PHY 712 Electrodynamics
10-10:50 AM MWF in Olin 103

Notes for Lecture 25:

Complete reading of Chap. 9 & 10

A. Antenna radiation

B. Superposition of radiation from multiple sources

C. Scattered radiation

	Fri: 03/21/2025	No class	Take-home exam		
24	Mon: 03/24/2025	Chap. 9	Radiation from time harmonic sources	#20	03/26/2025
25	Wed: 03/26/2025	Chap. 9 & 10	Radiation from scattering	#21	03/28/2025
26	Fri: 03/28/2025	Chap. 11	Special Theory of Relativity		
27	Mon: 03/31/2025	Chap. 11	Special Theory of Relativity		
28	Wed: 04/2/2025	Chap. 11	Special Theory of Relativity		
29	Fri: 04/4/2024	Chap. 14	Radiation from accelerating charged particles		
30	Mon: 04/07/2025	Chap. 14	Radiation from accelerating charged particles		
31	Wed: 04/09/2025	Chap. 14	Synchrotron radiation and Compton scattering		
32	Fri: 04/11/2025	Chap. 13 & 15	Other radiation -- Cherenkov & bremsstrahlung		
33	Mon: 04/14/2025	Special Topics			
34	Wed: 04/16/2025	Special Topics			
35	Fri: 04/18/2025		Presentations I		
	Mon: 04/21/2025	Special topics			
	Wed: 04/23/2025		Presentations II		
	Fri: 04/25/2025		Presentations III		
36	Mon: 04/28/2025		Review		

PHY 712 -- Assignment #21

Assigned: 3/26/2025 Due: 3/28/2025

Continue reading Chapter 9 (Sec. 9.1-9.4) in **Jackson** .

1. Problem 9.16(a) in **Jackson** . In this case, "exactly" really means following the approach discussed in Sec. 9.4 using the current density given in this problem. You might want to draw a diagram to indicate how θ is defined.

Physics Colloquium

- Thursday -

March 27, 2025

Probing the Spectroscopy and Dynamics of Polarons and Excitons in Organic Materials

Polarons and excitons play a central role in the electronic, optical, and transport properties of molecular aggregates, thin films and crystals, semiconducting polymers, and hybrid organic-inorganic semiconductors. In such π-conjugated systems, charged (polarons) and neutral (excitons) excitations are strongly coupled to the nuclear degrees of freedom, underscoring the importance of electron-phonon interactions.

In the first half of my talk, I will present a theory describing the spatial coherence length of polarons in disordered organic materials, revealing a simple relationship between the oscillator strength of the infrared absorption band and the polaron coherence function. TheHolstein-type Hamiltonian, represented in a multiparticle basis set, has been successful in quantitatively reproducing several recently measured spectra recorded in doped and undoped polymer films, confirming the association of an enhanced peak ratio with extended polaron coherence. Emphasis will be placed on understanding the fundamental nature and origin of the components polarized along the intra and inter-chain directions and their dependence on structural and conformational disorder, vibronic coupling, and Coulomb binding.

In the second half of my talk, we will explore the similarities and differences in the spectral response of excitons and polarons in organic materials, with a particular focus on how simple optical probes like steady-state absorption and photoluminescence can be used to extract information about the exciton and polaron coherence lengths -quantities which are critical for understanding nanoscale energy and charge transport processes in emergent semiconducting materials.

Prof. Raja Ghosh
Department of Chemistry and Physics
North Carolina State University

Reception 3:30
Olin Lobby

Colloquium 4:00
Olin 101

Electromagnetic waves from time harmonic sources – review:

For scalar potential (Lorenz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

For vector potential (Lorenz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Previously, we made use of an expansion of the kernel in terms of Bessel funcsions and spherical harmonics; now we consider and alternative approach:

Fields from time harmonic source:

$$\tilde{\Phi}(\mathbf{r},\omega) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

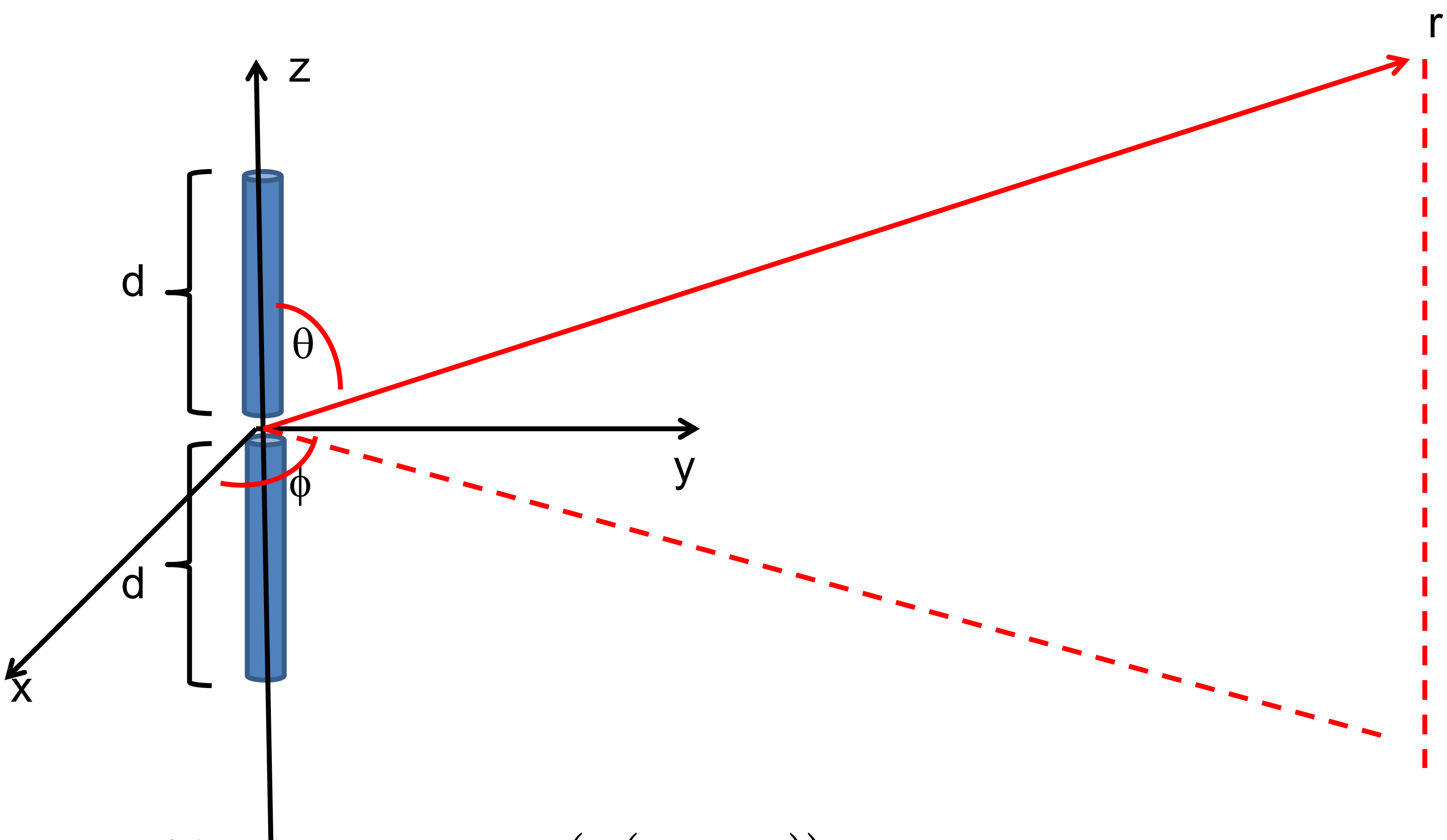
For $r >> r'$: $\quad |\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' + \ldots$

$$\tilde{\Phi}(\mathbf{r},\omega) \approx \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int d^3r' \, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \, \tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r' \, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \, \tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Consider antenna source (center-fed)

Note – these notes differ from previous formulation d/2 ←→ d



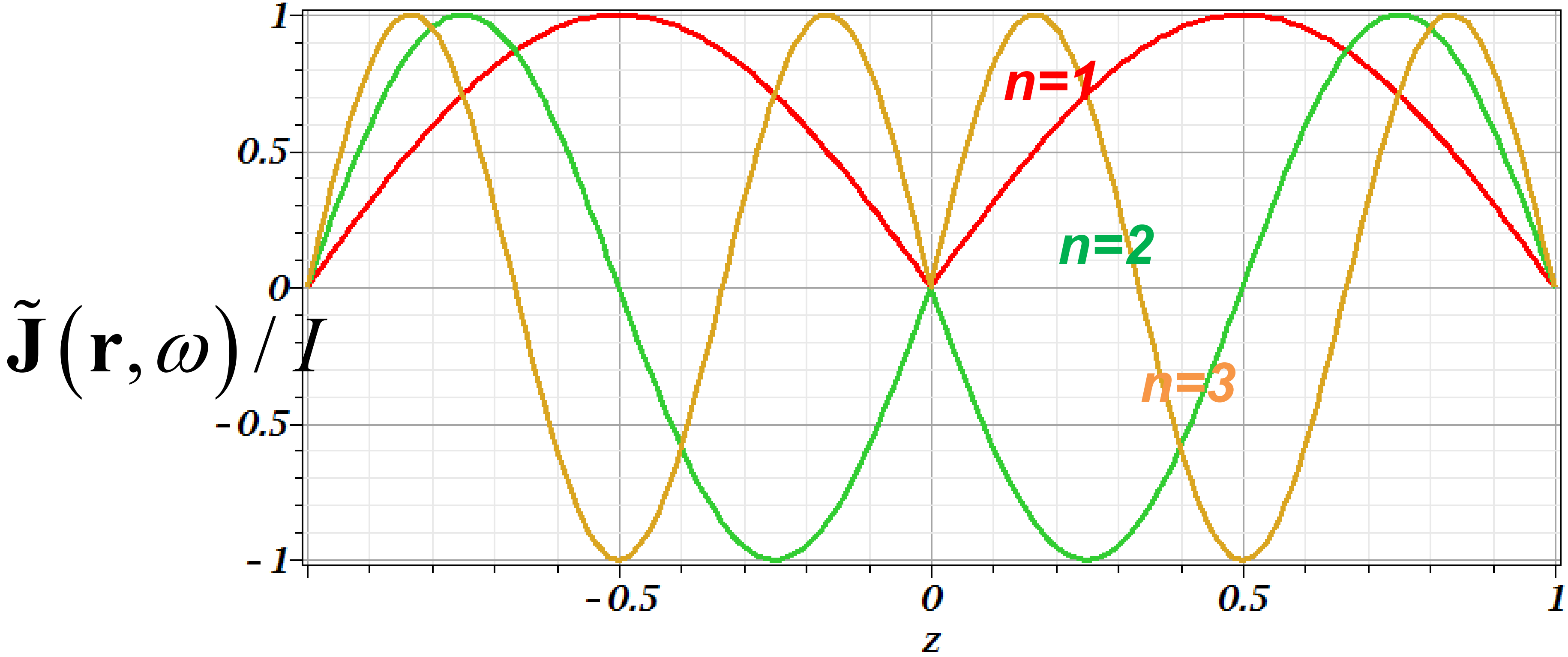
$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d - |z|\right)\right)\delta(x)\delta(y) \quad \text{for } -d \le z \le d$$

$$k \equiv \frac{\omega}{c}$$

Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d - |z|\right)\right)\delta(x)\delta(y) \qquad \text{for} \quad -d \le z \le d$$

$$\text{for} \quad k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1, 2, 3....$$



Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\Big(k\big(d - |z|\big)\Big)\delta(x)\delta(y) \quad \text{for} \quad -d \le z \le d$$

$$k \equiv \frac{\omega}{c}$$

Vector potential from source:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

For $r >> d$ $\quad \tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\, I\int_{-d}^{d} dz'\; e^{-ikz'\cos\theta} \quad \sin\Big(k\big(d - |z'|\big)\Big)$$

Consider antenna source -- continued

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} I \int_{-d}^{d} dz\, e^{-ikz\cos\theta} \sin\left(k\left(d-|z|\right)\right)$$

$$= \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{kr} 2I\left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin^2\theta}\right]$$

In the radiation zone:

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla\times\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx ik\hat{\mathbf{r}}\times\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) \approx -ikc\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\tilde{\mathbf{A}}(\mathbf{r},\omega)\right)$$

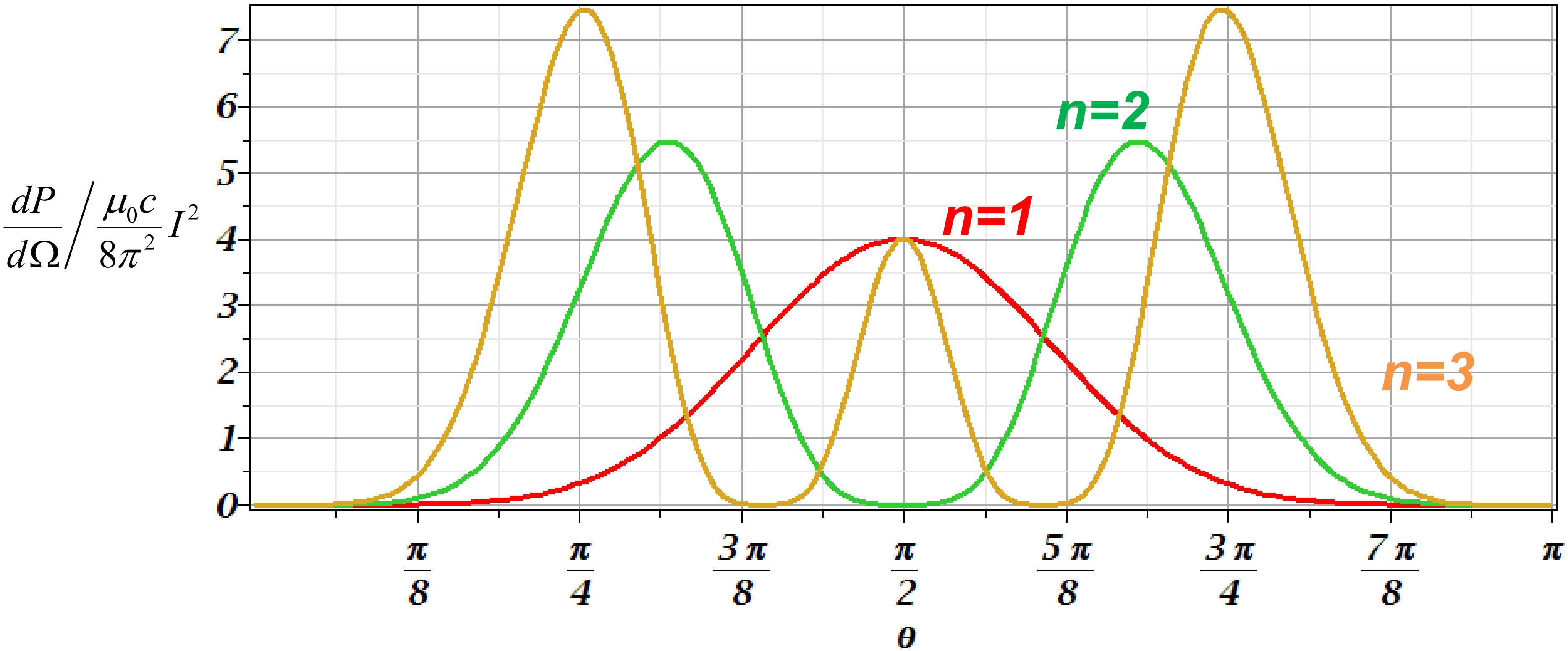
$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0} r^2 \hat{\mathbf{r}}\cdot\Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right) = \frac{k^2c}{2\mu_0} r^2\left(\left|\tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2 - \left|\hat{\mathbf{r}}\cdot\tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2\right)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2\left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin\theta}\right]^2$$

Consider antenna source -- continued

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2$$

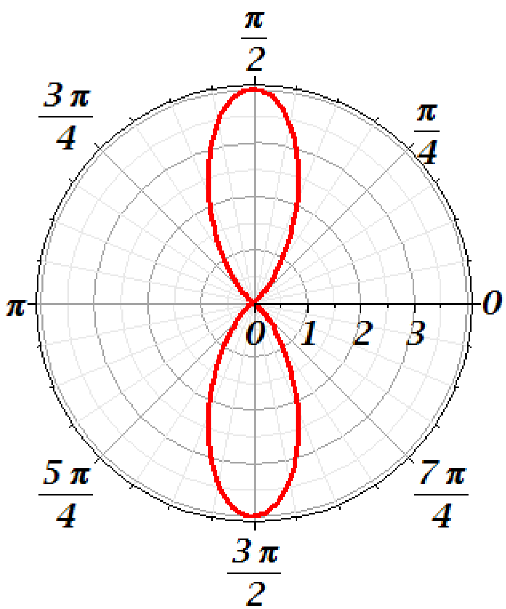
$$\text{for } k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1, 2, 3....$$



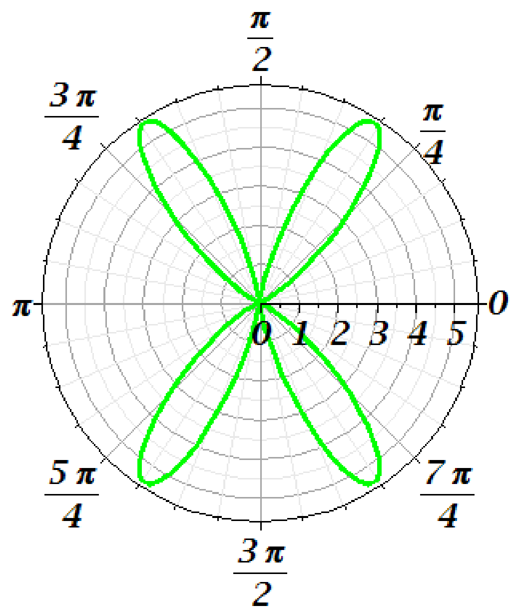
Consider antenna source -- continued

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2$$

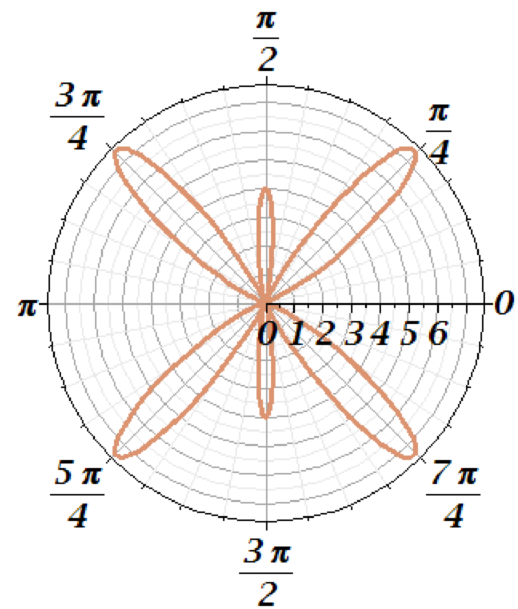
For $kd = n\pi$:



n=1

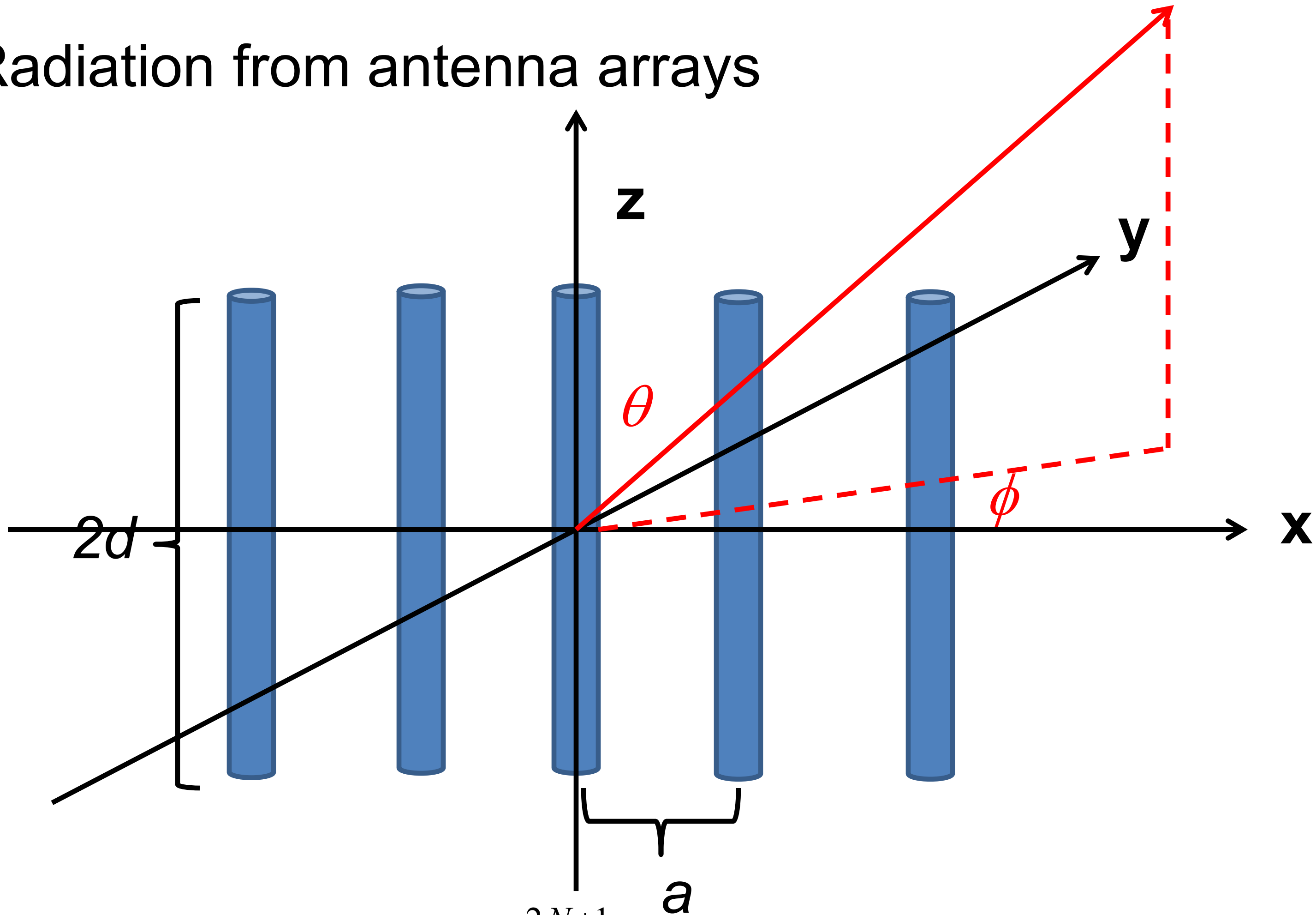


n=2



n=3

Radiation from antenna arrays



$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d - |z|\right)\right) \sum_{j=1}^{2N+1} \delta\left(x - (N+1-j)a\right)\delta(y) \quad \text{for } -d \le z \le d$$

$$k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1,2,3....$$

Note that these antennas are all "in phase".

Radiation from antenna arrays -- continued

Vector potential from array source:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\frac{\mu_0}{4\pi}\int d^3r'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)\approx\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}I\sin\left(k\left(d-|z|\right)\right)\sum_{j=1}^{2N+1}\delta\left(x-\left(N+1-j\right)a\right)\delta(y) \quad \text{for } -d\leq z\leq d$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)\approx\hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\sum_{j=-N}^{N}e^{-ikaj\sin\theta\cos\phi}\right)I\int_{-d}^{d}dz\, e^{-ikz\cos\theta}\ \sin\left(k\left(d-|z|\right)\right)$$

$$\sum_{j=-N}^{N}e^{-ikaj\sin\theta\cos\phi}=\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\phi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\phi\right)}$$

Digression – summation of a geometric series

$$\sum_{j=-N}^{N} e^{-iAj} = e^{-iA}\sum_{j=-N}^{N} e^{-iAj} + e^{iAN} - e^{-iA(N+1)}$$

$$\sum_{j=-N}^{N} e^{-iAj} = \frac{e^{iAN} - e^{-iA(N+1)}}{1-e^{-iA}} = \frac{e^{iA/2}}{e^{iA/2}}\frac{e^{iAN} - e^{-iA(N+1)}}{1-e^{-iA}}$$

$$= \frac{2i\sin\left(A\left(N+1/2\right)\right)}{2i\sin(A/2)}$$

$$= \frac{\sin\left(A\left(N+1/2\right)\right)}{\sin(A/2)}$$

$$\sum_{j=-N}^{N} e^{-ikaj\sin\theta\cos\varphi} = \frac{\sin\left(\frac{1}{2}ka\left(2N+1\right)\sin\theta\cos\varphi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\varphi\right)}$$

Radiation from antenna arrays -- continued

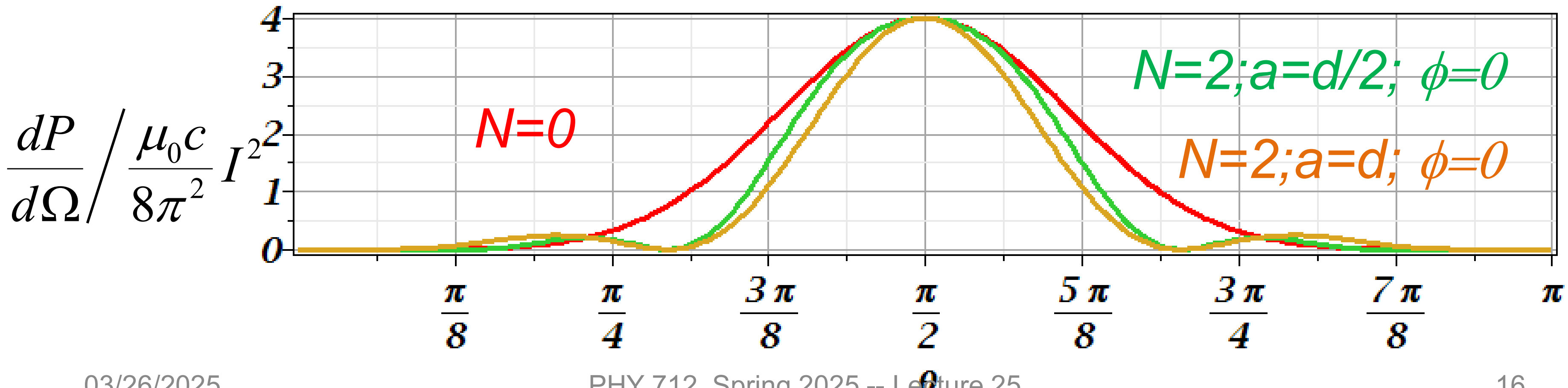
In the radiation zone:

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla \times \tilde{\mathbf{A}}(\mathbf{r},\omega) \approx ik\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) \approx -ikc\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r},\omega)\right)$$

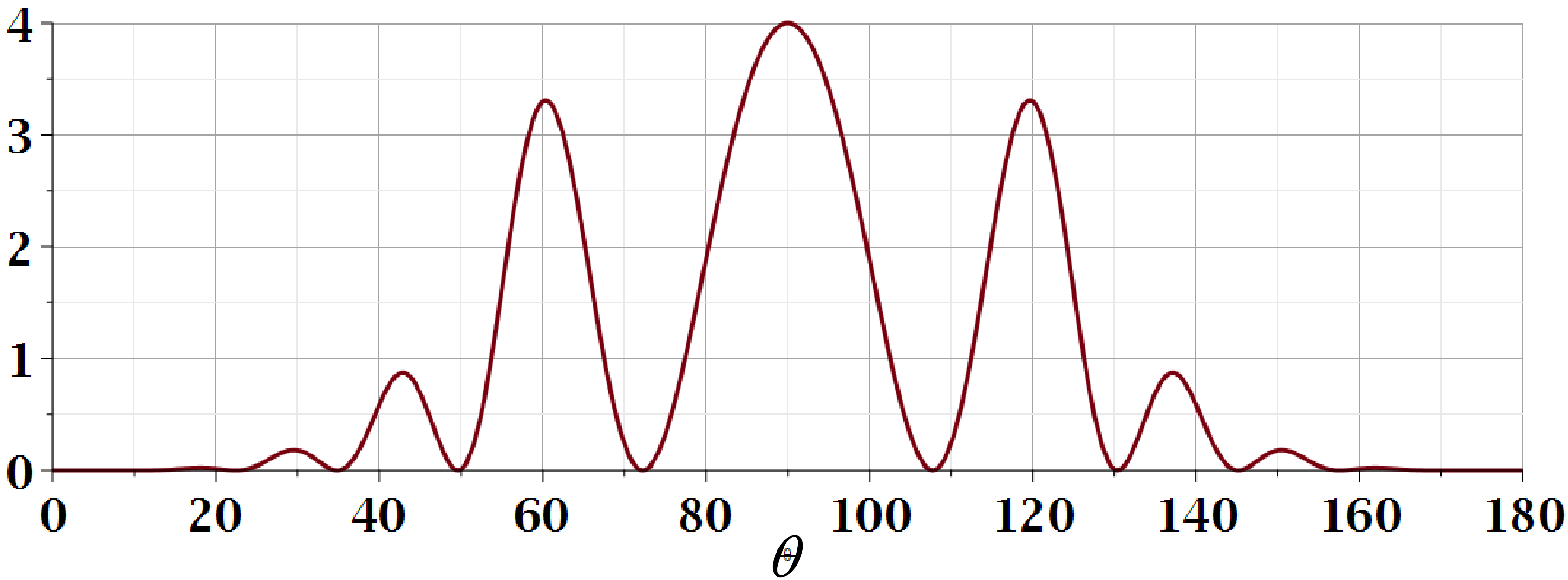
$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0} r^2 \hat{\mathbf{r}} \cdot \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega) \times \tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right) = \frac{k^2 c r^2}{2\mu_0}\left(\left|\tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2 - \left|\hat{\mathbf{r}} \cdot \tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2\right)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta}\right]^2 \left[\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\phi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\phi\right)}\right]^2$$



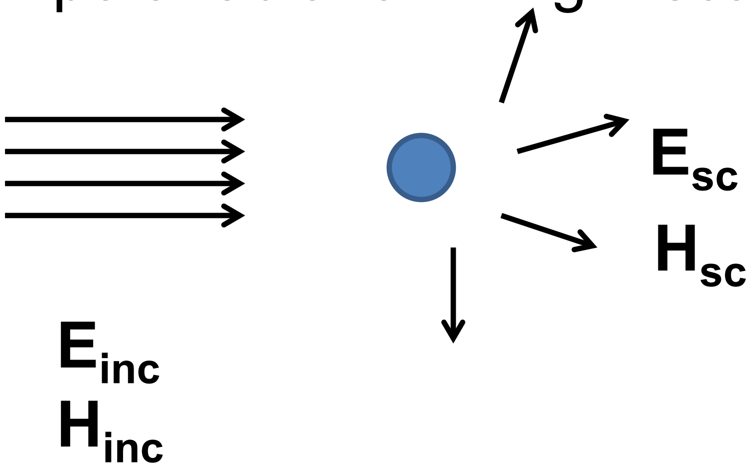
$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2 \left[\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\varphi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\varphi\right)} \right]^2$$

Example for $\phi = 0, N = 10, \quad kd = \pi = 2ka$



Additional amplitude patterns can be obtained by controlling relative phases of antennas.

Now – consider a different radiation source ---
Dipole radiation in light scattering by small (dielectric) particles

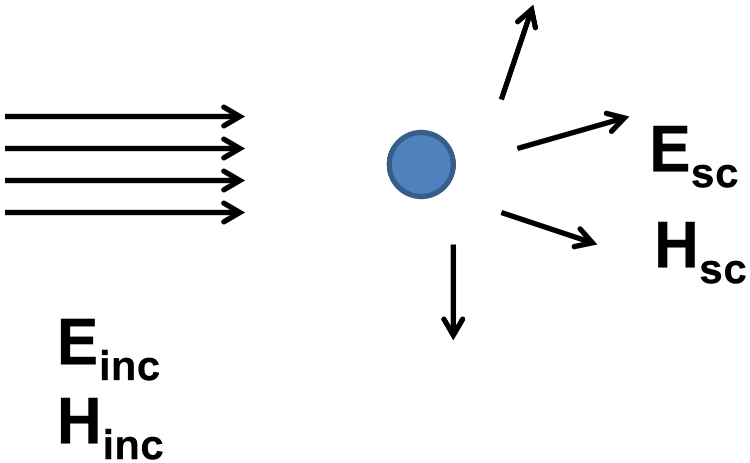


$$\mathbf{E}_{\text{inc}} = \hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_{\mathbf{0}}\cdot\mathbf{r}} \qquad \mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c}\hat{\mathbf{k}}_{\mathbf{0}} \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation :

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r}\left(\left(\hat{\mathbf{r}}\times\mathbf{p}\right)\times\hat{\mathbf{r}}\right) \qquad \mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c}\hat{\mathbf{r}}\times\mathbf{E}_{\text{sc}}$$

Dipole radiation in light scattering by small (dielectric) particles



$$\mathbf{E}_{\text{inc}} = \hat{\mathbf{v}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}}$$

$$\mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c} \hat{\mathbf{k}}_0 \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation:

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r} \left(\left(\hat{\mathbf{r}} \times \mathbf{p} \right) \times \hat{\mathbf{r}} \right)$$

$$\mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c} \hat{\mathbf{r}} \times \mathbf{E}_{\text{sc}}$$

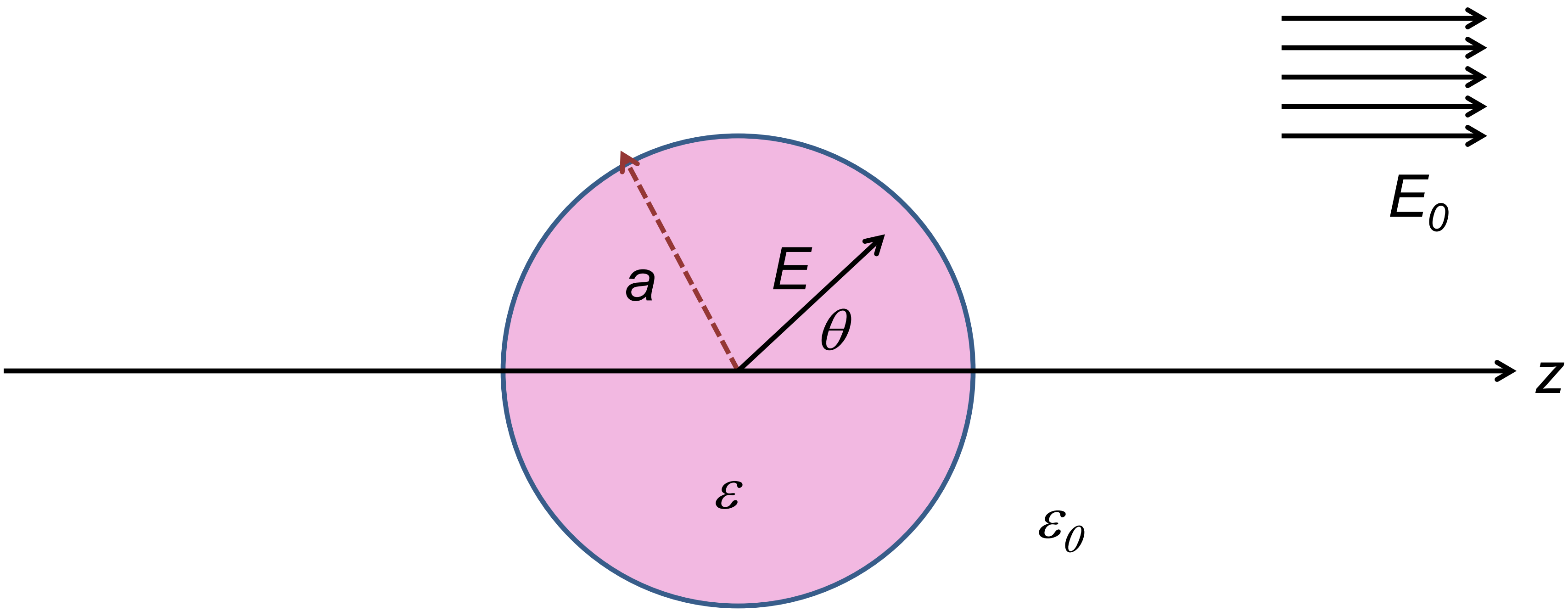
Scattering cross section:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}}, \hat{\mathbf{v}}; \hat{\mathbf{k}}_0, \hat{\mathbf{v}}_0\right) = \frac{r^2 \hat{\mathbf{r}} \cdot \left\langle \mathbf{S}_{sc} \right\rangle_{avg}}{\hat{\mathbf{k}}_0 \cdot \left\langle \mathbf{S}_{inc} \right\rangle_{avg}}$$

$$= \frac{r^2 \left| \hat{\mathbf{v}} \cdot \mathbf{E}_{sc} \right|^2}{\left| \hat{\boldsymbol{\varepsilon}}_0 \cdot \mathbf{E}_{inc} \right|^2} = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2} \left| \hat{\mathbf{v}} \cdot \mathbf{p} \right|^2$$

Recall previous analysis for electrostatic case:

Boundary value problems in the presence of dielectrics – example:



$$\text{At } r = a: \quad \varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi_>(\mathbf{r}) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos\theta)$$

At $r = a$:
$$\varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

For $r \to \infty$ $\quad \Phi_>(\mathbf{r}) = -E_0 r \cos\theta$

Solution -- only $l = 1$ contributes

$$B_1 = -E_0$$

$$A_1 = -\left(\frac{3}{2 + \varepsilon/\varepsilon_0} \right) E_0 \qquad C_1 = \left(\frac{\varepsilon/\varepsilon_0 - 1}{2 + \varepsilon/\varepsilon_0} \right) a^3 E_0$$

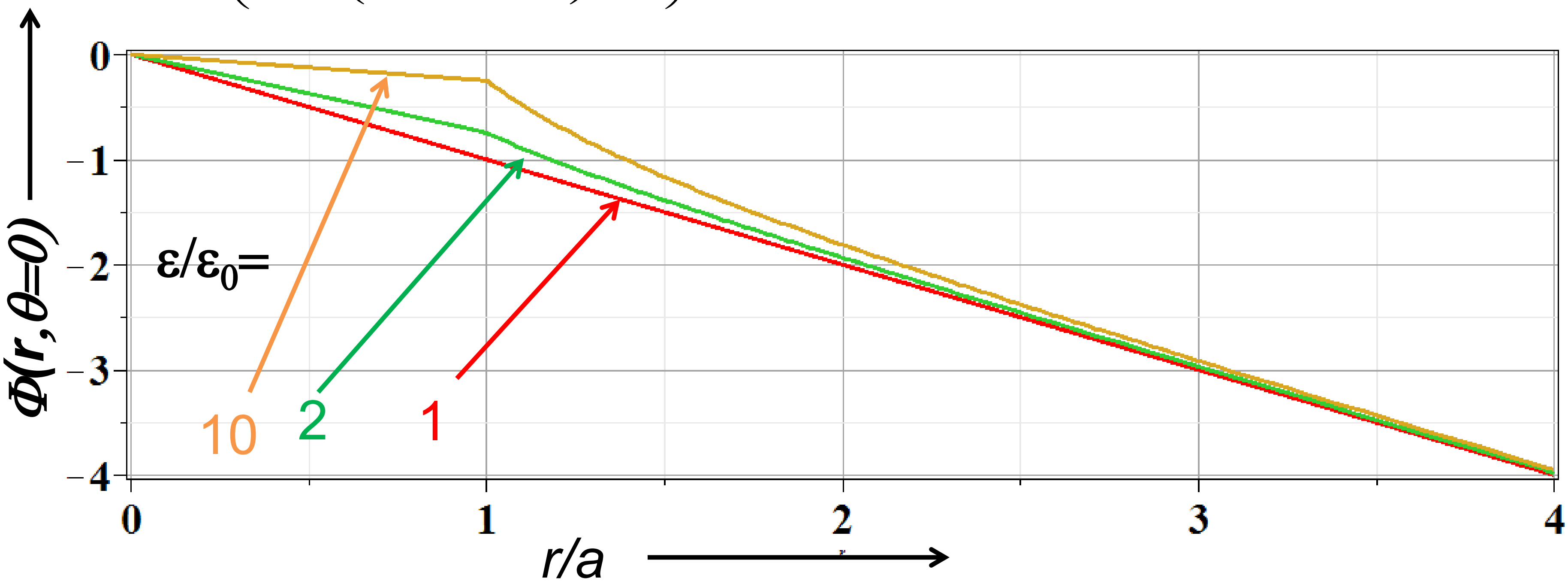
Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = -\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right)E_0\, r\cos\theta$$

$$\Phi_>(\mathbf{r}) = -\left(r - \left(\frac{\varepsilon/\varepsilon_0 - 1}{2+\varepsilon/\varepsilon_0}\right)\frac{a^3}{r^2}\right)E_0 \cos\theta$$

Induced dipole moment:

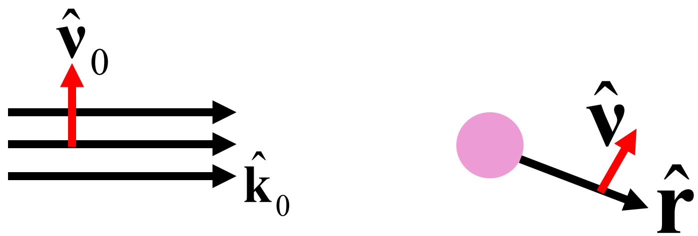
$$\mathbf{p} = 4\pi a^3 \varepsilon_0 \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right)\mathbf{E}_0$$



Estimation of scattering dipole moment:

Suppose the scattering particle is a dielectric sphere with permittivity ε and radius *a*:

Note polarization notation change for clarity.



$$\mathbf{p} = 4\pi a^3 \varepsilon_0 \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right) \mathbf{E}_{inc}$$

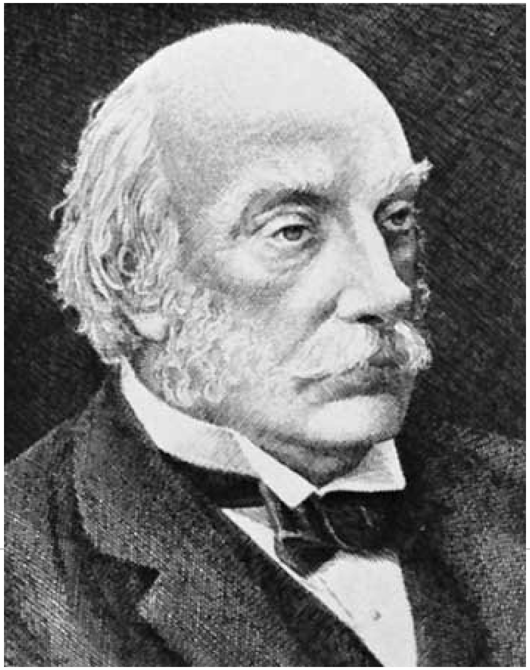
$$\mathbf{E}_{\text{inc}} = \hat{\mathbf{v}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}}$$

Scattering cross section:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}}, \hat{\mathbf{v}}; \hat{\mathbf{k}}_0, \hat{\mathbf{v}}_0\right) = \frac{r^2 \left|\hat{\mathbf{v}} \cdot \mathbf{E}_{sc}\right|^2}{\left|\hat{\mathbf{v}}_0 \cdot \mathbf{E}_{inc}\right|^2} = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2} \left|\hat{\mathbf{v}} \cdot \mathbf{p}\right|^2$$

$$= k^4 a^6 \left| \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right|^2 \left|\hat{\mathbf{v}} \cdot \hat{\mathbf{v}}_0\right|^2$$

https://www.britannica.com/biography/John-William-Strutt-3rd-Baron-Rayleigh

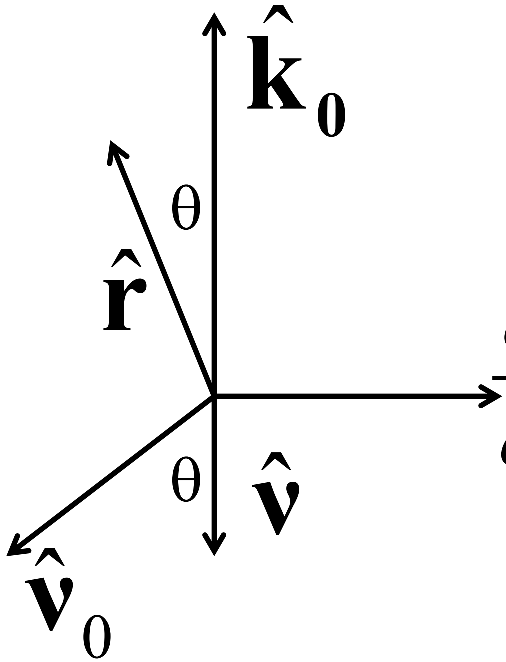


WRITTEN BY: R. Bruce Lindsay
See Article History

Alternative Titles: John William Strutt, 3rd Baron Rayleigh of Terling Place

Lord Rayleigh, in full **John William Strutt, 3rd Baron Rayleigh of Terling Place**, (born November 12, 1842, Langford Grove, Maldon, Essex, England—died June 30, 1919, Terling Place, Witham, Essex), English physical scientist who made fundamental discoveries in the fields of acoustics and optics that are basic to the theory of wave propagation in fluids. He received the Nobel Prize for Physics in 1904 for his successful isolation of argon, an inert atmospheric gas.

Scattering by dielectric sphere with permittivity ε and radius *a*:

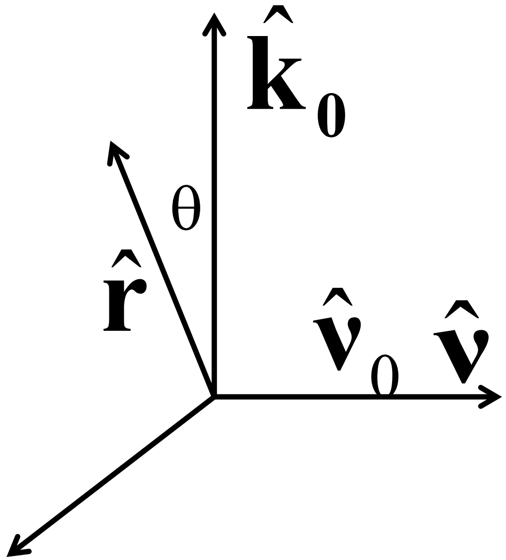


For $\mathbf{E}_{\mathbf{inc}}$ polarized in scattering plane:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right) == k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left|\hat{\mathbf{v}}\cdot\hat{\mathbf{v}}_0\right|^2$$

$$= k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\cos^2\theta$$

Scattering by dielectric sphere with permittivity ε and radius *a*:



For $\mathbf{E}_{inc}$ polarized perpendicular to scattering plane:

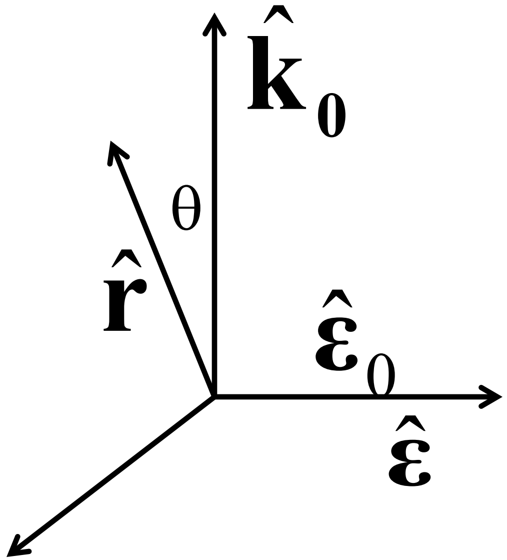
$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right)=k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left|\hat{\mathbf{v}}\cdot\hat{\mathbf{v}}_0\right|^2$$

$$=k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2$$

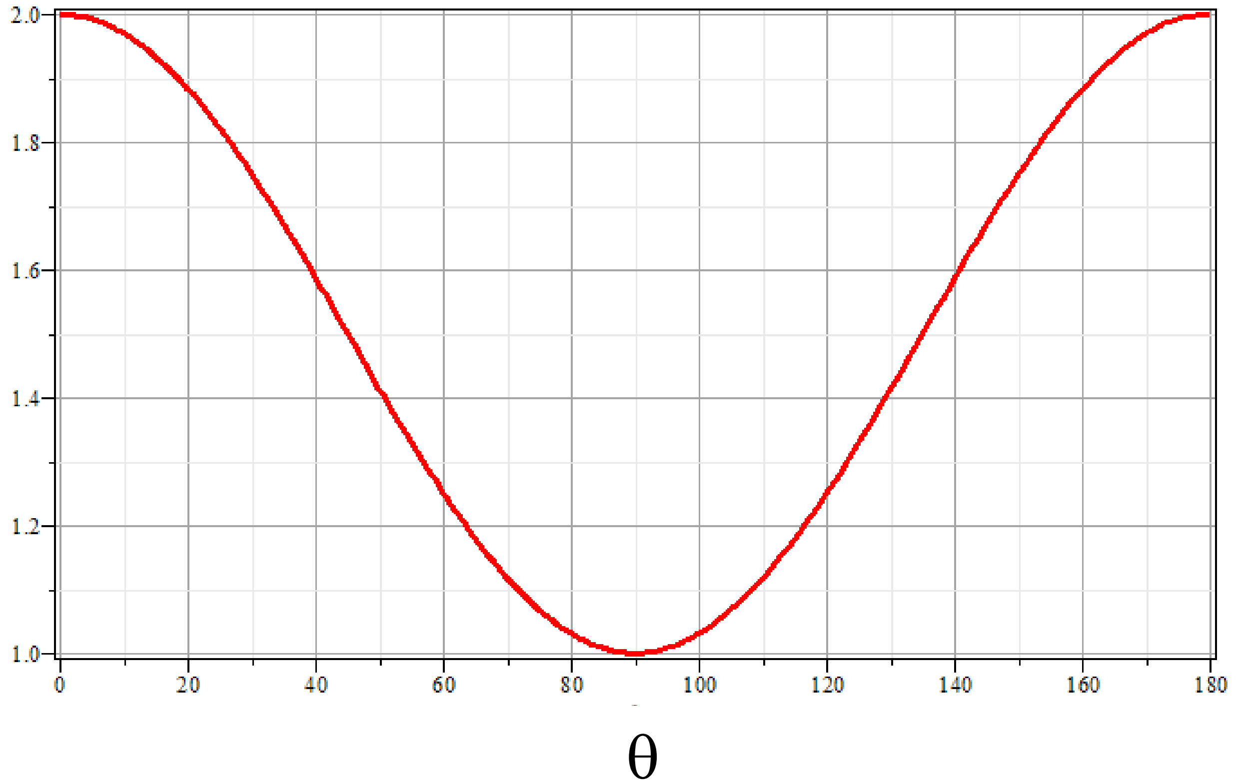
Assuming both incident polarizations are equally likely, average cross section is given by:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right)=\frac{k^4a^6}{2}\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left(\cos^2\theta+1\right)$$

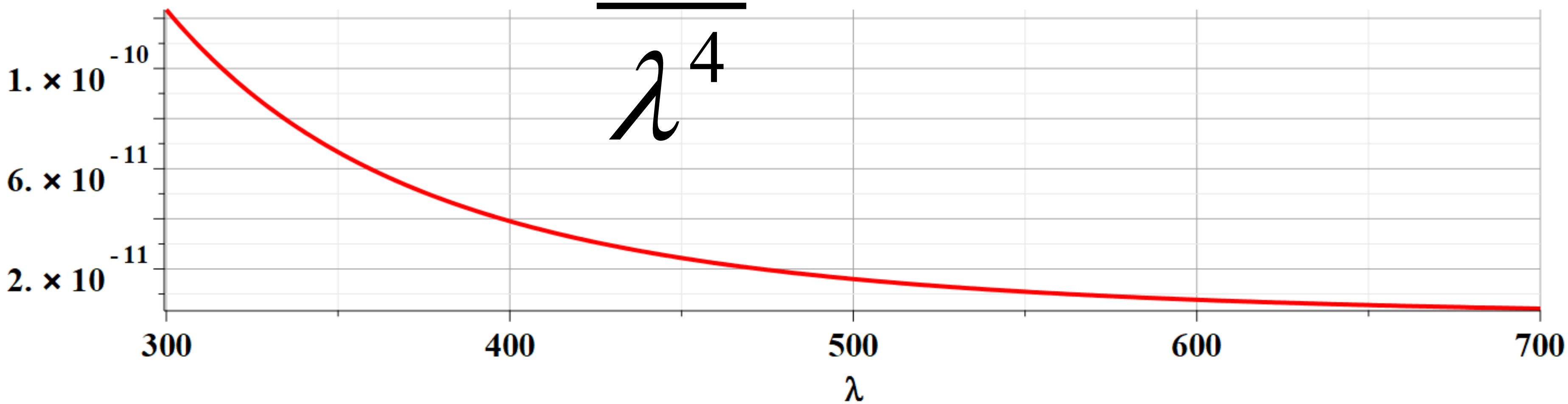
Scattering by dielectric sphere with permittivity ε and radius *a*:

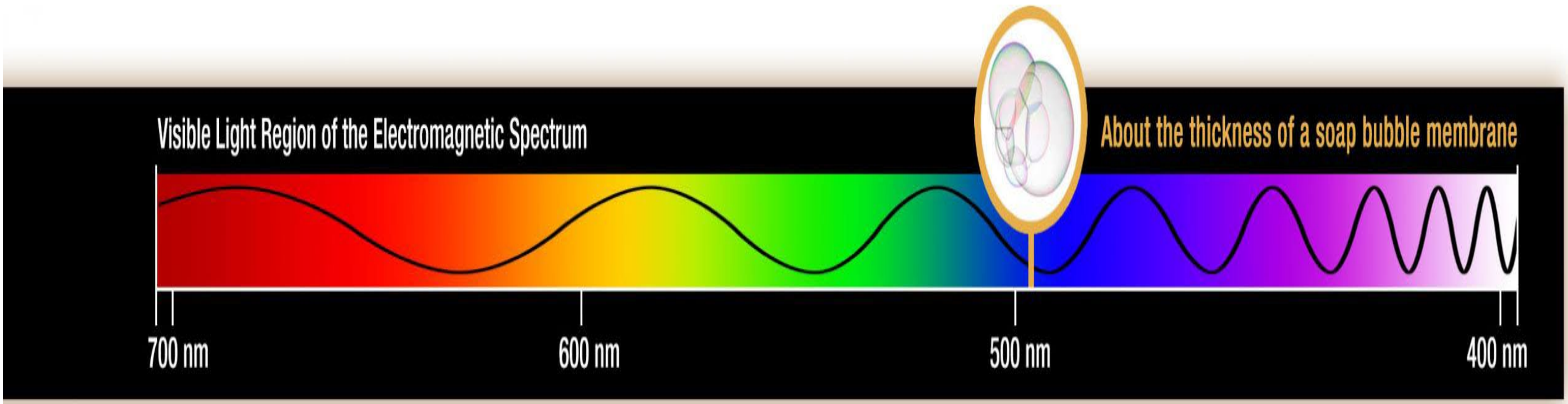


$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right) = \frac{k^4a^6}{2}\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left(\cos^2\theta+1\right)$$



$$\frac{1}{\lambda^4}$$





Brief introduction to multipole expansion of electromagnetic fields (Chap. 9.7)

Sourceless Maxwell's equations

in terms of $\mathbf{E}$ and $\mathbf{H}$ fields with time dependence $e^{-i\omega t}$:

$$\nabla \times \mathbf{E} = ikZ_0\mathbf{H} \qquad \nabla \times \mathbf{H} = -ik\mathbf{E} / Z_0$$

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$$

where $k \equiv \omega / c$ and $Z_0 \equiv \sqrt{\mu_0 / \epsilon_0}$

Decoupled equations:

$$\left(\nabla^2 + k^2\right)\mathbf{E} = 0 \qquad \left(\nabla^2 + k^2\right)\mathbf{H} = 0$$

$$\mathbf{H} = -\frac{i}{kZ_0}\nabla \times \mathbf{E} \qquad \mathbf{E} = \frac{iZ_0}{k}\nabla \times \mathbf{H}$$

Multipole expansion of electromagnetic fields -- continued

Note that:

$$\left(\nabla^2 + k^2\right)\left(\mathbf{r}\cdot\mathbf{E}\right) = 0 \qquad \left(\nabla^2 + k^2\right)\left(\mathbf{r}\cdot\mathbf{H}\right) = 0$$

Convenient operators for angular momentum analysis

Define: $\mathbf{L} \equiv \frac{1}{i}\left(\mathbf{r}\times\nabla\right)$

Note that $\mathbf{r} \cdot \mathbf{L} = 0$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2 r}{\partial r^2} - \frac{L^2}{r^2}$$

Eigenfunctions:

$$L^2 Y_{lm}(\theta,\phi) = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_{lm}(\theta,\phi) = l(l+1)Y_{lm}(\theta,\phi)$$

Multipole expansion of electromagnetic fields -- continued

Magnetic multipole field:

$$\mathbf{r} \cdot \mathbf{H}_{lm}^{M} \equiv \frac{l(l+1)}{k} g_l(kr) Y_{lm}(\theta,\phi)$$

$$\mathbf{r} \cdot \mathbf{E}_{lm}^{M} = 0$$

$$\mathbf{L} \cdot \mathbf{E}_{lm}^{M} = l(l+1) Z_0 g_l(kr) Y_{lm}(\theta,\phi)$$

spherical Bessel function

Electric multipole field:

$$\mathbf{r} \cdot \mathbf{E}_{lm}^{E} \equiv -Z_0 \frac{l(l+1)}{k} f_l(kr) Y_{lm}(\theta,\phi)$$

$$\mathbf{r} \cdot \mathbf{H}_{lm}^{E} = 0$$

$$\mathbf{L} \cdot \mathbf{H}_{lm}^{E} = l(l+1) f_l(kr) Y_{lm}(\theta,\phi)$$

spherical Bessel function

Multipole expansion of electromagnetic fields -- continued

Vector spherical harmonics: (for $l > 0$)

$$\mathbf{X}_{lm}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{lm}(\theta,\phi)$$

Orthogonality conditions:

$$\int d\Omega\ \mathbf{X}_{l'm'}{}^{*}(\theta,\phi)\cdot\mathbf{X}_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

$$\int d\Omega\ \mathbf{X}_{l'm'}{}^{*}(\theta,\phi)\cdot\left(\mathbf{r}\times\mathbf{X}_{lm}(\theta,\phi)\right) = 0$$

General expansion of fields:

$$\mathbf{H} = \sum_{lm}\left[a_{lm}^{E} f_l(kr)\mathbf{X}_{lm}(\theta,\phi) - \frac{i}{k}a_{lm}^{M}\nabla\times\left(g_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right)\right]$$

$$\mathbf{E} = \sum_{lm}\left[\frac{i}{k}a_{lm}^{E}\nabla\times\left(f_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right) + a_{lm}^{M} g_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right]$$

Multipole expansion of electromagnetic fields -- continued

Time averaged power distribution of radiation far from source:

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|\sum_{lm}(-i)^{l+1}\left[a_{lm}^E \mathbf{X}_{lm}(\theta,\phi)\times\hat{\mathbf{r}} + a_{lm}^M \mathbf{X}_{lm}(\theta,\phi)\right]\right|^2$$

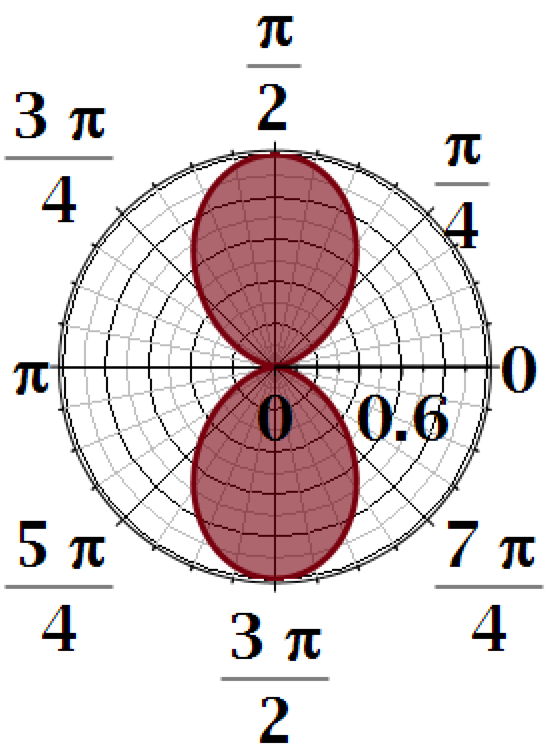
For a pure multipole radiation with either a_{lm}^E or a_{lm}^M :

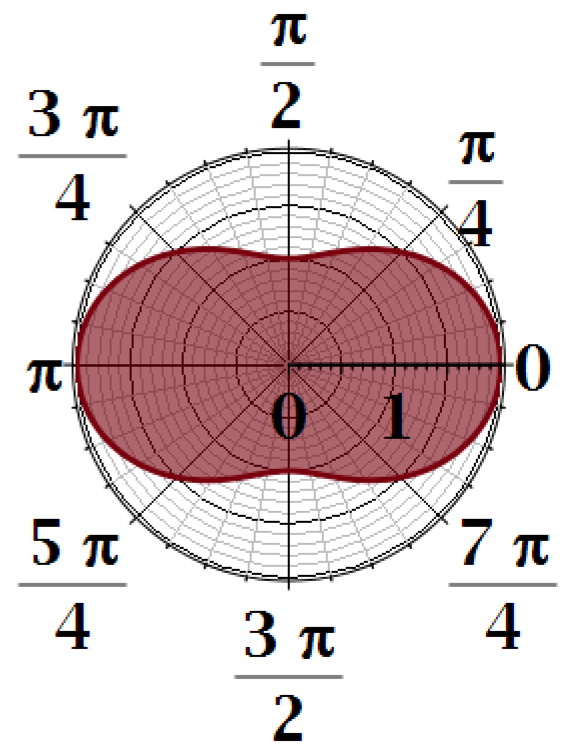
$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|a_{lm}\right|^2 \left|\mathbf{X}_{lm}(\theta,\phi)\right|^2$$

$$\left|\mathbf{X}_{lm}(\theta,\phi)\right|^2 = \frac{1}{2l(l+1)}\left(2m^2\left|Y_{lm}\right|^2 + (l+m)(l-m+1)\left|Y_{l(m-1)}\right|^2 + (l-m)(l+m+1)\left|Y_{l(m+1)}\right|^2\right)$$

For example: $l = 1$

$$\left|\mathbf{X}_{10}(\theta,\phi)\right|^2 = \frac{3}{8\pi}\sin^2\theta \qquad \left|\mathbf{X}_{11}(\theta,\phi)\right|^2 = \left|\mathbf{X}_{1-1}(\theta,\phi)\right|^2 = \frac{3}{16\pi}\left(1+\cos^2\theta\right)$$





For example: $l = 2$

$$\left|\mathbf{X}_{20}(\theta,\phi)\right|^2 = \frac{15}{8\pi}\sin^2\theta\,\cos^2\theta \qquad \left|\mathbf{X}_{21}(\theta,\phi)\right|^2 = \frac{5}{16\pi}\left(1-3\cos^2\theta+4\cos^4\theta\right) \qquad \left|\mathbf{X}_{22}(\theta,\phi)\right|^2 = \frac{5}{16\pi}\left(1-\cos^4\theta\right)$$