PHY 712 Electrodynamics
10-10:50 AM MWF Olin 103

Notes for Lecture 27: Theory of Special Relativity

Chap. 11 – Sec. 11.1-11.7,11.9 in JDJ)

A. Lorentz transformation relations
B. Electromagnetic field transformations
C. Connection to Liénard-Wiechert potentials for constant velocity sources

24	Mon: 03/24/2025	Chap. 9	Radiation from time harmonic sources	#20	03/26/2025
25	Wed: 03/26/2025	Chap. 9 & 10	Radiation from scattering	#21	03/28/2025
26	Fri: 03/28/2025	Chap. 11	Special Theory of Relativity	#22	03/31/2025
27	Mon: 03/31/2025	Chap. 11	Special Theory of Relativity	#23	04/02/2025
28	Wed: 04/02/2025	Chap. 11	Special Theory of Relativity		
29	Fri: 04/04/2024	Chap. 14	Radiation from accelerating charged particles		
30	Mon: 04/07/2025	Chap. 14	Radiation from accelerating charged particles		
31	Wed: 04/09/2025	Chap. 14	Synchrotron radiation and Compton scattering		
32	Fri: 04/11/2025	Chap. 13 & 15	Other radiation -- Cherenkov & bremsstrahlung		
33	Mon: 04/14/2025	Special Topics			
34	Wed: 04/16/2025	Special Topics			
35	Fri: 04/18/2025		Presentations I		
	Mon: 04/21/2025	Special topics			
	Wed: 04/23/2025		Presentations II		
	Fri: 04/25/2025		Presentations III		
36	Mon: 04/28/2025		Review		

Class time shifted to 9 AM

PHY 712 -- Assignment #23

Assigned: 3/31/2025 Due: 4/02/2025

Continue reading Chapters 11 (Especially Sec. 11.9) in **Jackson** .

1. Derive the relationships between the component of the electric and magnetic field components E_x, E_y, E_z, B_x, B_y, and B_z as measured in the stationary frame of reference and the components E'_x, E'_y, E'_z, B'_x, B'_y, and B'_z measured in a moving frame of reference which is moving at a constant relative velocity v along the x axis.

Crazy colloquium schedule for this week

Wed. Apr. 2, 2025 — Professor Fan Yang, Department of Computer Science, Wake Forest University– "Towards Conceptual Understanding of Large Language Models" (Host: N. Holzwarth)

Thurs. Apr. 3, 2025 — Ph.D. Defense: Ian Newsome — "Semiclassical Effects in Curved Spacetime and Quantum Electrodynamics" — Olin 107, 9:00 AM (Advisor: Prof. P. Anderson)

Thurs. Apr. 3, 2025 — Special Colloquium on Perspectives in Physics, Professors Paul Anderson and Natalie Holzwarth

Friday Apr. 4, 2025 — Ph.D. Defense: Leda Gao — "Extracting information from black hole merger simulations: The robustness of quasinormal modes" — Olin 107, 10:00 AM (Advisor: Prof. G. Cook)

Towards Conceptual Understanding of Large Language Models

Large Language Models (LLMs) have demonstrated impressive capabilities in understanding and generating human language, yet the internal mechanisms by which they encode semantic knowledge and conceptual structures remain largely opaque. In this talk, I will introduce my recent exploration in probing LLM representations, focusing on the depth-dependent emergence of conceptual understanding and the structure of learned knowledge. The first part of the talk will introduce the concept of Concept Depth, which posits that simpler concepts are captured in shallower layers while more abstract and complex ones require deeper processing. This framework is supported by probing experiments across multiple LLM architectures, shedding light on how different levels of semantic information are distributed within model layers. Building upon this, I will discuss a complementary perspective that extends beyond single vector representations to Gaussian Concept Subspaces (GCS), a novel framework that models conceptual knowledge as a distribution rather than a single fixed direction in representation space. This approach improves robustness in identifying learned concepts and has practical implications for interpretability and intervention in model behavior. Finally, I will present a vision for moving beyond empirical characterization toward physical interpretations of LLMs, drawing connections between deep learning representations and theoretical modeling of dynamical systems. This future perspective invites interdisciplinary collaboration in physics, neuroscience, and AI, aiming to ground the emergent behaviors of LLMs in a more principled framework.

Professor Fan Yang
Department of Computer Science
Wake Forest University

Wednesday 4/2/2025

Reception 3:30
Olin Lobby

Colloquium 4:00
Olin 101

Reminder – L. V. Lorenz and H. A. Lorentz

Ludvig Valentin Lorenz
1829-1891
➔Lorenz gauge

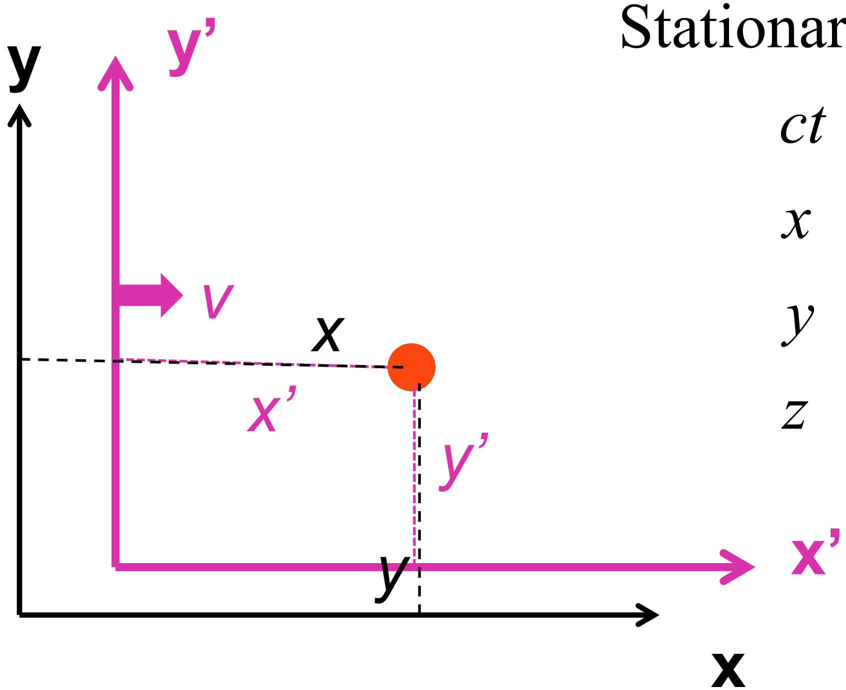
Hendrik Antoon Lorentz
1853-1928
➔Lorentz transformation

Lorentz transformations

Convenient notation :

$$\beta_v \equiv \frac{v}{c}$$

$$\gamma_v \equiv \frac{1}{\sqrt{1-\beta_v{}^2}}$$



Stationary frame		Moving frame
ct	$=$	$\gamma(ct'+\beta x')$
x	$=$	$\gamma(x'+\beta ct')$
y	$=$	y'
z	$=$	z'

Lorentz transformations -- continued

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \mathcal{L}_v \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \mathcal{L}_v^{-1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice :

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

Velocity relationships

Consider: $u_x = \dfrac{u'_x + v}{1 + vu'_x / c^2} \qquad u_y = \dfrac{u'_y}{\gamma_v \left(1 + vu'_x / c^2\right)} \qquad u_z = \dfrac{u'_z}{\gamma_v \left(1 + vu'_x / c^2\right)}.$

Note that $\gamma_u \equiv \dfrac{1}{\sqrt{1-(u/c)^2}} = \dfrac{1 + vu'_x / c^2}{\sqrt{1-(u'/c)^2}\sqrt{1-(v/c)^2}} = \gamma_v \gamma_{u'} \left(1 + vu'_x / c^2\right)$

$$\Rightarrow \gamma_u c = \gamma_v \left(\gamma_{u'} c + \beta_v \gamma_{u'} u'_x\right)$$

$$\Rightarrow \gamma_u u_x = \gamma_v \left(\gamma_{u'} u'_x + \gamma_{u'} v\right) = \gamma_v \left(\gamma_{u'} u'_x + \beta_v \gamma_{u'} c\right)$$

$$\Rightarrow \gamma_u u_y = \gamma_{u'} u'_y \qquad \gamma_u u_z = \gamma_{u'} u'_z$$

$$\Rightarrow \quad \begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \mathcal{L}_v \begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} u'_x \\ \gamma_{u'} u'_y \\ \gamma_{u'} u'_z \end{pmatrix}$$

Special theory of relativity and Maxwell's equations

Maxwell's equations in any inertial reference frame:

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Lorenz gauge condition:

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$$

Potential equations:

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\rho$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{J}$$

Field relations:

$$\mathbf{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

More 4-vectors:

$$\alpha = \{0,1,2,3\}$$

Time and position : $\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Rightarrow x^\alpha$

Charge and current : $\begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \Rightarrow J^\alpha$

Vector and scalar potentials : $\begin{pmatrix} \Phi \\ A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow A^\alpha$

Lorentz transformations

$$\mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Time and space: $x^\alpha = \mathcal{L}_v x'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} x'^\beta$

Charge and current: $J^\alpha = \mathcal{L}_v J'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} J'^\beta$

Vector and scalar potential: $A^\alpha = \mathcal{L}_v A'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} A'^\beta$

Notation: $\mathcal{L}_v^{\alpha\beta} x'^\beta \equiv \sum_{\beta=0}^{3} \mathcal{L}_v^{\alpha\beta} x'^\beta$

Repeated index summation convention

4-vector relationships

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} \Leftrightarrow \left(A^0, \mathbf{A}\right): \text{ upper index 4-vector } \quad A^\alpha \text{ for } (\alpha = 0,1,2,3)$$

Keeping track of signs -- lower index 4-vector $A_\alpha = \left(A^0, -\mathbf{A}\right)$

Derivative operators (defined with different sign convention):

$$\partial^\alpha = \left(\frac{\partial}{c\partial t}, -\nabla\right) \qquad\qquad \partial_\alpha = \left(\frac{\partial}{c\partial t}, \nabla\right)$$

Special theory of relativity and Maxwell's equations

Continuity equation: $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad \Rightarrow \quad \partial_\alpha J^\alpha = 0$

Lorenz gauge condition: $\frac{1}{c} \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0 \quad \Rightarrow \quad \partial_\alpha A^\alpha = 0$

Potential equations:

$$\left.\begin{aligned} \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi &= 4\pi\rho \\ \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} &= \frac{4\pi}{c} \mathbf{J} \end{aligned}\right\} \partial_\alpha \partial^\alpha A^\beta = \frac{4\pi}{c} J^\beta$$

Field relations:

$$\mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$\Rightarrow$??

From the scalar and vector potentials, we can determine the E and B fields and then relate them to 4-vectors, finding --

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$E_x = -\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{c\partial t} = -\left(\partial^0 A^1 - \partial^1 A^0\right)$$

$$E_y = -\frac{\partial \Phi}{\partial y} - \frac{\partial A_y}{c\partial t} = -\left(\partial^0 A^2 - \partial^2 A^0\right)$$

$$E_z = -\frac{\partial \Phi}{\partial z} - \frac{\partial A_z}{c\partial t} = -\left(\partial^0 A^3 - \partial^3 A^0\right)$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -\left(\partial^2 A^3 - \partial^3 A^2\right)$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = -\left(\partial^3 A^1 - \partial^1 A^3\right)$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = -\left(\partial^1 A^2 - \partial^2 A^1\right)$$

Field strength tensor $F^{\alpha\beta} \equiv \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)$

For stationary frame

$$F^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

For moving frame

$$F'^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -B'_z & B'_y \\ E'_y & B'_z & 0 & -B'_x \\ E'_z & -B'_y & B'_x & 0 \end{pmatrix}$$

Summary --

Field strength tensor $F^{\alpha\beta} \equiv \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)$

$$F^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F'^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -B'_z & B'_y \\ E'_y & B'_z & 0 & -B'_x \\ E'_z & -B'_y & B'_x & 0 \end{pmatrix}$$

➔ This analysis shows that the E and B fields must be treated as components of the field strength tensor and that in order to transform between inertial frames, we need to use the tensor transformation relationships:

Transformation of field strength tensor

$$F^{\alpha\beta} = \mathcal{L}_v^{\alpha\gamma} F'^{\gamma\delta} \mathcal{L}_v^{\delta\beta} \qquad \mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E'_x & -\gamma_v\left(E'_y + \beta_v B'_z\right) & -\gamma_v\left(E'_z - \beta_v B'_y\right) \\ E'_x & 0 & -\gamma_v\left(B'_z + \beta_v E'_y\right) & \gamma_v\left(B'_y - \beta_v E'_z\right) \\ \gamma_v\left(E'_y + \beta_v B'_z\right) & \gamma_v\left(B'_z + \beta_v E'_y\right) & 0 & -B'_x \\ \gamma_v\left(E'_z - \beta_v B'_y\right) & -\gamma_v\left(B'_y - \beta_v E'_z\right) & B'_x & 0 \end{pmatrix}$$

Inverse transformation of field strength tensor

$$F'^{\alpha\beta} = \mathcal{L}_v^{-1\alpha\gamma} F^{\gamma\delta} \mathcal{L}_v^{-1\delta\beta} \qquad \mathcal{L}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F'^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -\gamma_v\left(E_y - \beta_v B_z\right) & -\gamma_v\left(E_z + \beta_v B_y\right) \\ E_x & 0 & -\gamma_v\left(B_z - \beta_v E_y\right) & \gamma_v\left(B_y + \beta_v E_z\right) \\ \gamma_v\left(E_y - \beta_v B_z\right) & \gamma_v\left(B_z - \beta_v E_y\right) & 0 & -B_x \\ \gamma_v\left(E_z + \beta_v B_y\right) & -\gamma_v\left(B_y + \beta_v E_z\right) & B_x & 0 \end{pmatrix}$$

Summary of results:

$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y = \gamma_v\left(E_y - \beta_v B_z\right) \qquad B'_y = \gamma_v\left(B_y + \beta_v E_z\right)$$

$$E'_z = \gamma_v\left(E_z + \beta_v B_y\right) \qquad B'_z = \gamma_v\left(B_z - \beta_v E_y\right)$$

Comparison of the two transformations

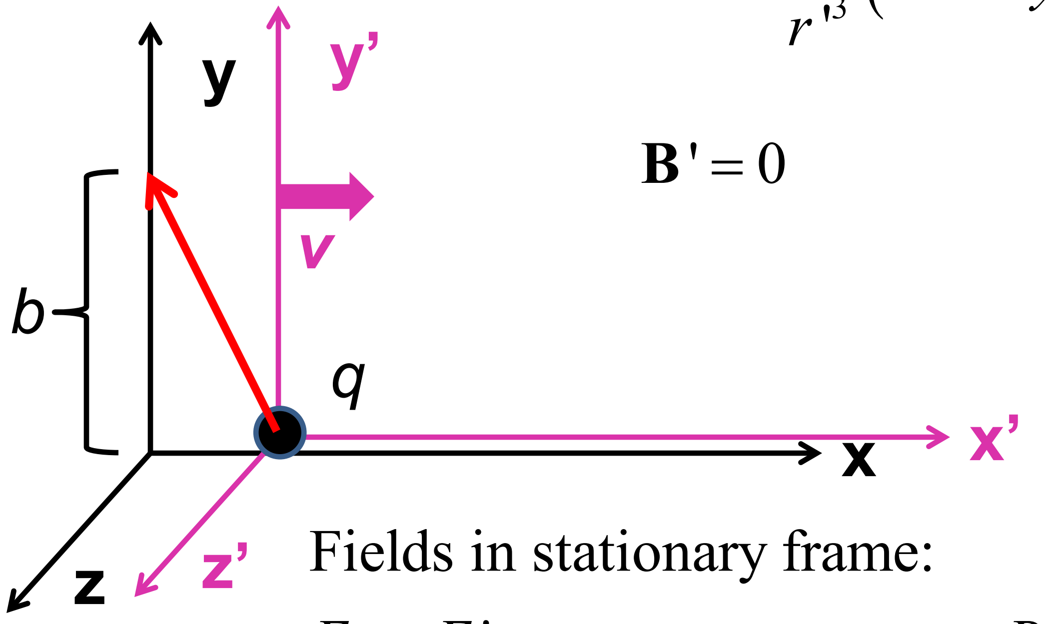
$$F^{\alpha\beta} = \mathcal{L}_v^{\alpha\gamma} F'^{\gamma\delta} \mathcal{L}_v^{\delta\beta} \qquad \mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E'_x & -\gamma_v\left(E'_y + \beta_v B'_z\right) & -\gamma_v\left(E'_z - \beta_v B'_y\right) \\ E'_x & 0 & -\gamma_v\left(B'_z + \beta_v E'_y\right) & \gamma_v\left(B'_y - \beta_v E'_z\right) \\ \gamma_v\left(E'_y + \beta_v B'_z\right) & \gamma_v\left(B'_z + \beta_v E'_y\right) & 0 & -B'_x \\ \gamma_v\left(E'_z - \beta_v B'_y\right) & -\gamma_v\left(B'_y - \beta_v E'_z\right) & B'_x & 0 \end{pmatrix}$$

$$F'^{\alpha\beta} = \mathcal{L}_v^{-1\alpha\gamma} F^{\gamma\delta} \mathcal{L}_v^{-1\delta\beta} \qquad \mathcal{L}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F'^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -\gamma_v\left(E_y - \beta_v B_z\right) & -\gamma_v\left(E_z + \beta_v B_y\right) \\ E_x & 0 & -\gamma_v\left(B_z - \beta_v E_y\right) & \gamma_v\left(B_y + \beta_v E_z\right) \\ \gamma_v\left(E_y - \beta_v B_z\right) & \gamma_v\left(B_z - \beta_v E_y\right) & 0 & -B_x \\ \gamma_v\left(E_z + \beta_v B_y\right) & -\gamma_v\left(B_y + \beta_v E_z\right) & B_x & 0 \end{pmatrix}$$

Example:



Fields in moving frame:

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$

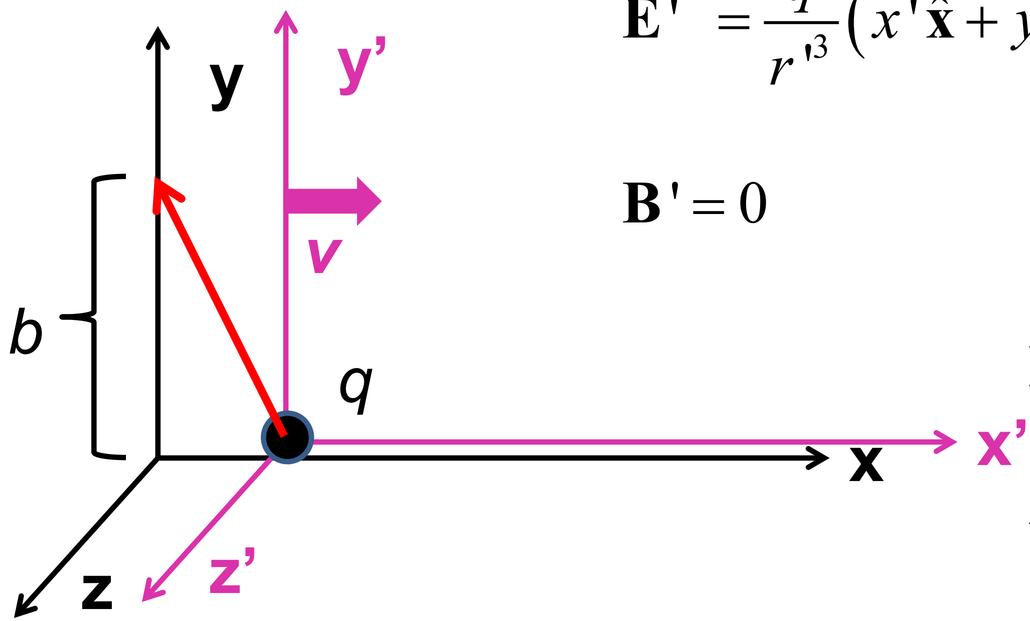
Fields in stationary frame:

$$E_x = E'_x \qquad B_x = B'_x$$

$$E_y = \gamma_v\left(E'_y + \beta_v B'_z\right) \qquad B_y = \gamma_v\left(B'_y - \beta_v E'_z\right)$$

$$E_z = \gamma_v\left(E'_z - \beta_v B'_y\right) \qquad B_z = \gamma_v\left(B'_z + \beta_v E'_y\right)$$

Example:



Fields in moving frame:

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame:

$$E_x = E'_x \qquad\qquad B_x = B'_x$$

$$E_y = \gamma_v\left(E'_y + \beta_v B'_z\right) \qquad B_y = \gamma_v\left(B'_y - \beta_v E'_z\right)$$

$$E_z = \gamma_v\left(E'_z - \beta_v B'_y\right) \qquad B_z = \gamma_v\left(B'_z + \beta_v E'_y\right)$$

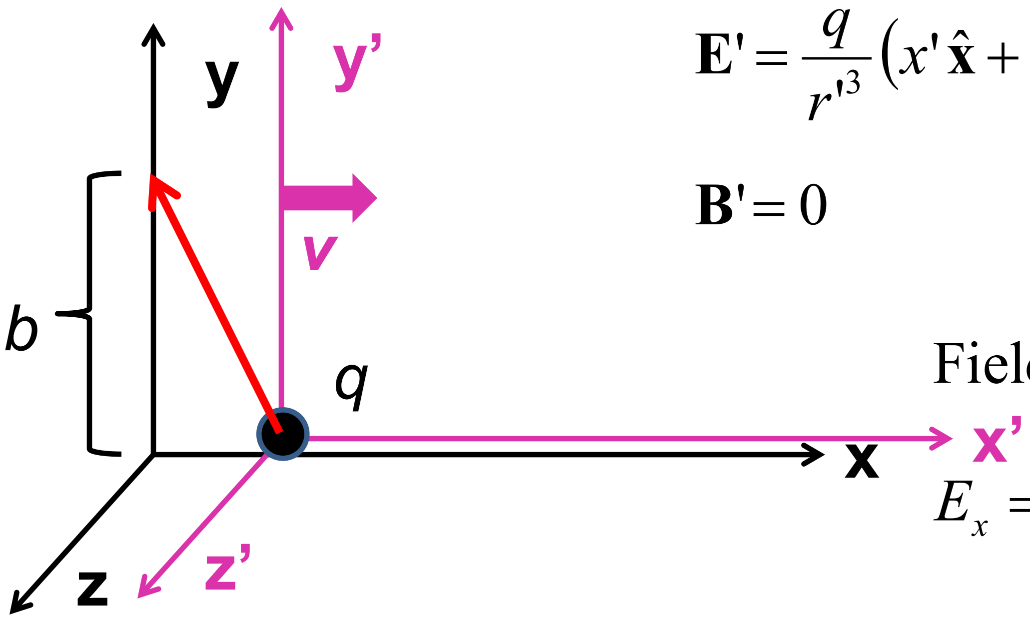
Fields in stationary frame:

$$E_x = E'_x = \frac{q\left(-vt'\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$E_y = \gamma_v\left(E'_y\right) = \frac{q\left(\gamma_v b\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$B_z = \gamma_v\left(\beta_v E'_y\right) = \frac{q\left(\gamma_v \beta_v b\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

Example:



Fields in moving frame:

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame:

$$E_x = E'_x \quad = \frac{q\left(-v\gamma_v t\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

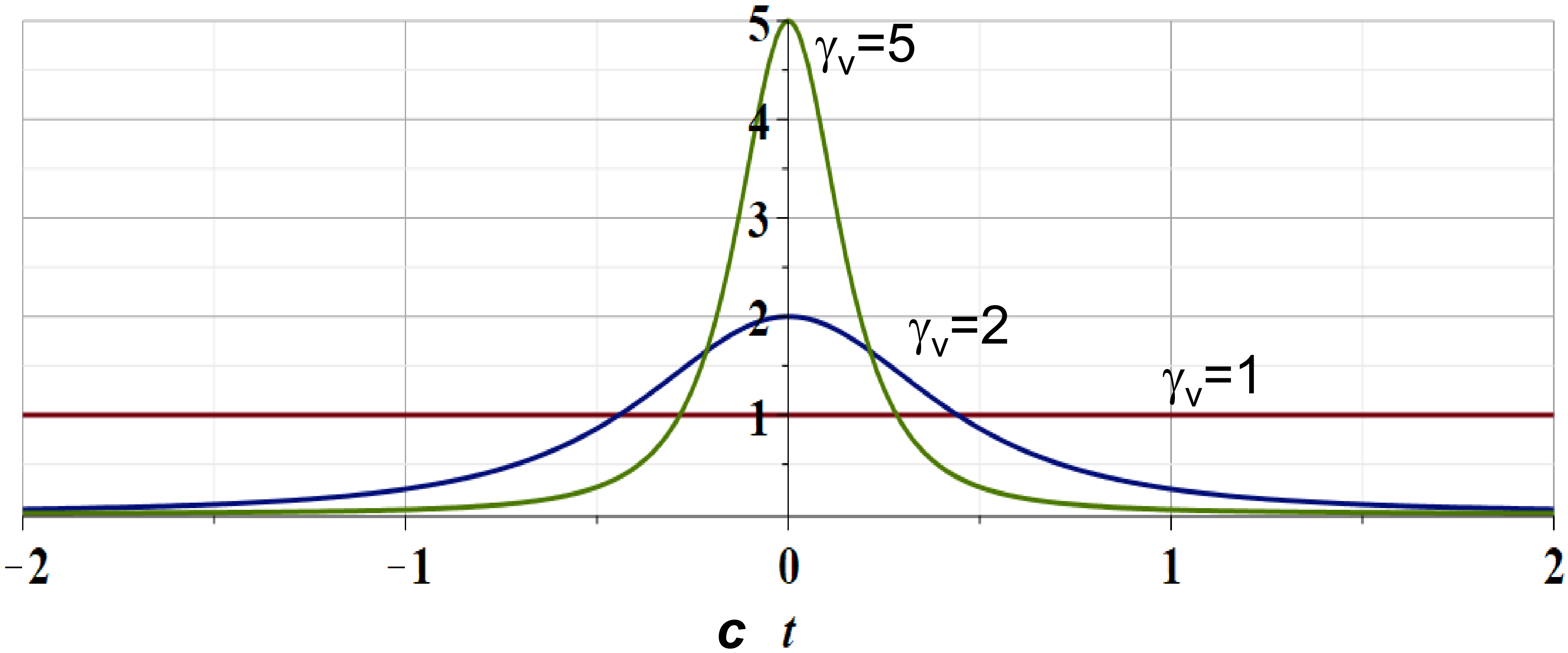
$$E_y = \gamma_v\left(E'_y\right) = \frac{q\left(\gamma_v b\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

$$B_z = \gamma_v\left(\beta_v E'_y\right) = \frac{q\left(\gamma_v \beta_v b\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

Expression in terms of consistent coordinates

$$t' = \gamma_v t$$

$$E_y = \frac{q(\gamma_v b)}{\left((-v\gamma_v t)^2 + b^2\right)^{3/2}} = \frac{q(\gamma_v b)}{\left((\gamma_v^2 - 1)c^2t^2 + b^2\right)^{3/2}}$$



Examination of this system from the viewpoint of the the Liènard-Wiechert potentials (temporarily keeping SI units)

$$\rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{R}_q(t)) \qquad \mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r}-\mathbf{R}_q(t)) \qquad \dot{\mathbf{R}}_q(t) = \frac{d\mathbf{R}_q(t)}{dt}$$

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\int d^3r'\, dt'\, \frac{\rho(\mathbf{r},t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0 c^2}\int\int d^3r'\, dt'\, \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right)$$

Evaluating integral over t':

$$\int_{-\infty}^{\infty} dt'\, f(t')\delta\left(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1-\dfrac{\mathbf{R}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

Examination of this system from the viewpoint of the the Liènard-Wiechert potentials – continued (SI units)

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{R - \dfrac{\mathbf{v}\cdot\mathbf{R}}{c}}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}}{R - \dfrac{\mathbf{v}\cdot\mathbf{R}}{c}}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{R}_q(t_r)$ $\qquad \mathbf{v} = \dfrac{d\mathbf{R}_q(t_r)}{dt_r}$

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t)$$

Examination of this system from the viewpoint of the the Liènard-Wiechert potentials – continued (SI units)

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\left(1-\frac{v^2}{c^2}\right)+\left(\mathbf{R}\times\left\{\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0 c^2}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1-\frac{v^2}{c^2}+\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right)-\frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]$$

$$\mathbf{B}(\mathbf{r},t)=\frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{cR}.$$

Examination of this system from the viewpoint of the the Liènard-Wiechert potentials – (**Gaussian units**)

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R}\times\left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}.$$

Examination of this system from the viewpoint of the the Liènard-Wiechert potentials – continued (**Gaussian units**)

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2}\right)\right]$$

For our example:

$$\mathbf{R}_q(t_r) = vt_r\hat{\mathbf{x}} \qquad \mathbf{r} = b\hat{\mathbf{y}}$$

$$\mathbf{R} = b\hat{\mathbf{y}} - vt_r\hat{\mathbf{x}} \qquad R = \sqrt{v^2 t_r^2 + b^2}$$

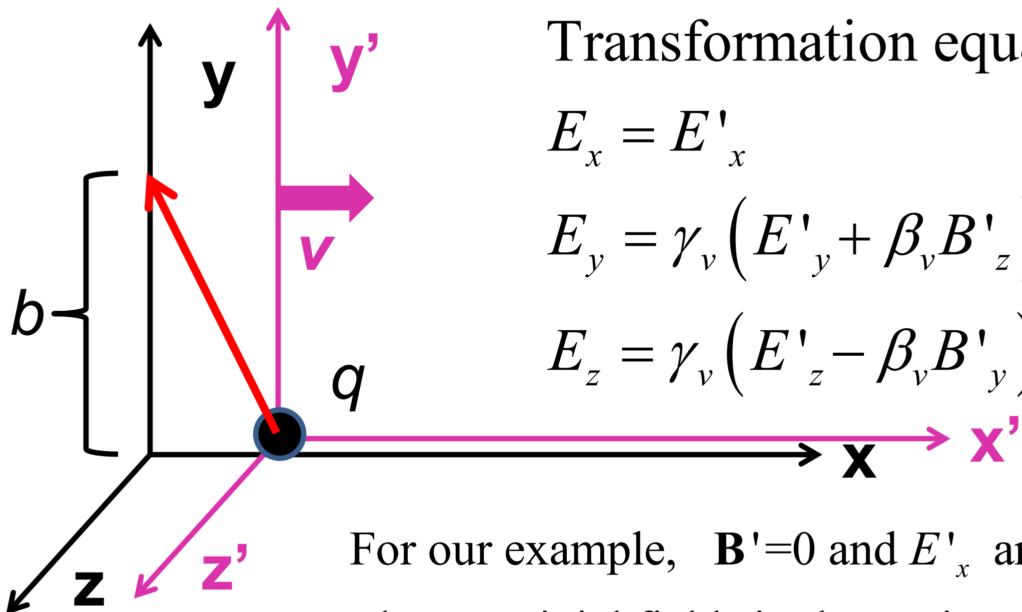
$$\mathbf{v} = v\hat{\mathbf{x}} \qquad t_r = t - \frac{R}{c}$$

This should be equivalent to the result given in Jackson (11.152):

$$\mathbf{E}(x,y,z,t) = \mathbf{E}(0,b,0,t) = q\frac{-v\gamma t\hat{\mathbf{x}} + \gamma b\hat{\mathbf{y}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

$$\mathbf{B}(x,y,z,t) = \mathbf{B}(0,b,0,t) = q\frac{\gamma\beta b\hat{\mathbf{z}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

Summary --



Transformation equations:

$$E_x = E'_x \qquad B_x = B'_x$$

$$E_y = \gamma_v \left(E'_y + \beta_v B'_z \right) \qquad B_y = \gamma_v \left(B'_y - \beta_v E'_z \right)$$

$$E_z = \gamma_v \left(E'_z - \beta_v B'_y \right) \qquad B_z = \gamma_v \left(B'_z + \beta_v E'_y \right)$$

For our example, $\mathbf{B}'=0$ and E'_x and E'_y are nontrivial

The nontrivial fields in the stationary frame are

$$E_x = E'_x$$

$$E_y = \gamma_v E'_y$$

$$B_z = \gamma_v \beta_v E'_y$$

Is this result consistent with the the Liènard-Wiechert analysis?