Results equivalent to Bessel function expansion in the limit $kr \rightarrow \infty$.

Other time harmonic radiation sources analyzed using the ``alternative approach''

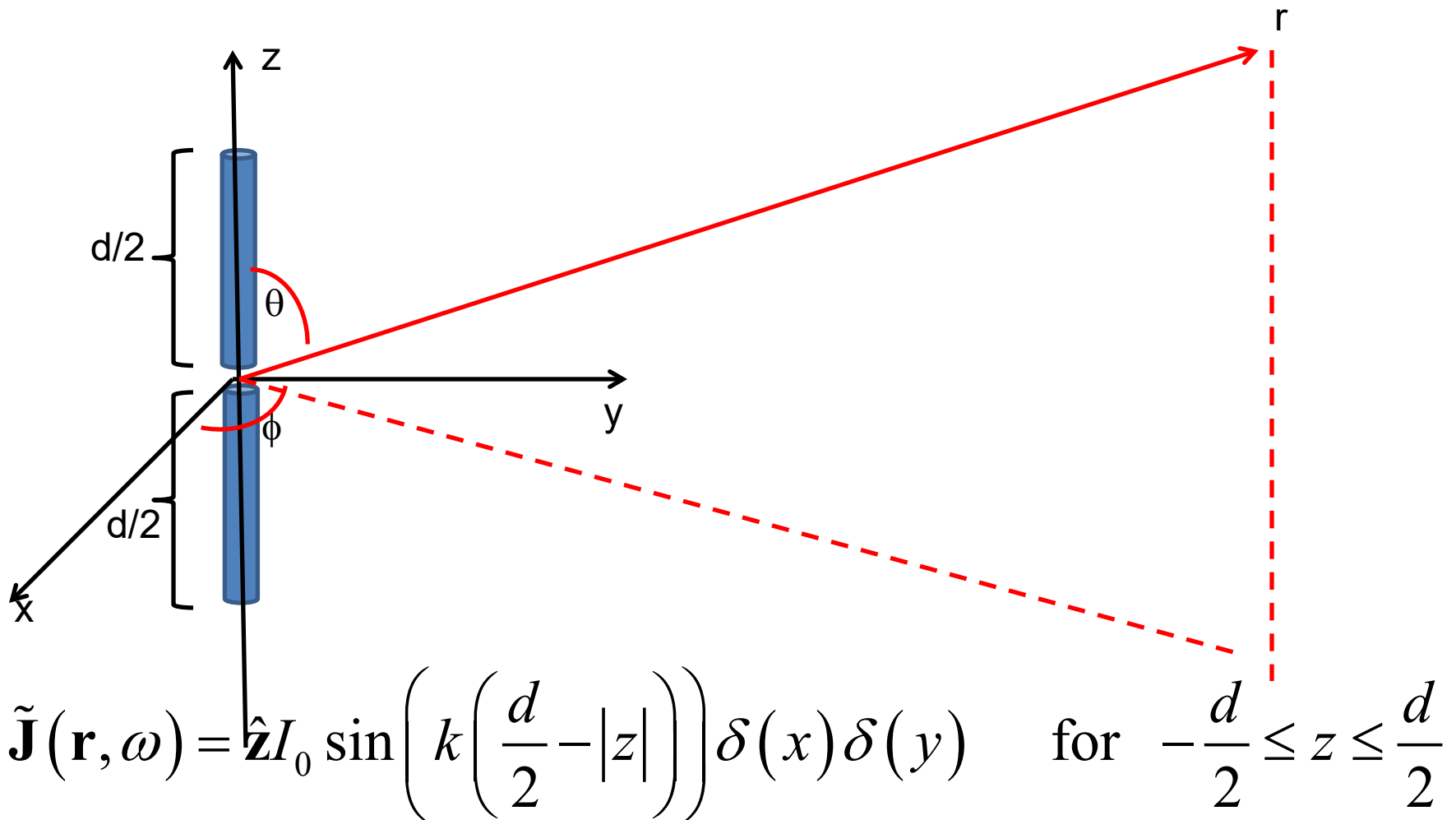


$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx$$

$$\frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r}', \omega) = I_0 \sin\left(\frac{kd}{2} - k|z|\right) \delta(x) \delta(y) \hat{\mathbf{z}}$$

Consider antenna source (center-fed)



$$k \equiv \frac{\omega}{c}$$

Alternative approach – linear center-fed antenna continued

$$\begin{aligned}\tilde{\mathbf{A}}(\mathbf{r}, \omega) &\approx \hat{\mathbf{z}} \frac{\mu_0 I_0}{4\pi} \frac{e^{ikr}}{r} \int_{-d/2}^{d/2} dz' e^{-ik \cos(\theta) z'} \sin\left(\frac{kd}{2} - k|z'|\right) \\ &= \hat{\mathbf{z}} \frac{\mu_0 I_0}{2\pi} \frac{e^{ikr}}{kr} \left(\frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2 \theta} \right)\end{aligned}$$

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin \theta} \right|^2$$

Some details

$$\frac{dP}{d\Omega} = r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S} \rangle_{avg} = \frac{r^2}{2\mu_0} \hat{\mathbf{r}} \cdot \Re(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{B}}^*(\mathbf{r}, \omega))$$

In the appropriate limit:

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) \approx i\omega \tilde{\mathbf{A}}(\mathbf{r}, \omega) \qquad \tilde{\mathbf{B}}(\mathbf{r}, \omega) \approx ik \hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r}, \omega)$$

$$\frac{dP}{d\Omega} \approx \frac{\omega k}{2\mu_0} \left| r \tilde{\mathbf{A}}(\mathbf{r}, \omega) \right|^2 \hat{\mathbf{r}} \cdot (\hat{\mathbf{z}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{z}})) = \frac{\omega k}{2\mu_0} \left| r \tilde{\mathbf{A}}(\mathbf{r}, \omega) \right|^2 \sin^2 \theta$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \hat{\mathbf{z}} \frac{\mu_0 I_0}{2\pi} \frac{e^{ikr}}{kr} \left(\frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2 \theta} \right)$$

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin \theta} \right|^2$$

Alternative approach – linear center-fed antenna continued

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin \theta} \right|^2$$

for $kd = \pi$:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \frac{\cos^2\left(\frac{\pi}{2} \cos \theta\right)}{\sin^2 \theta}$$

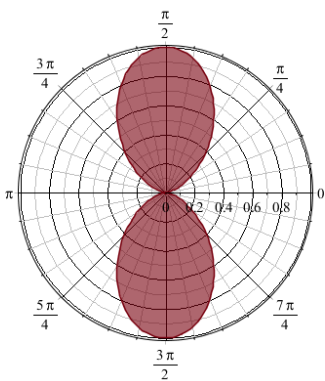
for $kd = 2\pi$:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{4}{8\pi^2} \frac{\cos^4\left(\frac{\pi}{2} \cos \theta\right)}{\sin^2 \theta}$$

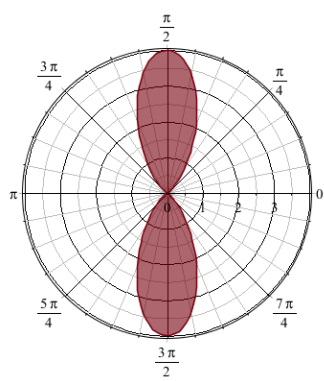
Alternative approach – linear center-fed antenna continued

Time averaged power:

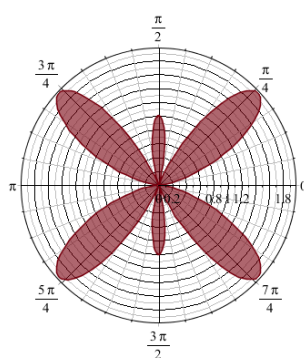
$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin \theta} \right|^2$$



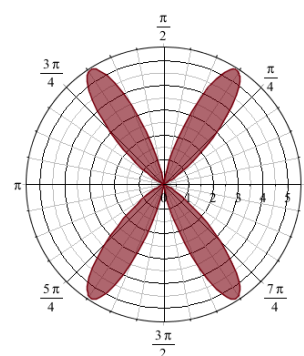
$kd = \pi$



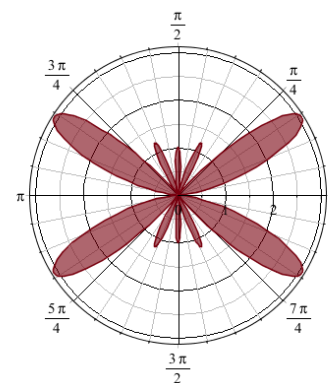
2π



3π



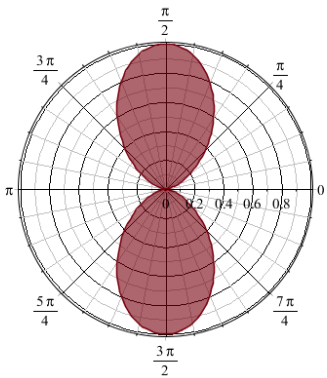
4π



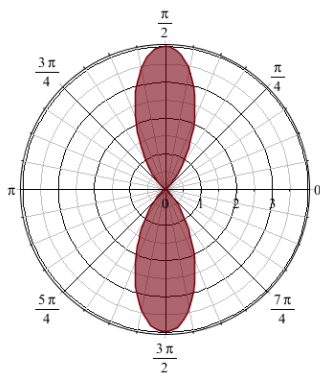
5π

Interesting patterns for special values of kd

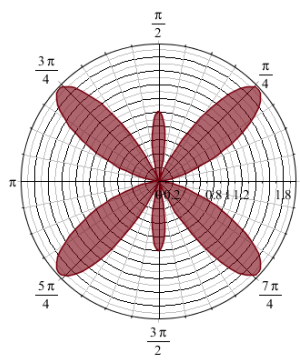
Some details --



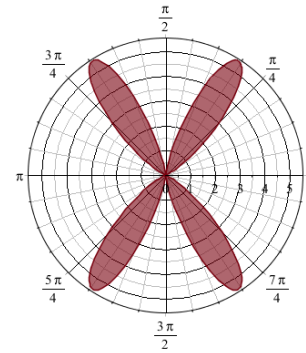
$$kd = \pi$$



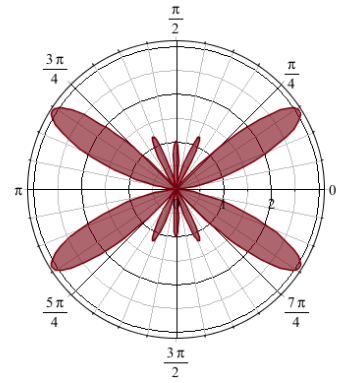
$$2\pi$$



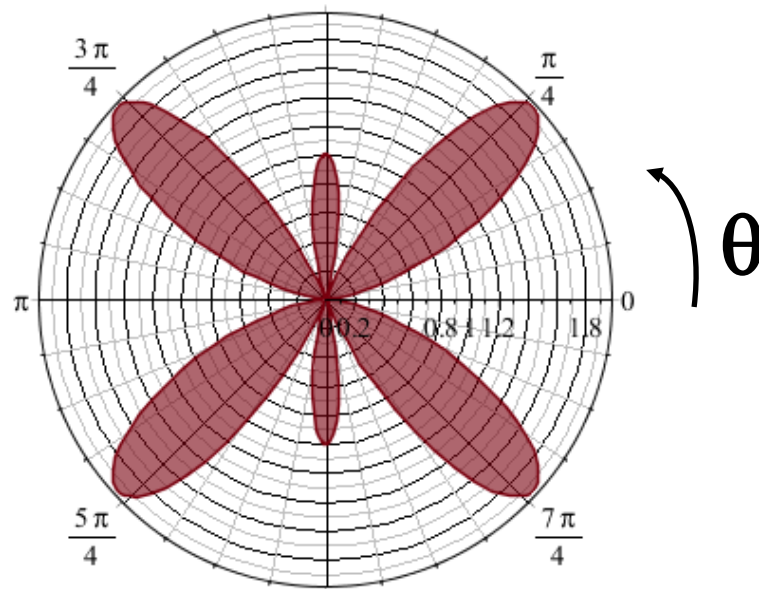
$$\frac{3\pi}{2}$$



$$4\pi$$



$$5\pi$$



Polar plot:
Angle indicates
values of theta

Radius indicates
value scaled to 1.