PHY 712 Electrodynamics
10-10:50 AM MWF Olin 103

Lecture 30: Continue reading Chap. 14 – (Especially 14.6)

Radiation by accelerating charged particles –

particularly the analysis of of synchrotron radiation

1. **Review of the basic equations**
2. **Detailed analysis of synchrotron radiation generated from man-made facilities**
3. **Synchrotron radiation from astronomical sources**

24	Mon: 03/24/2025	Chap. 9	Radiation from time harmonic sources	#20	03/26/2025
25	Wed: 03/26/2025	Chap. 9 & 10	Radiation from scattering	#21	03/28/2025
26	Fri: 03/28/2025	Chap. 11	Special Theory of Relativity	#22	03/31/2025
27	Mon: 03/31/2025	Chap. 11	Special Theory of Relativity	#23	04/02/2025
28	Wed: 04/02/2025	Chap. 11	Special Theory of Relativity	#24	04/04/2025
29	Fri: 04/04/2024	Chap. 14	Radiation from accelerating charged particles	#25	04/07/2025
30	Mon: 04/07/2025	Chap. 14	Analysis of synchroton radiation	#26	04/09/2025
31	Wed: 04/09/2025	Chap. 14	Synchrotron radiation and Compton scattering		
32	Fri: 04/11/2025	Chap. 13 & 15	Other radiation -- Cherenkov & bremsstrahlung		
33	Mon: 04/14/2025	Special Topics			
34	Wed: 04/16/2025	Special Topics			
35	Fri: 04/18/2025		Presentations I		
	Mon: 04/21/2025	Special topics			
	Wed: 04/23/2025		Presentations II		
	Fri: 04/25/2025		Presentations III		
36	Mon: 04/28/2025		Review		

PHY 712 -- Assignment #26

Assigned: 4/07/2025 Due: 4/09/2025

Continue reading Chap. 14 in **Jackson** .

In class, we showed how the synchotron radiation spectrum is scaled by the critical frequency ω_c or critical energy $E_c = \hbar\omega_c$. Using the intensity formula for radiation in the parallel plane at θ=0, for a beam with E_c=10 GeV, estimate the intensity relative to peak intensity for the following types of radiation (noting your choice of wavelength or frequency for each range)

1. Infrared
2. Visable
3. Xray

The results discussed in today's lecture come from the following 1949 paper of Julian Schwinger --

PHYSICAL REVIEW VOLUME 75, NUMBER 12 JUNE 15, 1949

On the Classical Radiation of Accelerated Electrons

JULIAN SCHWINGER
Harvard University, Cambridge, Massachusetts

(Received March 8, 1949)

This paper is concerned with the properties of the radiation from a high energy accelerated electron, as recently observed in the General Electric synchrotron. An elementary derivation of the total rate of radiation is first presented, based on Larmor's formula for a slowly moving electron, and arguments of relativistic invariance. We then construct an expression for the instantaneous power radiated by an electron moving along an arbitrary, prescribed path. By casting this result into various forms, one obtains the angular distribution, the spectral distribution, or the combined angular and spectral distributions of the radiation. The method is based on an examination of the rate at which the electron irreversibly transfers energy to the electromagnetic field, as determined by half the difference of retarded and advanced electric field intensities. Formulas are obtained for an arbitrary charge-current distribution and then specialized to a point charge. The total radiated power and its angular distribution are obtained for an arbitrary trajectory. It is found that the direction of motion is a strongly preferred direction of emission at high energies. The spectral distribution of the radiation depends upon the detailed motion over a time interval large compared to the period of the radiation. However, the narrow cone of radiation generated by an energetic electron indicates that only a small part of the trajectory is effective in producing radiation observed in a given direction, which also implies that very high frequencies are emitted. Accordingly, we evaluate the spectral and angular distributions of the high frequency radiation by an energetic electron, in their dependence upon the parameters characterizing the instantaneous orbit. The average spectral distribution, as observed in the synchrotron measurements, is obtained by averaging the electron energy over an acceleration cycle. The entire spectrum emitted by an electron moving with constant speed in a circular path is also discussed. Finally, it is observed that quantum effects will modify the classical results here obtained only at extraordinarily large energies.

EARLY in 1945, much attention was focused on the design of accelerators for the production of very high energy electrons and other charged particles.[1] In connection with this activity, the author investigated in some detail the limitations to the is instantaneously at rest is

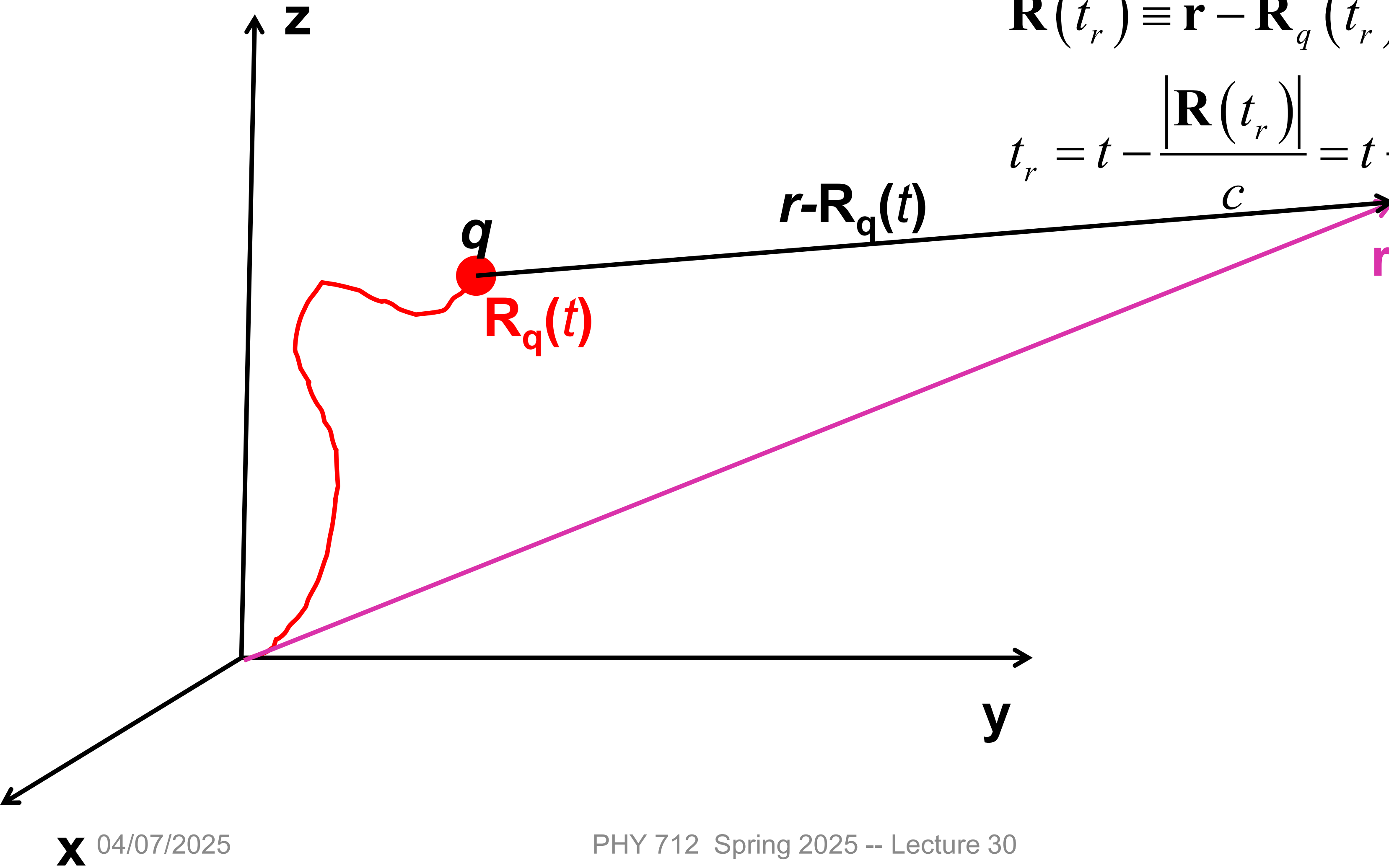
$$P=\frac{2}{3}\frac{e^2}{c^3}\left(\frac{d\mathbf{v}}{dt}\right)^2=\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{d\mathbf{p}}{dt}\right)^2. \tag{I.1}$$

Radiation from a moving charged particle analyzed by Liénard-Wiechard --

Variables (notation): $$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

$$t_r = t - \frac{|\mathbf{R}(t_r)|}{c} = t - \frac{R(t_r)}{c}$$



Alfred-Marie Liénard

Born	Alfred-Marie Liénard 2 April 1869 Amiens, France
Died	29 April 1958 (aged 89) Paris, France
Known for	Liénard equation Liénard–Chipart criterion Liénard–Wiechert potential
Awards	Poncelet Prize (1929)
Scientific career	
Institutions	École des Mines de Saint-Étienne

Emil Johann Wiechert

Emil Johann Wiechert

Born	26 December 1861 Tilsit, Province of Prussia, Kingdom of Prussia
Died	19 March 1928 (aged 66) Göttingen, Germany
Nationality	German
Citizenship	German
Alma mater	University of Königsberg, University of Göttingen
Known for	Liénard–Wiechert potential Maxwell–Wiechert model

From Liénard-Wiechard analysis, the electric and magnetic fields far from source are given by --

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left\{\mathbf{R}\times\left[\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t)=\frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}$$

Note that all of the variables on the right hand side of the equations depend on *t_r* .

$$\dot{\mathbf{R}}_q(t_r)\equiv\frac{d\mathbf{R}_q(t_r)}{dt_r}\equiv\mathbf{v} \qquad \mathbf{R}(t_r)\equiv\mathbf{r}-\mathbf{R}_q(t_r)\equiv\mathbf{R} \qquad t_r=t-\frac{|\mathbf{R}(t_r)|}{c}=t-\frac{R(t_r)}{c}$$

Let $\hat{\mathbf{R}}\equiv\frac{\mathbf{R}}{R}$ $\quad \boldsymbol{\beta}\equiv\frac{\mathbf{v}}{c}$ $\quad \dot{\boldsymbol{\beta}}\equiv\frac{\dot{\mathbf{v}}}{c}$

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{cR\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\} \qquad \mathbf{B}(\mathbf{r},t)=\hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

Poynting vector:

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}(\mathbf{E}\times\mathbf{B})$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2 = \frac{q^2}{4\pi cR^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

Power radiated

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2 = \frac{q^2}{4\pi c R^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

Spectral composition of electromagnetic radiation

Previously we determined the power distribution from a charged particle:

$$\frac{dP(t)}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\left.\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}\right|_{t_r=t-R/c}$$

$$\equiv\left|\boldsymbol{a}(t)\right|^2$$

where $$\boldsymbol{a}(t)\equiv\sqrt{\frac{q^2}{4\pi c}}\left.\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\right|_{t_r=t-R/c}$$

Time integrated power per solid angle:

$$\frac{dW}{d\Omega}=\int_{-\infty}^{\infty}dt\,\frac{dP(t)}{d\Omega}=\int_{-\infty}^{\infty}dt\left|\boldsymbol{a}(t)\right|^2=\int_{-\infty}^{\infty}d\omega\left|\tilde{\boldsymbol{a}}(\omega)\right|^2$$

Spectral composition of electromagnetic radiation -- continued

Time integrated power per solid angle:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \, \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt \left|\mathbf{a}(t)\right|^2 = \int_{-\infty}^{\infty} d\omega \left|\tilde{\mathbf{a}}(\omega)\right|^2$$

Fourier amplitude:

$$\tilde{\mathbf{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, \mathbf{a}(t) \, e^{i\omega t} \qquad \mathbf{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, \tilde{\mathbf{a}}(\omega) \, e^{-i\omega t}$$

Parseval's theorem

Marc-Antoine Parseval des Chênes 1755-1836

http://www-history.mcs.st-andrews.ac.uk/Biographies/Parseval.html

Spectral composition of electromagnetic radiation -- continued

Consequences of Parseval's analysis:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt \left|\mathbf{a}(t)\right|^2 = \int_{-\infty}^{\infty} d\omega \left|\tilde{\mathbf{a}}(\omega)\right|^2$$

Note that: $\tilde{\mathbf{a}}(\omega) = \tilde{\mathbf{a}}^*(-\omega)$

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} d\omega \left|\tilde{\mathbf{a}}(\omega)\right|^2 = \int_{0}^{\infty} d\omega \left(\left|\tilde{\mathbf{a}}(\omega)\right|^2 + \left|\tilde{\mathbf{a}}(-\omega)\right|^2 \right) \equiv \int_{0}^{\infty} d\omega \frac{\partial^2 I}{\partial\Omega\partial\omega}$$

$$\frac{\partial^2 I}{\partial\Omega\partial\omega} \equiv 2\left|\tilde{\mathbf{a}}(\omega)\right|^2$$

Spectral composition of electromagnetic radiation -- continued

For our case: $$\boldsymbol{a}(t) \equiv \sqrt{\frac{q^2}{4\pi c}} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^3} \right|_{t_r = t - R/c}$$

Fourier amplitude:

$$\tilde{\boldsymbol{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \; e^{i\omega t} \; \boldsymbol{a}(t)$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt \; e^{i\omega t} \; \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^3} \right|_{t_r = t - R/c}$$

Spectral composition of electromagnetic radiation -- continued

Fourier amplitude:

$$\tilde{\mathbf{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, \mathbf{a}(t)\, e^{i\omega t}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt \left. \frac{\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right]}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^3} \right|_{t_r = t - R/c} e^{i\omega t}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \frac{dt}{dt_r} \left. \frac{\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right]}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^3} \right|_{t_r = t - R/c} e^{i\omega \left(t_r + R(t_r)/c \right)}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \left. \frac{\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right]}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^2} \right|_{t_r = t - R/c} e^{i\omega \left(t_r + R(t_r)/c \right)}$$

Spectral composition of electromagnetic radiation -- continued

Working expression :

$$\tilde{\mathbf{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^2} \right|_{t_r = t - R/c} e^{i\omega\left(t_r + R(t_r)/c\right)}$$

Recall: $\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \qquad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$

For $r >> R_q(t_r) \qquad R(t_r) \approx r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)$ where $\hat{\mathbf{r}} \equiv \frac{\mathbf{r}}{r}$

At the same level of approximation: $\hat{\mathbf{R}} \approx \hat{\mathbf{r}}$

Spectral composition of electromagnetic radiation -- continued

Working expression:

$$\tilde{a}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \left. \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^2} \right|_{t_r = t - R/c} e^{i\omega\left(t_r + R(t_r)/c\right)}$$

For $\mathbf{R} \approx \mathbf{r}$:

$$\tilde{a}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} e^{i\omega(r/c)} \int_{-\infty}^{\infty} dt_r \left. \frac{\left|\hat{\mathbf{r}} \times \left[\left(\hat{\mathbf{r}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}}\right)^2} \right|_{t_r = t - R/c} e^{i\omega\left(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c\right)}$$

Resulting spectral intensity expression:

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \left. \frac{\left|\hat{\mathbf{r}} \times \left[\left(\hat{\mathbf{r}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}}\right)^2} \right|_{t_r = t - R/c} e^{i\omega\left(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c\right)} \right|^2$$

➔Spectral form of radiation far from source:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c}\left| \int_{-\infty}^{\infty} dt_r \left. \frac{\left|\hat{\mathbf{r}} \times \left[\left(\hat{\mathbf{r}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2} \right|_{t_r = t - R/c} e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q\left(t_r\right)/c\right)} \right|^2$$

In order to analyze this expression, we need to know the particle trajectory $\mathbf{R}_q(t_r)$, its velocity $\boldsymbol{\beta}c = \frac{d\mathbf{R}_q(t_r)}{dt_r}$,

and its acceleration $\dot{\boldsymbol{\beta}}c = \frac{d^2\mathbf{R}_q(t_r)}{dt_r^2}$.

Spectral composition of electromagnetic radiation – more detailed treatment --

Alternative expression --

It can be shown that:

$$\frac{\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}=\frac{d}{dt_r}\left(\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\right)}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)}\right)$$

Integration by parts and assumptions about the integration limit behaviors shows that the spectral intensity depends on the following integral:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega}=\frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty}dt_r\left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\left(t_r\right)\right)\right]e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q\left(t_r\right)/c\right)}\right|^2$$

Some details --

Spectral intensity expression that needs to be evaluated:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r\; e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \left.\frac{\left|\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}\right|_{t_r=t-R/c}\right|^2$$

It can be shown that: $$\frac{\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2} = \frac{d}{dt_r}\left(\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\right)}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)}\right)$$

$$\int_{-\infty}^{\infty} dt_r\; e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \left.\frac{\left|\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}\right|_{t_r=t-R/c} = \int_{-\infty}^{\infty} dt_r\; e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \frac{d}{dt_r}\left(\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)}{\left(1-\boldsymbol{\beta}(t_r)\cdot\hat{\mathbf{r}}\right)}\right)$$

$$= \int_{-\infty}^{\infty} dt_r\; \frac{d}{dt_r}\left(e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\left(\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)}{\left(1-\boldsymbol{\beta}(t_r)\cdot\hat{\mathbf{r}}\right)}\right)\right) - i\omega\int_{-\infty}^{\infty} dt_r\; \left(e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\left(\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)\right)\right)$$

0

More details

$$\int_{-\infty}^{\infty} dt_r \; e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \left.\frac{\left|\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}\right|_{t_r = t - R/c} = \int_{-\infty}^{\infty} dt_r \; e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \frac{d}{dt_r}\left(\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)}{\left(1-\boldsymbol{\beta}(t_r)\cdot\hat{\mathbf{r}}\right)}\right)$$

$$= \int_{-\infty}^{\infty} dt_r \; \frac{d}{dt_r}\left(e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\left(\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)}{\left(1-\boldsymbol{\beta}(t_r)\cdot\hat{\mathbf{r}}\right)}\right)\right) - i\omega\int_{-\infty}^{\infty} dt_r \; \left(e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\left(\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)\right)\right)$$

Comes from integration by parts --

$$\int_{-\infty}^{\infty} dx \; F(x)\frac{dG(x)}{dx} = \int_{-\infty}^{\infty} dx \; \frac{d}{dx}\left(F(x)G(x)\right) - \int_{-\infty}^{\infty} dx \; \frac{dF(x)}{dx}G(x)$$

Spectral composition of electromagnetic radiation -- continued

When the dust clears, the spectral intensity depends

on the following integral:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega}=\frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty}dt_r\; e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\;\left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\left(t_r\right)\right)\right]\right|^2$$

In order to analyze this expression, we need to know

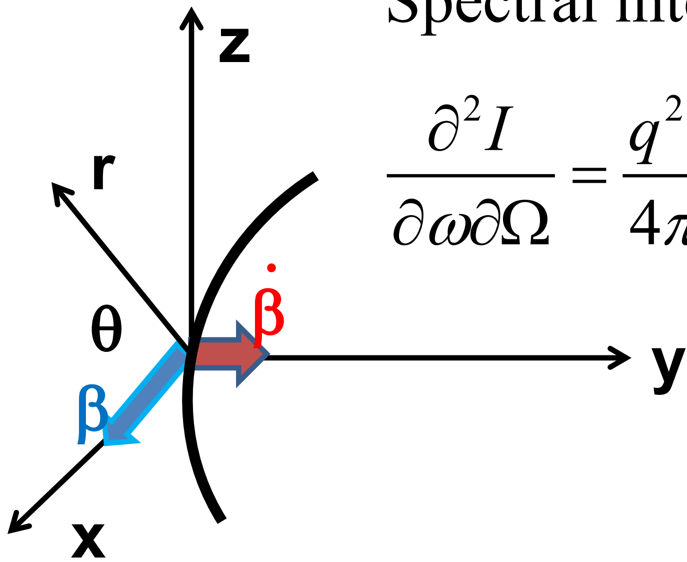
the particle trajectory $\mathbf{R}_q(t_r)$, its velocity $\boldsymbol{\beta}c = d\mathbf{R}_q(t_r)/dt_r$.

Recall that the spectral intensity is related

to the time integrated power:

$$\int_{-\infty}^{\infty}dt\,\frac{dP(t)}{d\Omega}=\int_{-\infty}^{\infty}d\omega\,\frac{\partial^2 I}{\partial\omega\partial\Omega}$$

Specific evaluation for particle moving in a circular path at a constant speed *v* --



Spectral intensity relationship:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r\ e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\ \left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)\right]\right|^2$$

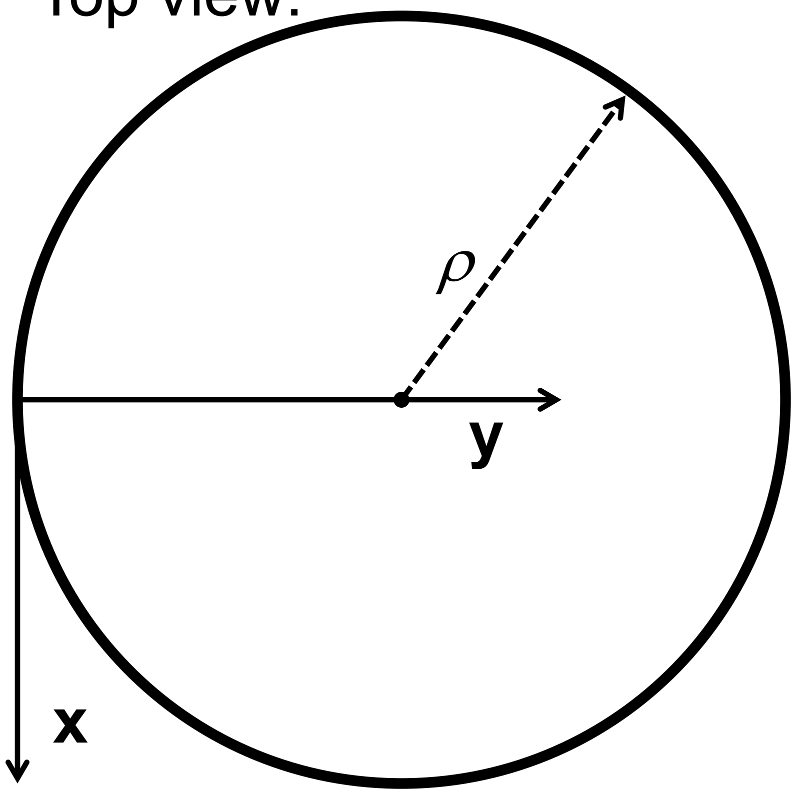
$$\mathbf{R}_q(t_r) = \rho\hat{\mathbf{x}}\sin\left(vt_r/\rho\right) + \rho\hat{\mathbf{y}}\left(1-\cos\left(vt_r/\rho\right)\right)$$

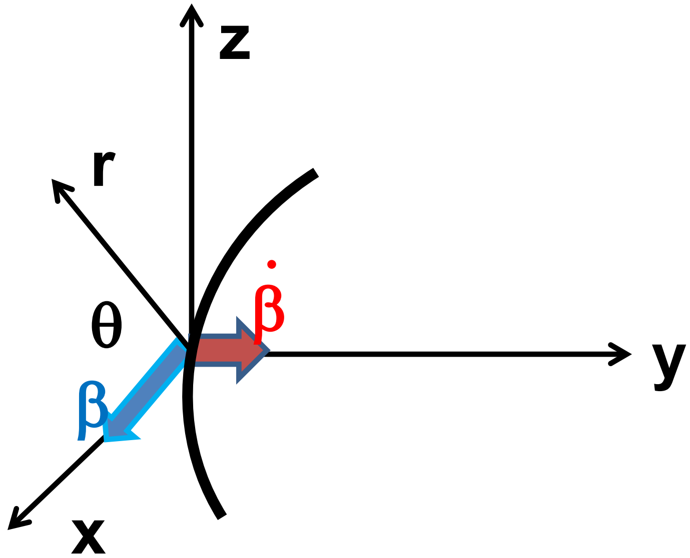
$$\boldsymbol{\beta}(t_r) = \beta\left(\hat{\mathbf{x}}\cos\left(vt_r/\rho\right) + \hat{\mathbf{y}}\sin\left(vt_r/\rho\right)\right)$$

For convenience, choose:

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta$$

Top view:





$$\mathbf{R}_q(t_r) = \rho\hat{\mathbf{x}}\sin(vt_r/\rho) + \rho\hat{\mathbf{y}}\left(1-\cos(vt_r/\rho)\right)$$

$$\boldsymbol{\beta}(t_r) = \beta\left(\hat{\mathbf{x}}\cos(vt_r/\rho) + \hat{\mathbf{y}}\sin(vt_r/\rho)\right)$$

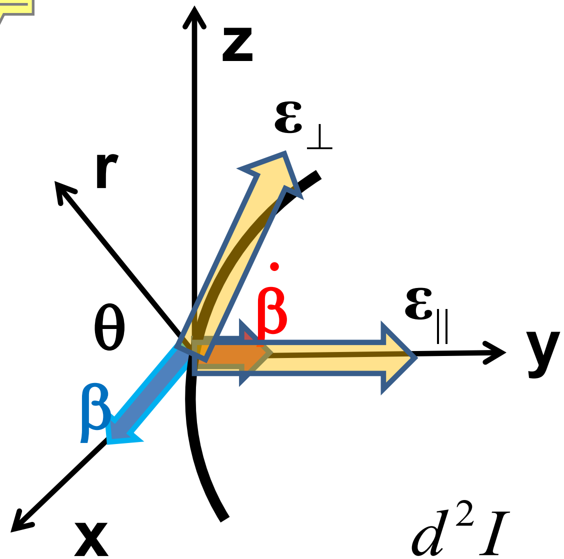
For convenience, choose:

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta$$

Note that we have previous shown that in the radiation zone, the Poynting vector is in the $\hat{\mathbf{r}}$ direction; we can then choose to analyze two orthogonal polarization directions:

$$\boldsymbol{\varepsilon}_{\parallel} = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_{\perp} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\boldsymbol{\beta}) = \beta\left(-\boldsymbol{\varepsilon}_{\parallel}\sin(vt_r/\rho) + \boldsymbol{\varepsilon}_{\perp}\sin\theta\cos(vt_r/\rho)\right)$$



$$\boldsymbol{\varepsilon}_{||} = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_\perp = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \boldsymbol{\beta}\right) =$$

$$\beta\left(-\boldsymbol{\varepsilon}_{||}\sin\left(vt_r/\rho\right) + \boldsymbol{\varepsilon}_\perp \sin\theta\cos\left(vt_r/\rho\right)\right)$$

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \beta) e^{i\omega(t - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c)} dt\right|^2$$

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2\omega^2\beta^2}{4\pi^2 c}\left\{|C_{||}(\omega)|^2 + |C_\perp(\omega)|^2\right\}$$

$$C_{||}(\omega) = \int_{-\infty}^{\infty} dt \sin(vt/\rho) e^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$C_\perp(\omega) = \int_{-\infty}^{\infty} dt \sin\theta\cos(vt/\rho) e^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

We will analyze this expression for two different cases. The first case, is appropriate for man-made synchrotrons used as light sources. In this case, the light is produced by short bursts of electrons moving close to the speed of light ($v \approx c(1-1/(2\gamma^2))$) passing a beam line port. In addition, because of the design of the radiation ports, $\theta \approx 0$, and the relevant integration times t are close to $t \approx 0$. This results in the form shown in Eq. 14.79 of your text. It is convenient to rewrite this form in terms of a critical frequency $\omega_c \equiv \dfrac{3c\gamma^3}{2\rho}$.

$$\frac{d^2I}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left\{\left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2\right\}$$

Some details:

Modified Bessel functions

$$K_{1/3}(\xi)=\sqrt{3}\int_0^\infty dx\cos\left[\tfrac{3}{2}\xi\left(x+\tfrac{1}{3}x^3\right)\right] \qquad K_{2/3}(\xi)=\sqrt{3}\int_0^\infty dx\, x\sin\left[\tfrac{3}{2}\xi\left(x+\tfrac{1}{3}x^3\right)\right]$$

Exponential factor

$$\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)=\omega\left(t_r-\frac{\rho}{c}\cos\theta\sin\left(vt_r/\rho\right)\right)$$

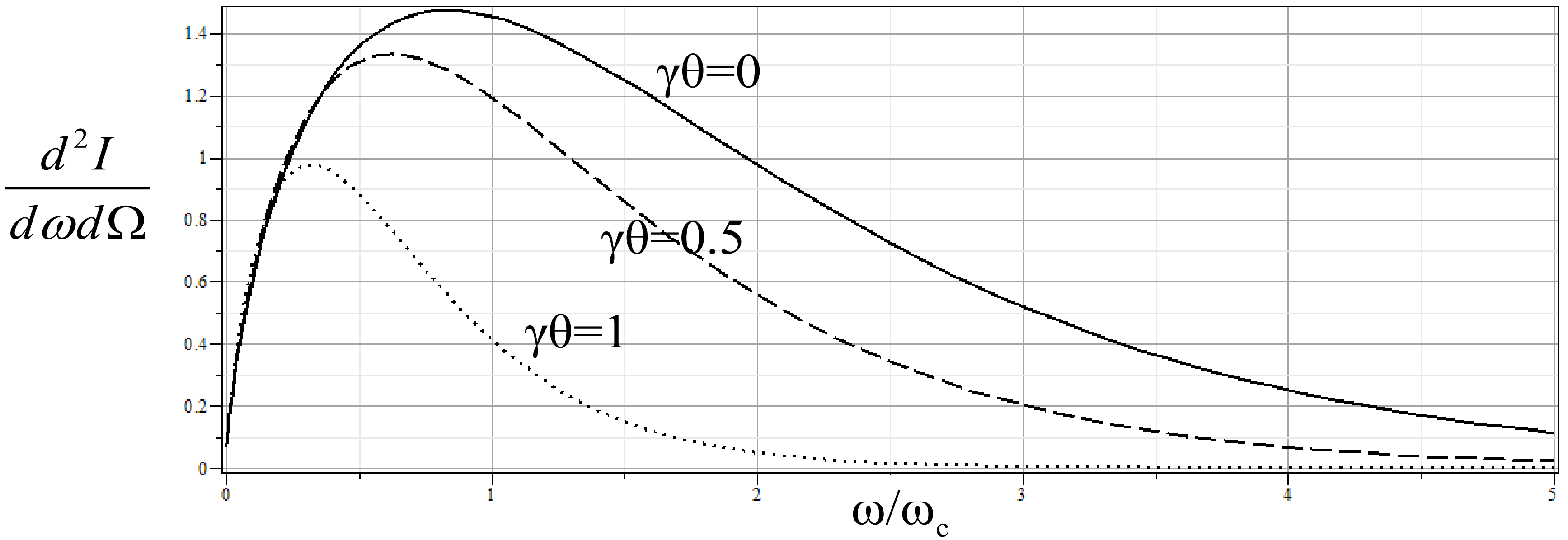
In the limit of $t_r\approx 0,\quad \theta\approx 0,\quad v\approx c\left(1-\dfrac{1}{2\gamma^2}\right)$

$$\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)\approx\frac{\omega t_r}{2\gamma^2}\left(1+\gamma^2\theta^2\right)+\frac{\omega c^2 t_r{}^3}{6\rho^2}=\frac{3}{2}\xi\left(x+\frac{1}{3}x^3\right)$$

where $\quad \xi=\dfrac{\omega\rho}{3c\gamma^3}\left(1+\gamma^2\theta^2\right)^{3/2}\quad$ and $\quad x=\dfrac{c\gamma t_r}{\rho\left(1+\gamma^2\theta^2\right)^{1/2}}$

$$\frac{d^2 I}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left\{\left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2\right\}$$

By plotting the intensity as a function of ω, we see that the intensity is largest near $\omega \approx \omega_c$. The plot below shows the intensity as a function of ω/ω_c for $\gamma\theta=0$, 0.5 and 1:

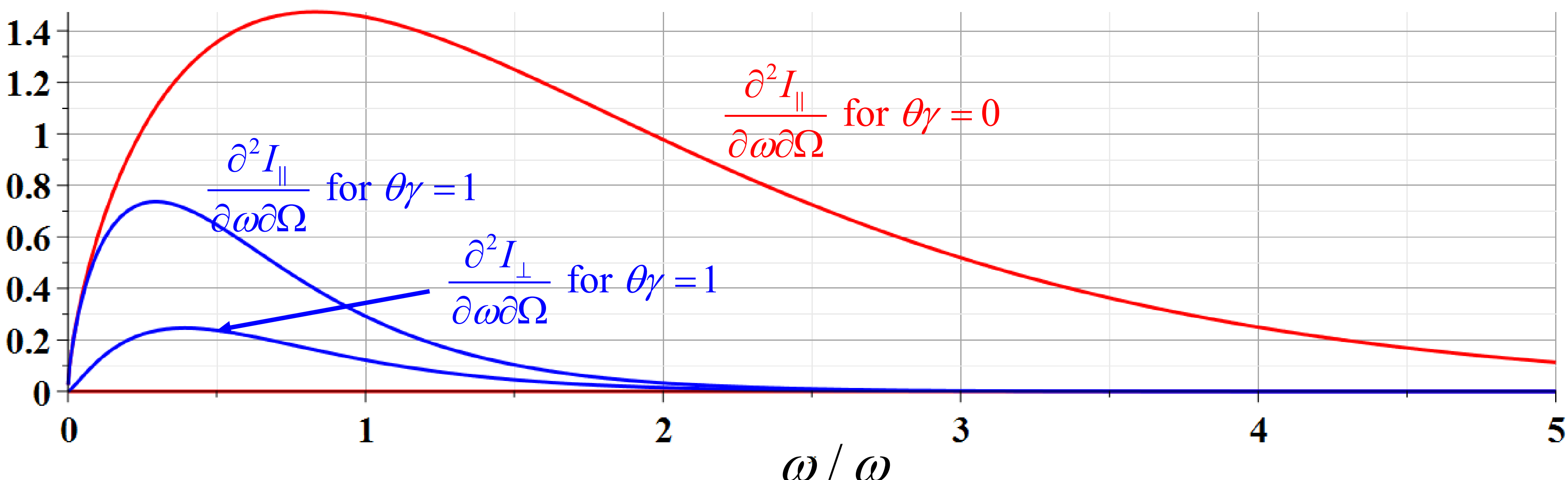


More details

$$\frac{d^2 I}{d\omega d\Omega} = \frac{d^2 I_{\parallel}}{d\omega d\Omega} + \frac{d^2 I_{\perp}}{d\omega d\Omega}$$

$$\frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2$$

$$\frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2$$



Why assume

$$t_r \approx 0, \quad \theta \approx 0, \quad v \approx c\left(1 - \frac{1}{2\gamma^2}\right) \quad ???$$

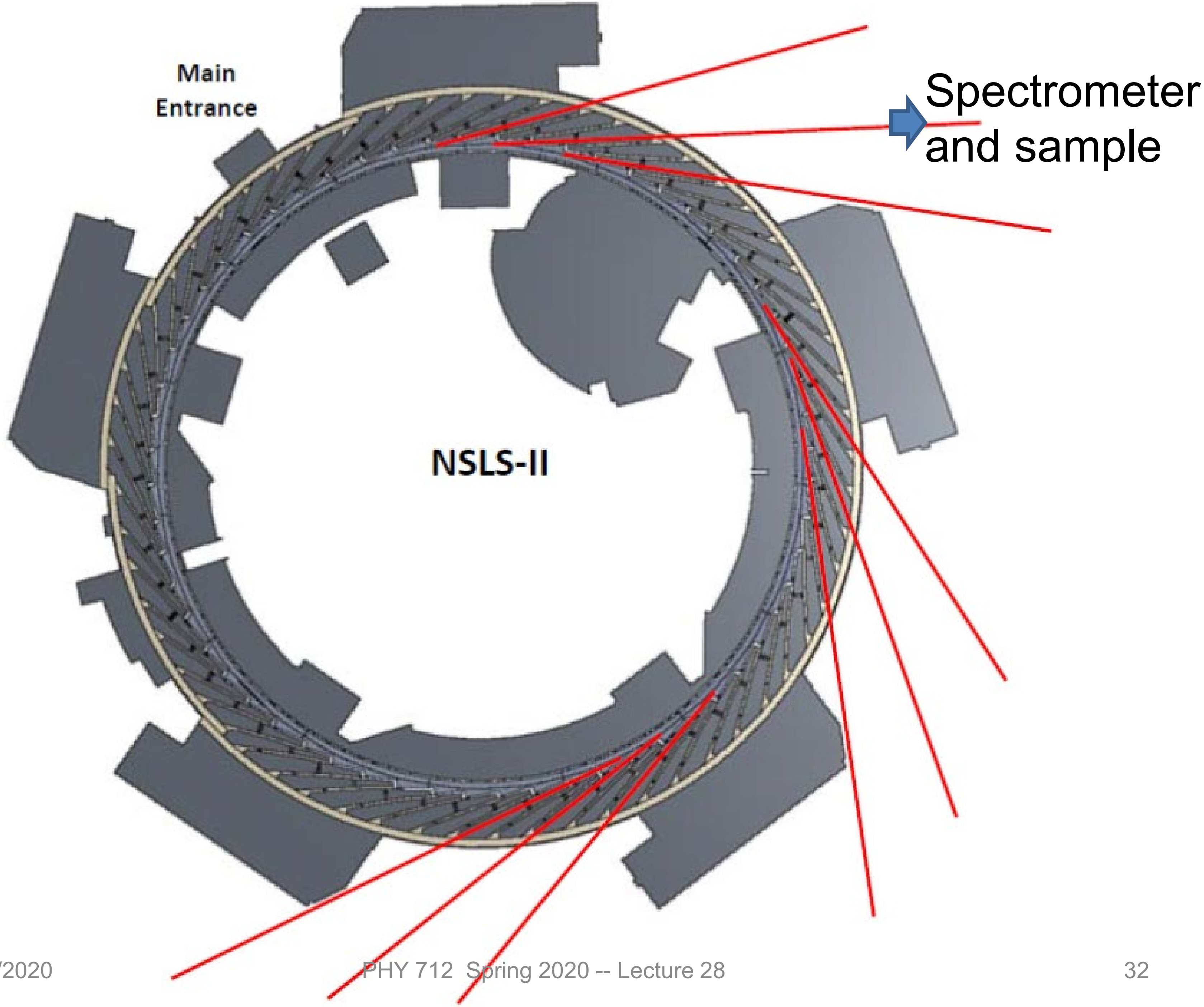
The above analysis applies to a class of man-made facilities dedicated to producing intense radiation in the continuous spectrum. For more specific information on man-made synchrotron sources, the following web page is useful: http://www.als.lbl.gov/als/synchrotron_sources.html.

Synchrotron radiation light source installations

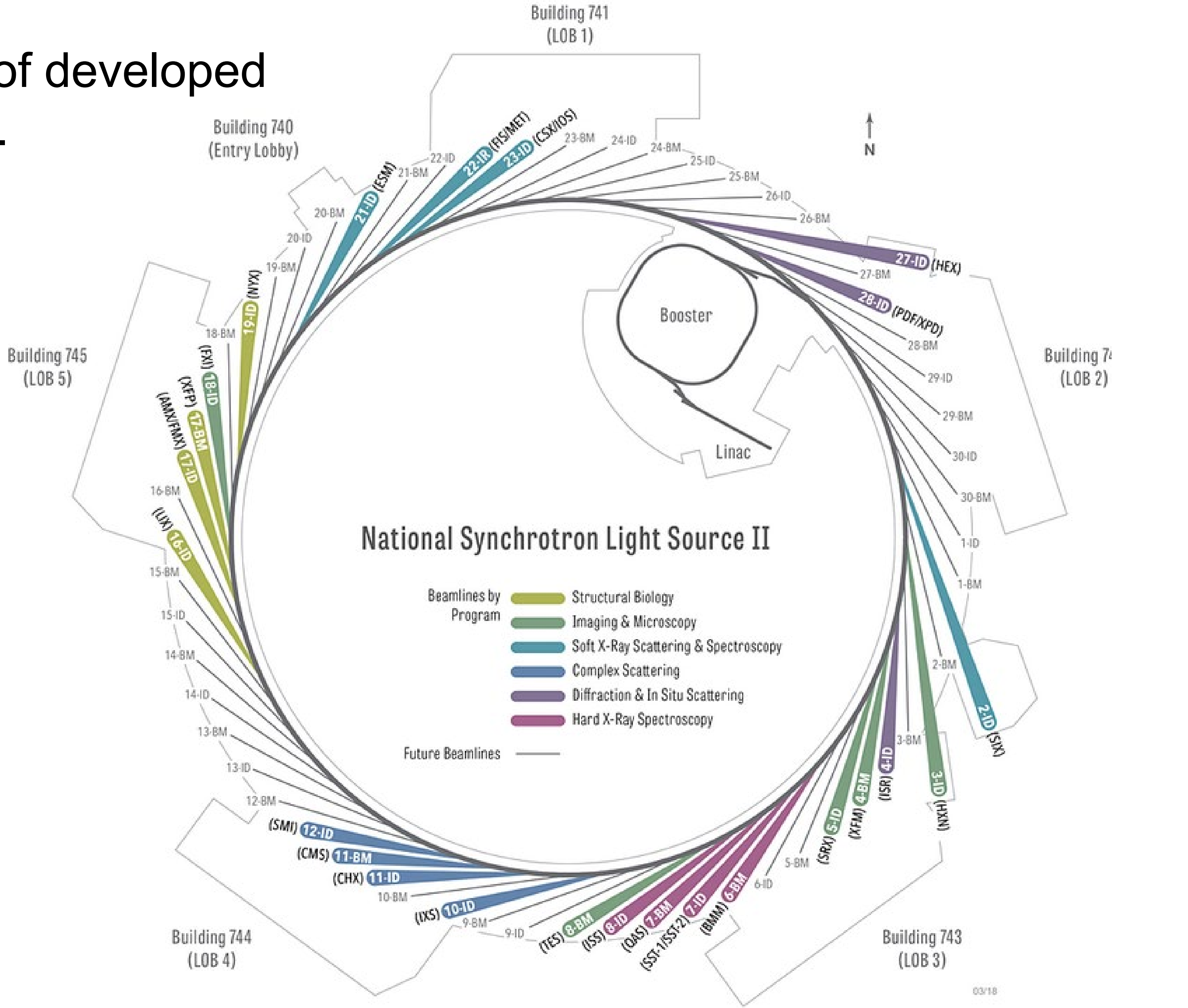
Synchrotron at Brookhaven National Lab, NY

E_c= 3 GeV X-ray radiation

https://www.bnl.gov/ps/


Main
Entrance
NSLS-II
Spectrometer
and sample

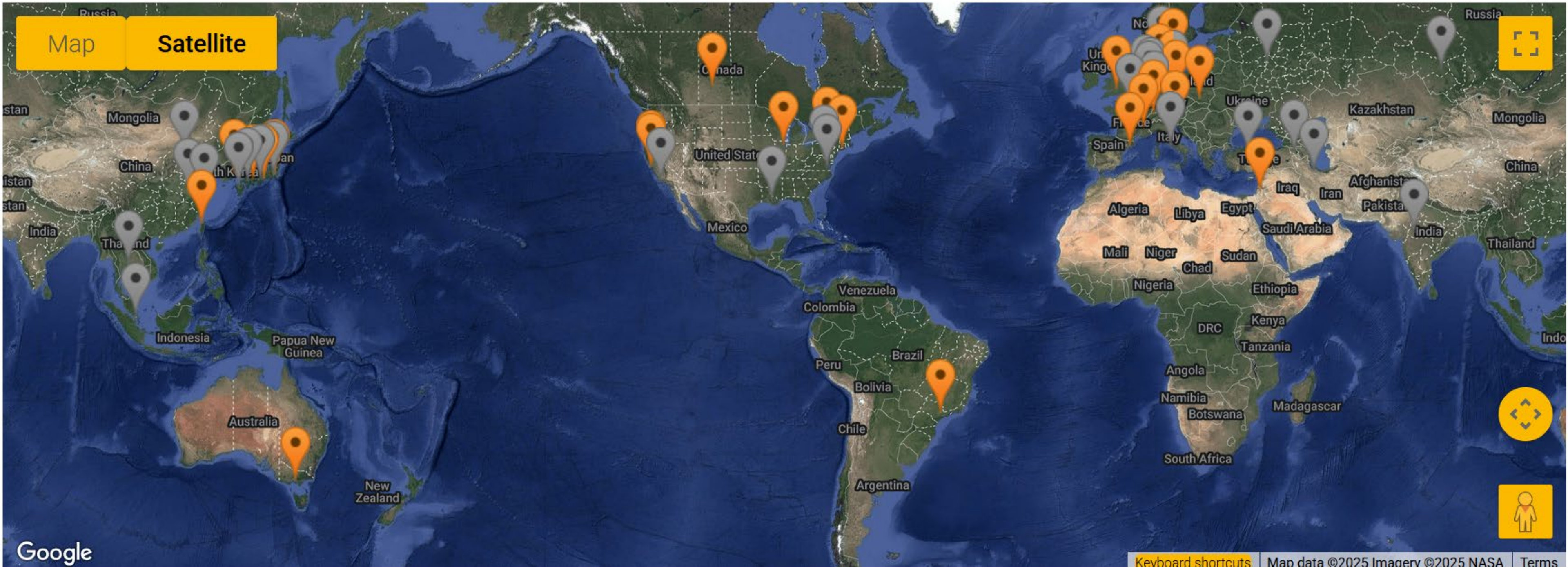
Overview of developed beamlines.



Advanced photon source, Argonne National Laboratory

https://www.aps.anl.gov/

https://lightsources.org/lightsources-of-the-world/



The second example of synchrotron radiation comes from a distant charged particle moving in a circular trajectory such that the spectrum represents a superposition of light generated over many complete circles. In this case, there is an interference effect which results in the spectrum consisting of discrete multiples of v/ρ. For this case we need to reconsider the analysis. There is a very convenient Bessel function identity of the form:

$$e^{-ix\sin u} = \sum_{m=-\infty}^{\infty} J_m(x)e^{-imu} \quad \text{Here } J_m(x) \text{ is a Bessel function of integer order } m.$$

$$\text{In our case } x = \frac{\omega\rho}{c}\cos\theta \text{ and } u = \frac{vt}{\rho}.$$

$$C_{\|}(\omega) = \int_{-\infty}^{\infty} dt \sin(vt/\rho) e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))} = \frac{c}{-i\omega\rho}\frac{\partial}{\partial\cos\theta}\int_{-\infty}^{\infty} dt e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$= \frac{c}{-i\omega\rho}\frac{\partial}{\partial\cos\theta}\sum_{m=-\infty}^{\infty} J_m\left(\frac{\omega\rho}{c}\cos\theta\right)2\pi\delta(\omega - m\frac{v}{\rho}).$$

Astronomical synchrotron radiation -- continued:

Note that:

$$\int_{-\infty}^{\infty} dt e^{i(\omega - m\frac{v}{\rho})t} = 2\pi\delta(\omega - m\frac{v}{\rho}).$$

$$\Rightarrow C_{\parallel}(\omega) = 2\pi i \sum_{m=-\infty}^{\infty} J_m^{'}\left(\frac{\omega\rho}{c}\cos\theta\right)\delta(\omega - m\frac{v}{\rho}),$$

where $J_m^{'}(x) \equiv \dfrac{dJ_m(x)}{dx}$

Similarly:

$$C_{\perp}(\omega) = \int_{-\infty}^{\infty} dt \sin\theta \cos(vt/\rho) e^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$= 2\pi \frac{\tan\theta}{v/c} \sum_{m=-\infty}^{\infty} J_m\left(\frac{\omega\rho}{c}\cos\theta\right)\delta(\omega - m\frac{v}{\rho}).$$

Astronomical synchrotron radiation -- continued:

In both of the expressions, the sum over *m* includes both negative and positive values. However, only the positive values of ω and therefore positive values of *m* are of interest. Using the identity: $J_{-m}(x) = (-1)^m J_m(x)$, the result becomes:

$$\frac{d^2 I}{d\omega d\Omega} =$$

$$\frac{q^2\omega^2\beta^2}{c}\sum_{m=0}^{\infty}\delta(\omega - m\frac{v}{\rho})\left\{\left[J_m'\left(\frac{\omega\rho}{c}\cos\theta\right)\right]^2 + \frac{\tan^2\theta}{v^2/c^2}\left[J_m\left(\frac{\omega\rho}{c}\cos\theta\right)\right]^2\right\}.$$

These results were derived by Julian Schwinger (Phys. Rev. **75**, 1912-1925 (1949)). The discrete case is similar to the result quoted in Problem 14.15 in Jackson's text.