PHY 712 Electrodynamics
10-10:50 AM MWF Olin 103

Discussion for Lecture 32:

Including topics from Chaps. 13 & 15 in JDJ

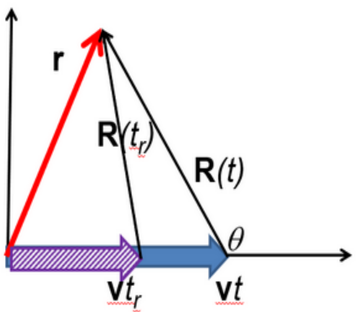
1. **Radiation from collisions of charged particles**
   a. **X-ray tube**
   b. **Radiation from Rutherford scattering**
2. **Cherenkov radiation**

24	Mon: 03/24/2025	Chap. 9	Radiation from time harmonic sources	#20	03/26/2025
25	Wed: 03/26/2025	Chap. 9 & 10	Radiation from scattering	#21	03/28/2025
26	Fri: 03/28/2025	Chap. 11	Special Theory of Relativity	#22	03/31/2025
27	Mon: 03/31/2025	Chap. 11	Special Theory of Relativity	#23	04/02/2025
28	Wed: 04/02/2025	Chap. 11	Special Theory of Relativity	#24	04/04/2025
29	Fri: 04/04/2024	Chap. 14	Radiation from accelerating charged particles	#25	04/07/2025
30	Mon: 04/07/2025	Chap. 14	Analysis of synchroton radiation	#26	04/09/2025
31	Wed: 04/09/2025	Chap. 14	Synchrotron radiation and Compton scattering	#27	04/11/2025
32	Fri: 04/11/2025	Chap. 13 & 15	Other radiation -- Cherenkov & bremsstrahlung	#28	04/14/2025
33	Mon: 04/14/2025		Special topic:E & M aspects of superconductivity		
34	Wed: 04/16/2025	Special Topics			
35	Fri: 04/18/2025		Presentations I		
	Mon: 04/21/2025	Special topics			
	Wed: 04/23/2025		Presentations II		
	Fri: 04/25/2025		Presentations III		
36	Mon: 04/28/2025		Review		

PHY 712 -- Assignment #28

Assigned: 4/11/2025 Due: 4/14/2025

1. In the context of analyzing Cherenkov radiation , we considered a particle of charge q moving at constant velocity **v** along the x axis producing electric and magnetic fields at time t at a position **r** in the x-y plane. The fields can be evaluated using the Liénard-Wiechert analysis given in Lecture 32 for example, evaluated at the retarded time t_r.



The analysis depends on the ratio of the speed of the particle v to the speed c_n of electromagnetic waves with the medium -- β_n. We showed the following relationship between the lengths $R(t)$, $R(t_r)$, and the angle θ with the x axis.

$$R(t_r) = \frac{R(t)}{\beta_n^2 - 1}\left(-\beta_n \cos\theta \pm \sqrt{1 - \beta_n^2 \sin^2\theta}\right)$$

a. First consider the case of the particle moving in vacuum so that $\beta_n < 1$. Determine the physical solutions in terms of choice of signs and range of angles θ for this case.
b. Now consider the case of the particle moving in a dielectric medium and moving fast enough so that $\beta_n > 1$. Determine the physical solutions in terms of choice of signs and range of angles θ for this case.

PHY 712 Presentation Schedule

Friday 4/18/2025

	Presenter Name	Topic
10:00-10:24	Edoardo Levati	Self-force
10:25-10:50	Pablo	Polarization in Kerr geometry

Wednesday 4/23/2025

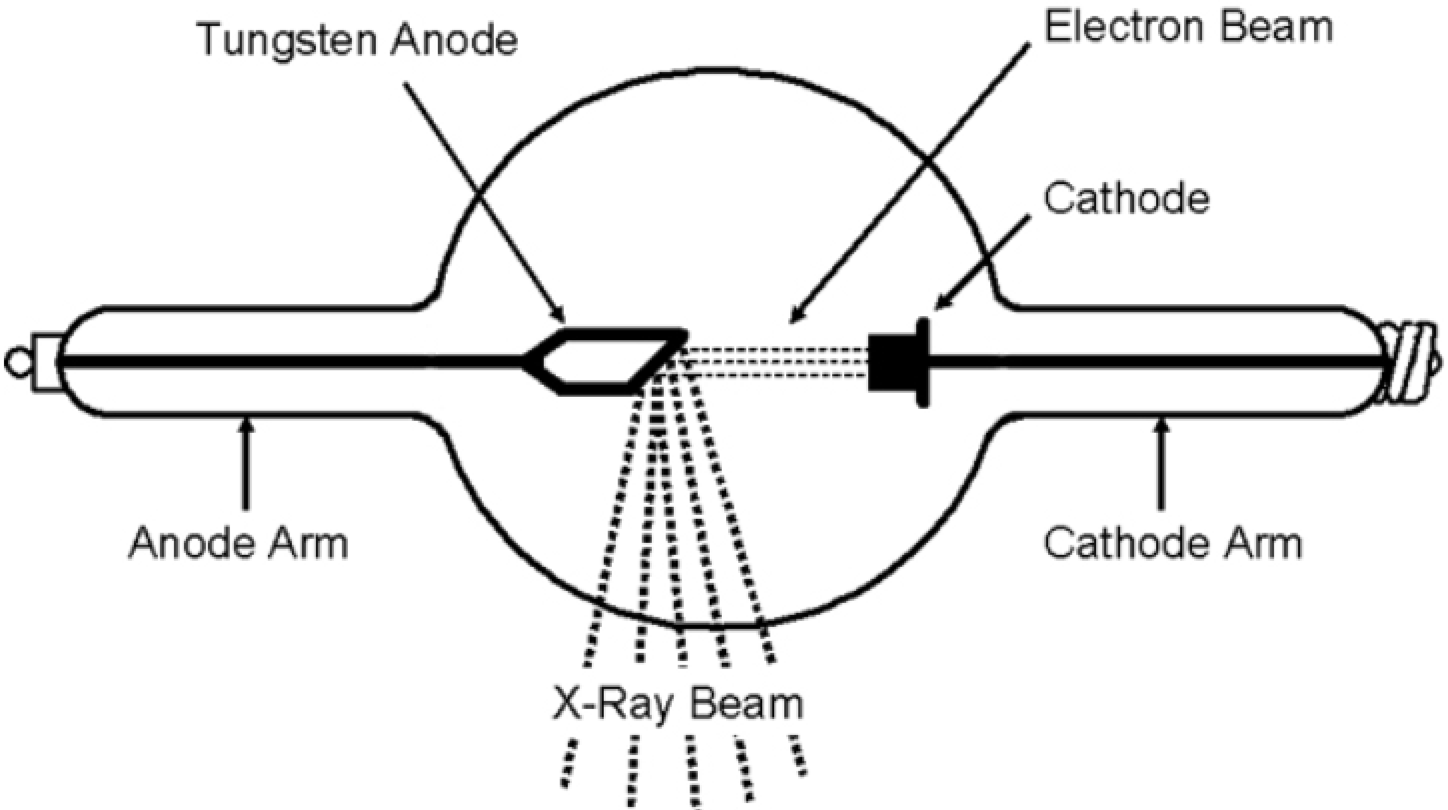
	Presenter Name	Topic
10:00-10:24	Thomas Myers	Ising Model
10:25-10:50	Conall O'Leary	

Friday 4/25/2025

	Presenter Name	Topic
10:00-10:24	Julia Radtke	
10:25-10:50	Bhargava Jogi R	

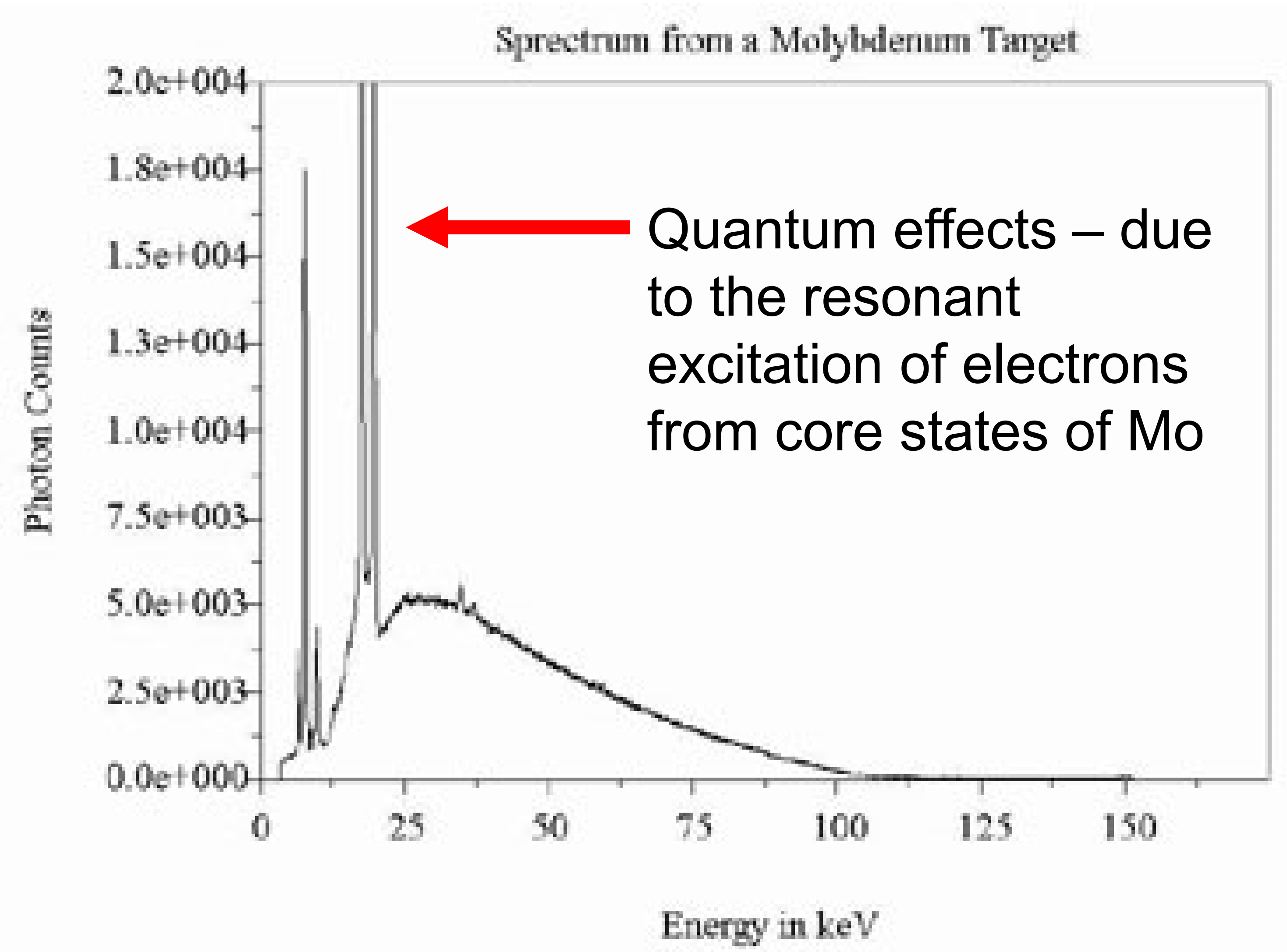
Generation of X-rays in a Coolidge tube

https://www.orau.org/ptp/collection/xraytubescoolidge/coolidgeinformation.htm



Invented in 1913. Associated with the German word “bremsstrahlung” – meaning breaking radiation.

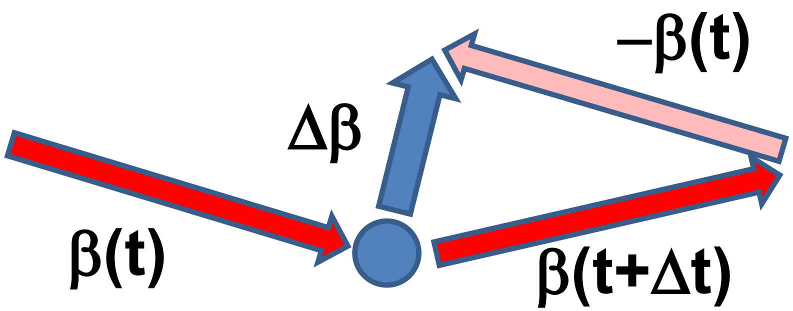
http://www.ndt-ed.org/EducationResources/CommunityCollege/Radiography/Physics/xrays.htm

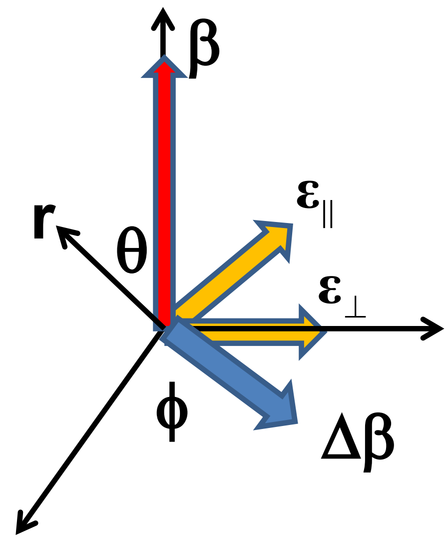


Quantum effects – due to the resonant excitation of electrons from core states of Mo

Radiation during collisions of charged particles

These ideas are presented in Chapter 15 of JDJ and were developed by many famous scientists during the early 1900's.





$\boldsymbol{\varepsilon}_{\parallel}$ is in the plane of $\boldsymbol{\beta}$ and $\mathbf{r}$

$\boldsymbol{\varepsilon}_{\perp}$ is perpendicular to the plane of $\boldsymbol{\beta}$ and $\mathbf{r}$

Results from previous analyses:

Spectral intensity of radiation from accelerating charged particle :

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int dt\ e^{i\omega\left(t-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c\right)} \frac{d}{dt}\left[\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\right)}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}}\right]\right|^2$$

Note that in the following slides we are taking the limit $\omega \rightarrow 0$ but keeping the notation of the differential intensity….

For a collision of duration τ emitting radiation with polarization $\boldsymbol{\varepsilon}$ and frequency $\omega \rightarrow 0$;

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\boldsymbol{\varepsilon}\cdot\left(\frac{\boldsymbol{\beta}(t+\tau)}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}(t+\tau)} - \frac{\boldsymbol{\beta}(t)}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}(t)}\right)\right|^2$$

Note that $\boldsymbol{\varepsilon}$ is perpendicular to **r**.

Radiation during collisions -- continued

For a collision of duration τ emitting radiation with polarization $\boldsymbol{\varepsilon}$ and frequency $\omega \to 0$:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\boldsymbol{\varepsilon}\cdot\left(\frac{\boldsymbol{\beta}(t+\tau)}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}(t+\tau)} - \frac{\boldsymbol{\beta}(t)}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}(t)}\right)\right|^2$$

We will evaluate this expression for two cases:

Non-relativistic limit:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\boldsymbol{\varepsilon}\cdot(\Delta\boldsymbol{\beta})\right|^2 \qquad \Delta\boldsymbol{\beta} \equiv \boldsymbol{\beta}(t+\tau) - \boldsymbol{\beta}(t)$$

Relativistic collision with small $\left|\Delta\boldsymbol{\beta}\right| \equiv \boldsymbol{\beta}(t+\tau) - \boldsymbol{\beta}(t)$:

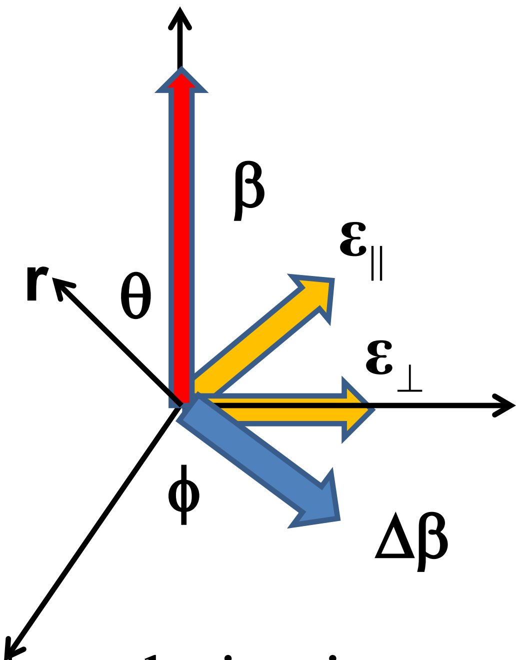
$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\boldsymbol{\varepsilon}\cdot\left(\frac{\Delta\boldsymbol{\beta} + \hat{\mathbf{r}}\times(\boldsymbol{\beta}\times\Delta\boldsymbol{\beta})}{(1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta})^2}\right)\right|^2$$

In the limit $\beta \rightarrow 0$, this is the same as the non-relativistic case.

Radiation during collisions -- continued

Relativistic collision with small $|\Delta\boldsymbol{\beta}|$:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\boldsymbol{\varepsilon}\cdot\left(\frac{\Delta\boldsymbol{\beta}+\hat{\mathbf{r}}\times(\boldsymbol{\beta}\times\Delta\boldsymbol{\beta})}{(1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta})^2}\right)\right|^2$$



Also assume $\Delta\boldsymbol{\beta}$ is perpendicular to $\boldsymbol{\beta}$ direction

Expressions (averaging over φ) for || or $\perp$ polarization:

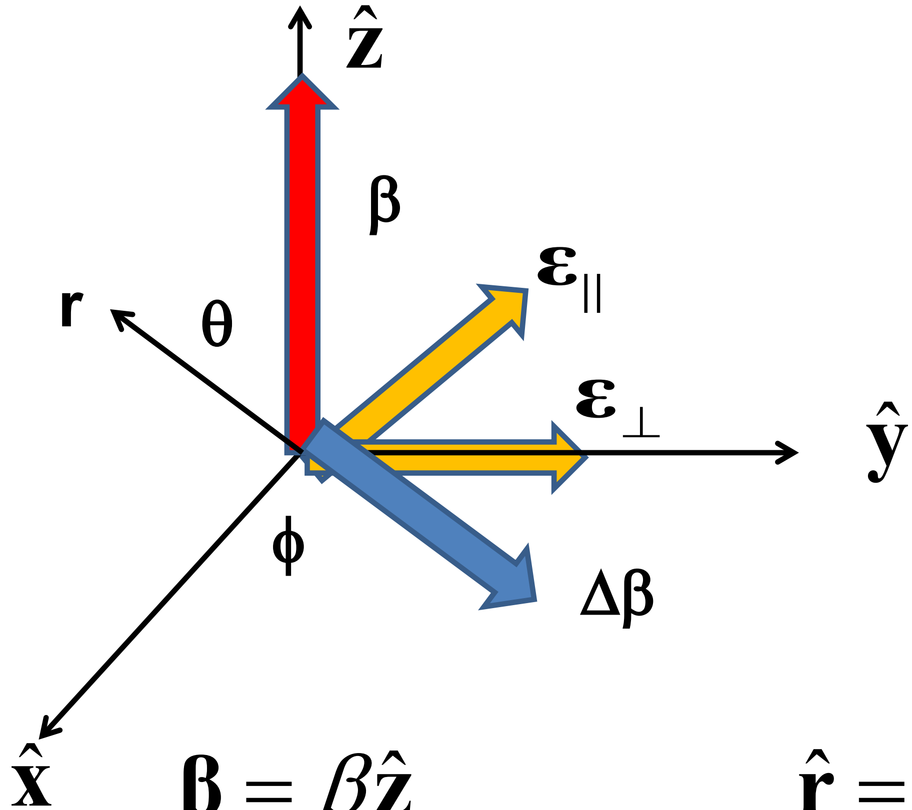
$$\frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c}|\Delta\boldsymbol{\beta}|^2 \frac{(\beta-\cos\theta)^2}{(1-\beta\cos\theta)^4}$$

polarization in ***r*** and ***β*** plane

$$\frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c}|\Delta\boldsymbol{\beta}|^2 \frac{1}{(1-\beta\cos\theta)^2}$$

polarization perpendicular to ***r*** and ***β*** plane

Some details:



(using geometry of Fig. 15.2 in Jackson)

$$\boldsymbol{\beta} = \beta\hat{\mathbf{z}} \qquad \hat{\mathbf{r}} = \sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}$$

$$\boldsymbol{\varepsilon}_{||} = -\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{z}} \qquad \boldsymbol{\varepsilon}_{\perp} = \hat{\mathbf{y}}$$

$$\boldsymbol{\Delta\beta} = \Delta\beta\left(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}\right)$$

Note: This is a wild assumption!

Some details -- continued:

$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}$$

$$\boldsymbol{\varepsilon}_{\perp} = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_{\|} = -\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{z}}$$

Consistent with radiation from charged particles.

$$\boldsymbol{\beta} = \beta\hat{\mathbf{z}}$$

Convenient geometry

$$\Delta\boldsymbol{\beta} = \Delta\beta\left(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}\right)$$

Wild guess

$$\Delta\boldsymbol{\beta} + \hat{\mathbf{r}} \times \left(\boldsymbol{\beta} \times \Delta\boldsymbol{\beta}\right) = \Delta\boldsymbol{\beta}\left(1 - \hat{\mathbf{r}} \cdot \boldsymbol{\beta}\right) + \boldsymbol{\beta}\left(\hat{\mathbf{r}} \cdot \Delta\boldsymbol{\beta}\right)$$

$$\boldsymbol{\varepsilon}_{\perp} \cdot \left(\Delta\boldsymbol{\beta} + \hat{\mathbf{r}} \times \left(\boldsymbol{\beta} \times \Delta\boldsymbol{\beta}\right)\right) = \Delta\beta\sin\phi\left(1 - \beta\cos\theta\right)$$

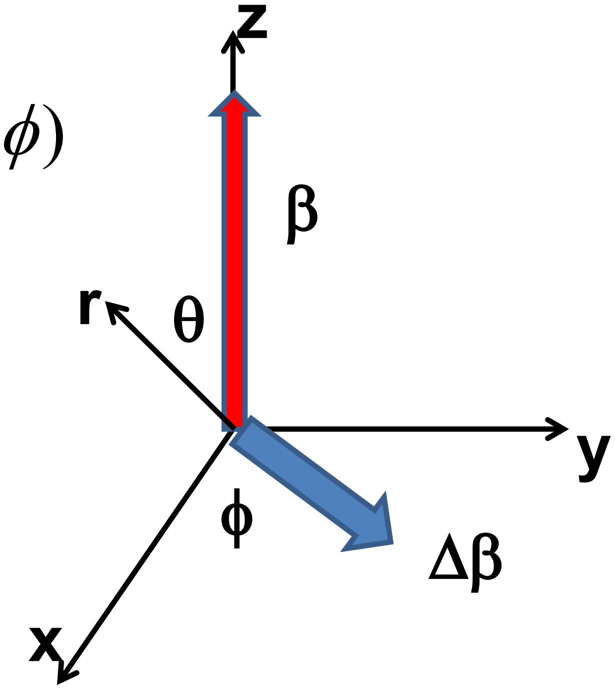
$$\boldsymbol{\varepsilon}_{\|} \cdot \left(\Delta\boldsymbol{\beta} + \hat{\mathbf{r}} \times \left(\boldsymbol{\beta} \times \Delta\boldsymbol{\beta}\right)\right) = \Delta\beta\cos\phi\left(\beta - \cos\theta\right)$$

Radiation during collisions -- continued

Intensity expressions: (averaging over ϕ)

$$\frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c}\left|\Delta\boldsymbol{\beta}\right|^2 \frac{\left(\beta - \cos\theta\right)^2}{\left(1-\beta\cos\theta\right)^4}$$

$$\frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c}\left|\Delta\boldsymbol{\beta}\right|^2 \frac{1}{\left(1-\beta\cos\theta\right)^2}$$



Relativistic collision at low ω and with small $\left|\Delta\boldsymbol{\beta}\right|$ and $\Delta\boldsymbol{\beta}$ perpendicular to plane of $\hat{\mathbf{r}}$ and $\boldsymbol{\beta}$, as a function of θ where $\hat{\mathbf{r}}\cdot\boldsymbol{\beta} = \beta\cos\theta$;

Integrating over solid angle:

$$\frac{dI}{d\omega} = \int d\Omega\left(\frac{d^2 I_{\parallel}}{d\omega d\Omega} + \frac{d^2 I_{\perp}}{d\omega d\Omega}\right) = \frac{2}{3\pi}\frac{q^2}{c}\gamma^2 \left|\Delta\boldsymbol{\beta}\right|^2$$

Some more details:

$$\int d\Omega \frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c} |\Delta\boldsymbol{\beta}|^2 \, 2\pi \int_{-1}^{1} d\cos\theta \frac{(\beta - \cos\theta)^2}{(1-\beta\cos\theta)^4}$$

$$= \frac{q^2}{4\pi c} |\Delta\boldsymbol{\beta}|^2 \frac{2}{3} \frac{1}{(1-\beta^2)}$$

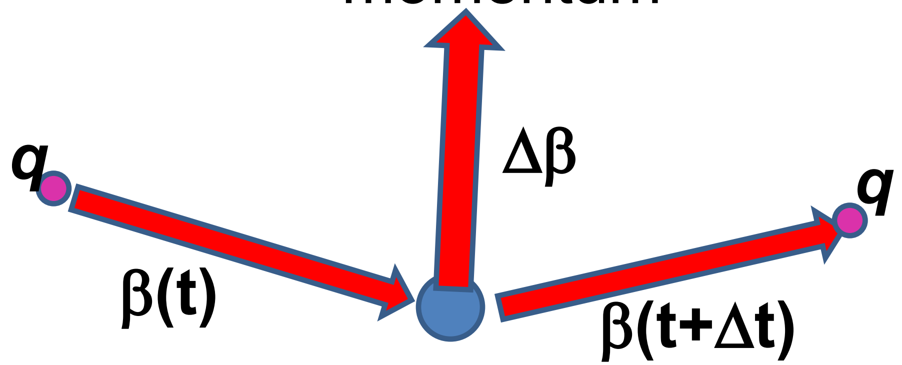
$$\int d\Omega \frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c} |\Delta\boldsymbol{\beta}|^2 \int_{-1}^{1} d\cos\theta \frac{1}{(1-\beta\cos\theta)^2}$$

$$= \frac{q^2}{4\pi c} |\Delta\boldsymbol{\beta}|^2 \frac{2}{(1-\beta^2)}$$

$$\frac{dI}{d\omega} = \int d\Omega \left(\frac{d^2 I_{\parallel}}{d\omega d\Omega} + \frac{d^2 I_{\perp}}{d\omega d\Omega} \right) = \frac{2}{3\pi} \frac{q^2}{c} \gamma^2 \; |\Delta\boldsymbol{\beta}|^2$$

Estimation of $\Delta\beta$

Need to consider the mechanics of collision; it is convenient to parameterize in terms of momentum --



Momentum transfer:

$$Qc \equiv \left|\mathbf{p}(t+\tau) - \mathbf{p}(t)\right| c \approx \gamma M c^2 \left|\Delta\boldsymbol{\beta}\right|$$

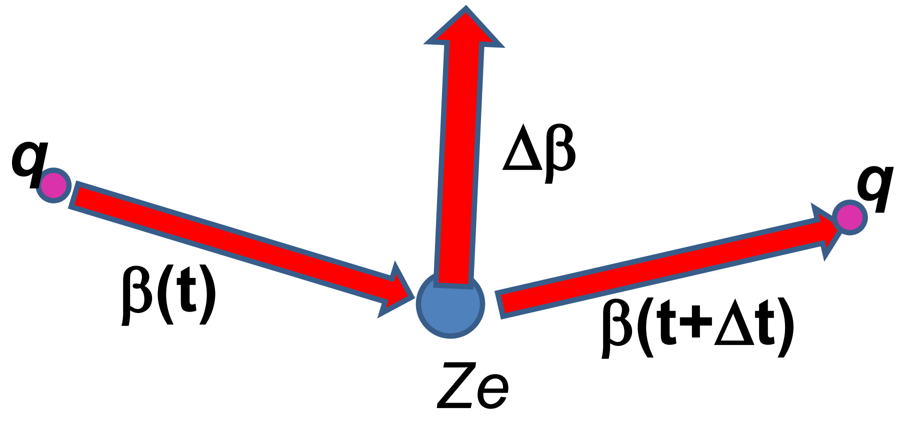
mass of particle having charge *q*

$$\frac{dI}{d\omega} = \frac{2}{3\pi} \; \frac{q^2}{c} \gamma^2 \left|\Delta\boldsymbol{\beta}\right|^2 \approx \frac{2}{3\pi} \; \frac{q^2}{M^2 c^3} Q^2$$

What are the conditions for the validity of this result?

What are possible mechanisms for the momentum transfer Q?

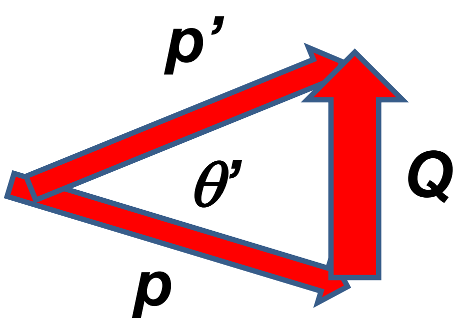
Estimation of $\Delta\beta$ or Q -- for the case of Rutherford scattering



Assume that target nucleus (charge Ze) has mass >>M;

Rutherford scattering cross-section in center of mass analysis:

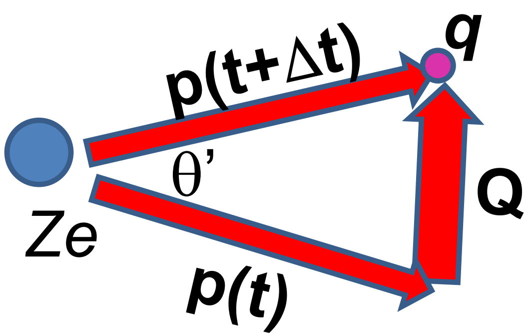
$$\frac{d\sigma}{d\Omega} = \left(\frac{2Zeq}{pv}\right)^2 \frac{1}{\left(2\sin\left(\theta'/2\right)\right)^4}$$



Assuming elastic scattering:

$$Q^2 = \left(2p\sin\left(\theta'/2\right)\right)^2 = 2p^2\left(1-\cos\theta'\right)$$

Case of Rutherford scattering -- continued



Rutherford scattering cross-section:

$$\frac{d\sigma}{d\Omega} = \left(\frac{2Zeq}{pv}\right)^2 \frac{1}{\left(2\sin\left(\theta'/2\right)\right)^4}$$

$$\frac{d\sigma}{dQ} = \int_{\varphi'} \frac{d\sigma}{d\Omega}\left|\frac{d\Omega}{dQ}\right| d\varphi'$$

$$d\Omega = d\varphi' d\cos\theta'$$

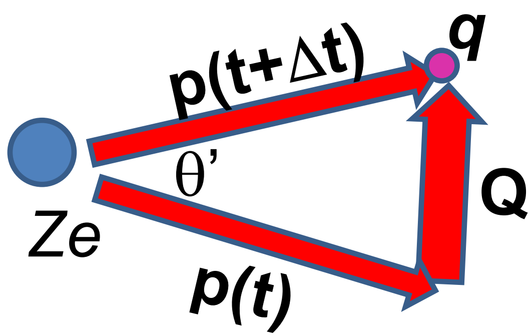
$$Q^2 = \left(2p\sin\left(\theta'/2\right)\right)^2 = 2p^2\left(1-\cos\theta'\right)$$

$$dQ = -\frac{p^2}{Q} d\cos\theta'$$

$$\Rightarrow \quad \frac{d\sigma}{dQ} = 8\pi\left(\frac{Zeq}{\beta c}\right)^2 \frac{1}{Q^3}$$

Does the algebra work out?

Case of Rutherford scattering -- continued



Differential radiation cross section :

$$\frac{d^2\chi}{d\omega dQ} = \frac{dI}{d\omega}\frac{d\sigma}{dQ} = \left(\frac{2}{3\pi}\frac{q^2}{M^2c^3}Q^2\right)\left(8\pi\left(\frac{Zeq}{\beta c}\right)^2\frac{1}{Q^3}\right)$$

$$= \frac{16}{3}\frac{(Ze)^2}{c}\left(\frac{q^2}{Mc^2}\right)^2\frac{1}{\beta^2}\frac{1}{Q}$$

Differential radiation cross section -- continued

Integrating over momentum transfer

$$\frac{d\chi}{d\omega} = \int_{Q_{\min}}^{Q_{\max}} dQ \frac{d^2\chi}{d\omega dQ} = \frac{16}{3}\frac{(Ze)^2}{c}\left(\frac{q^2}{Mc^2}\right)^2 \frac{1}{\beta^2} \ln\left(\frac{Q_{\max}}{Q_{\min}}\right)$$

How do the limits of Q occur?

Jackson suggests that these come from the limits of validity of the analysis.

1. Seems like cheating?
2. Perhaps fair?

Comment on frequency dependence --

Original expression for radiation intensity:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \int dt\; e^{i\omega\left(t - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c\right)} \frac{d}{dt}\left[\frac{\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \boldsymbol{\beta}\right)}{1 - \hat{\mathbf{r}}\cdot\boldsymbol{\beta}} \right] \right|^2$$

In the previous derivations, we have assumed that

$$\omega\left(t - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c\right) << 1.$$

$$\omega\left(t - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c\right) = \omega\left(t - \hat{\mathbf{r}}\cdot\int_0^t dt'\boldsymbol{\beta}(t') \right) \approx \omega\tau\left(1 - \hat{\mathbf{r}}\cdot\langle\boldsymbol{\beta}\rangle\right)$$

In the non-relativistic case, this means $\omega\tau \ll 1.$

Here τ is the effective collision time.

How to estimate the collision time?

Jackson uses the following analysis in terms of the impact parameter *b:*

Using classical mechanics and assuming $v \ll c$:

$$\tau \approx \frac{b}{v} \ll \frac{1}{\omega} \quad \text{and} \quad Q \approx \frac{2Zeq}{bv}$$

$$\text{Assume that } Q_{\min} = \frac{2Zeq}{b_{\max}v} = \frac{2Zeq\omega}{v^2}$$

Differential radiation cross section -- continued

Radiation cross section in terms of momentum transfer

$$\frac{d\chi}{d\omega} = \int_{Q_{\min}}^{Q_{\max}} dQ \frac{d^2\chi}{d\omega dQ} = \frac{16}{3}\frac{(Ze)^2}{c}\left(\frac{q^2}{Mc^2}\right)^2 \frac{1}{\beta^2} \ln\left(\frac{Q_{\max}}{Q_{\min}}\right)$$

Note that: $Q^2 = 2p^2(1-\cos\theta') \quad \Rightarrow Q_{\max} = 2p$

In general, $Q_{\min}$ is determined by the collision time

condition $\omega\tau < 1 \quad \Rightarrow Q_{\min} \approx \frac{2Zeq\omega}{v^2}$

Radiation cross section for classical non - relativistic process

$$\frac{d\chi}{d\omega} = \frac{16}{3}\frac{(Ze)^2}{c}\left(\frac{q^2}{Mc^2}\right)^2 \frac{1}{\beta^2} \ln\left(\frac{\lambda M v^3}{Zeq\omega}\right)$$

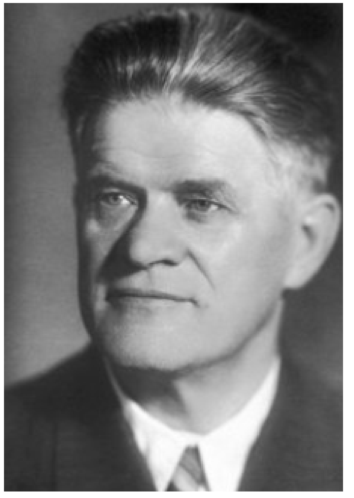
λ= “fudge factor” of order unity

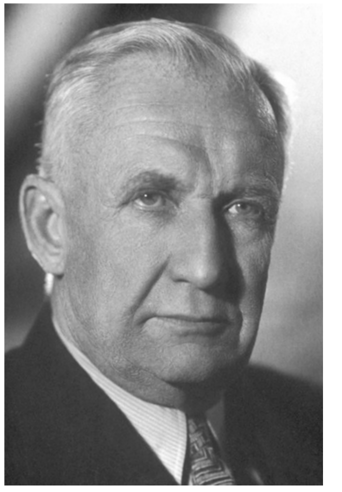
Cherenkov radiation

Cherenkov radiation emitted by the core of the Reed Research Reactor located at Reed College in Portland, Oregon, U.S. *Cherenkov radiation*. Photograph. *Encyclopædia Britannica Online*. Web. 12 Apr. 2013.
http://www.britannica.com/EBchecked/media/174732

The Nobel Prize in Physics 1958

Pavel A. Cherenkov
Il´ja M. Frank
Igor Y. Tamm





Affiliation at the time of the award: P.N. Lebedev Physical Institute, Moscow, USSR

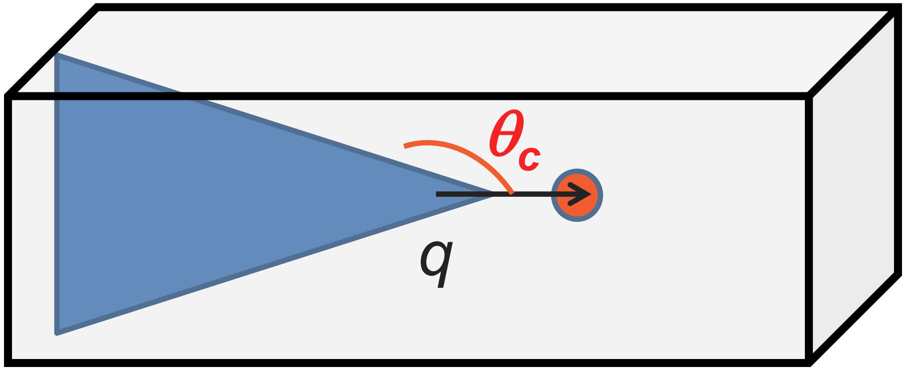
Prize motivation: "for the discovery and the interpretation of the Cherenkov effect."

https://www.nobelprize.org/prizes/physics/1958/ceremony-speech/

References for notes: Glenn S. Smith, *An Introduction to Electromagnetic Radiation* (Cambridge UP, 1997), Andrew Zangwill, Modern Electrodynamics (Cambridge UP, 2013)

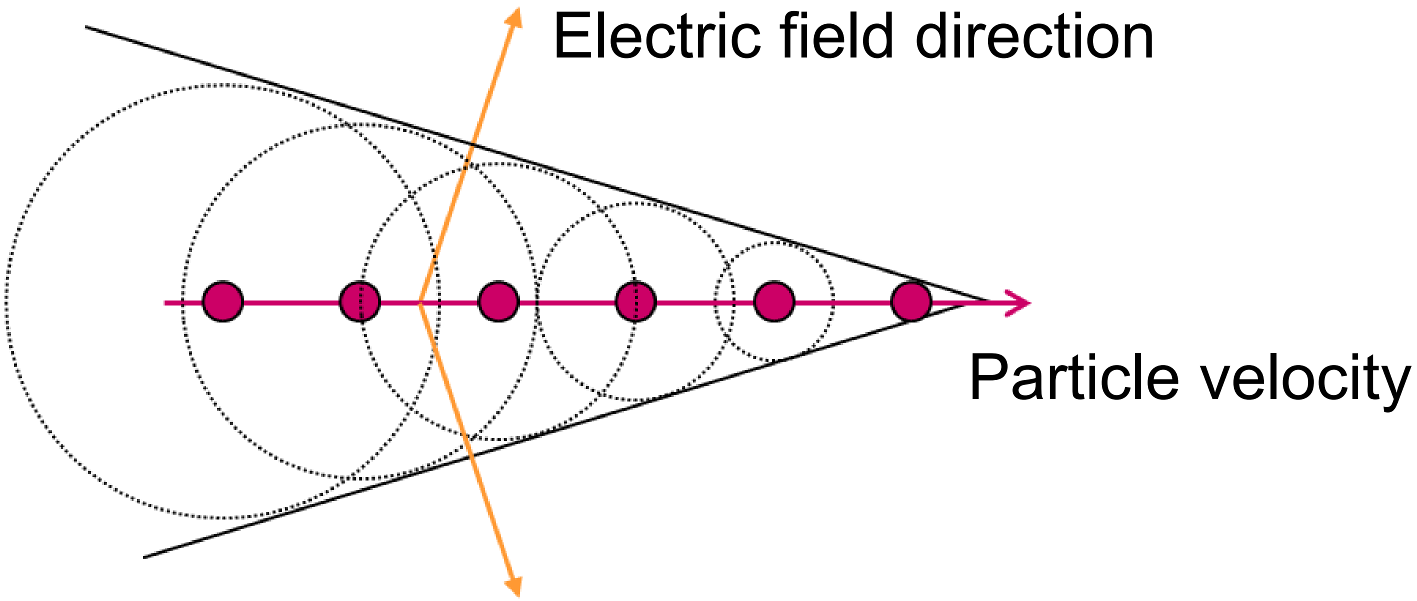
Cherenkov radiation

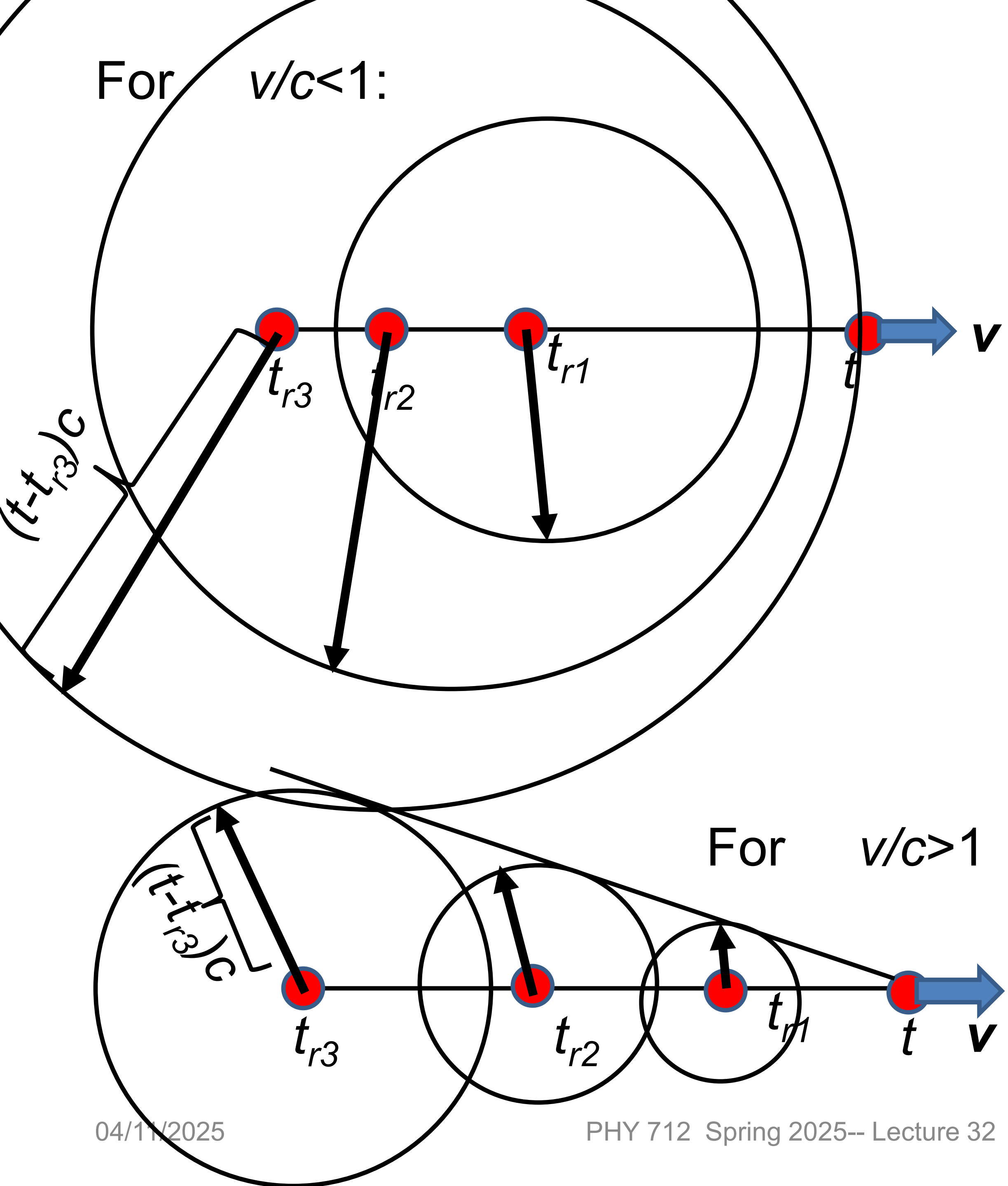
Discovered ~1930; bluish light emitted by energetic charged particles traveling within dielectric materials



Note that some treatments give a different definition of the critical angle θ_c

From: http://large.stanford.edu/courses/2014/ph241/alaeian2/




For $v/c<1$:
t_{r3}
t_{r2}
t_{r1}
t
$\mathbf{v}$
$(t-t_{r3})c$
For $v/c>1$
t_{r3}
t_{r2}
t_{r1}
t
$\mathbf{v}$
$(t-t_{r3})c$

Maxwell's potential equations within a material having permittivity and permeability (Lorentz gauge; cgs Gaussian units)

$$\nabla^2\Phi - \mu\varepsilon\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{4\pi}{\varepsilon}\rho$$

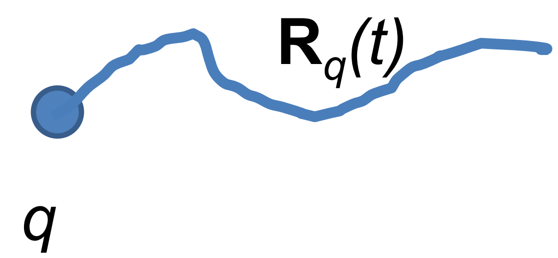
$$\nabla^2\mathbf{A} - \mu\varepsilon\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi\mu}{c}\mathbf{J}$$

Here the values of μ and ε depend on the material and on frequency.

Source: charged particle moving on trajectory $\mathbf{R}_q(t)$:

$$\rho(\mathbf{r},t) = q\delta\left(\mathbf{r}-\mathbf{R}_q(t)\right)$$

$$\mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta\left(\mathbf{r}-\mathbf{R}_q(t)\right)$$



Liénard-Wiechert potential solutions for charged particle moving within a material with refractive index *n*:

$$\Phi(\mathbf{r},t) = \frac{q}{\varepsilon} \frac{1}{\left| R(t_r) - \boldsymbol{\beta}_n \cdot \mathbf{R}(t_r) \right|}$$

$$\mathbf{A}(\mathbf{r},t) = q\mu \frac{\boldsymbol{\beta}_n}{\left| R(t_r) - \boldsymbol{\beta}_n \cdot \mathbf{R}(t_r) \right|}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r)$$

$$\boldsymbol{\beta}_n(t_r) \equiv \frac{\dot{\mathbf{R}}_q(t_r)}{c_n} \qquad c_n \equiv \frac{c}{\sqrt{\mu\varepsilon}} \equiv \frac{c}{n}$$

$$t_r = t - \frac{R(t_r)}{c_n}$$

Example --

$$\beta_n \equiv \frac{v}{c_n} \qquad c_n \equiv \frac{c}{\sqrt{\mu\varepsilon}} \equiv \frac{c}{n} \qquad \beta_n \equiv \frac{vn}{c}$$

Consider water with $n \approx 1.3$

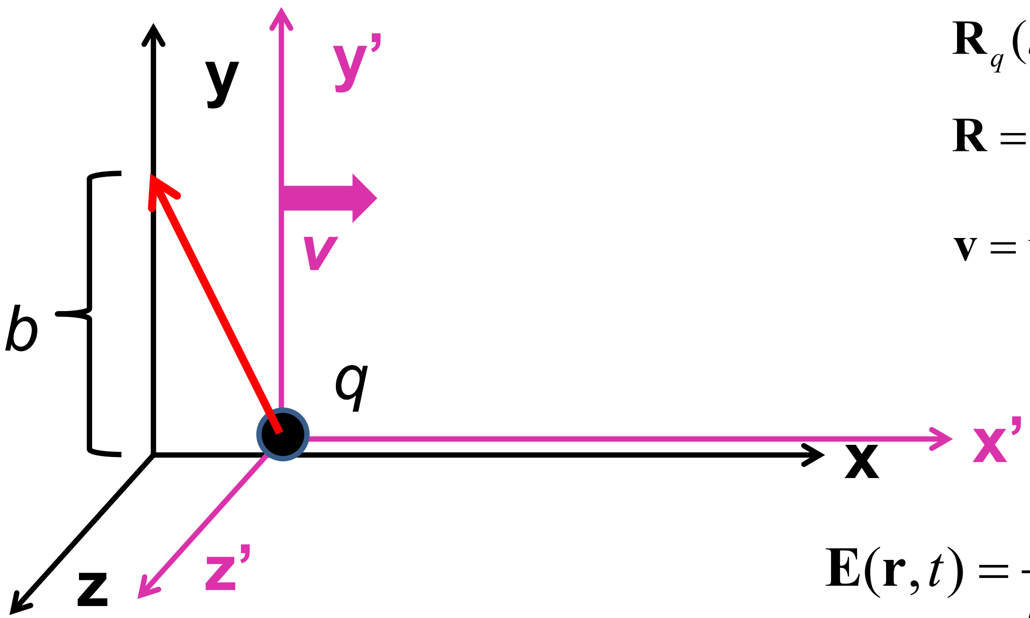
Which of these particles could produce Cherenkov radiation?

1. A neutron with speed c?
2. An electron with speed 0.6c?
3. A proton with speed 0.6c?
4. An electron with speed 0.8c?
5. An alpha particle with speed 0.8c?
6. None of these?

Further comment –

As discussed particularly in Chap. 13 of Jackson, a particle moving within a medium is likely to be slowed down so that the Cherenkov effect will only happen while $\beta_n>1$.

Recall – in Lecture 29, we considered a particle moving at constant velocity v in vacuum:



$$\mathbf{R}_q(t_r) = vt_r\hat{\mathbf{x}} \qquad \mathbf{r} = b\hat{\mathbf{y}}$$

$$\mathbf{R} = b\hat{\mathbf{y}} - vt_r\hat{\mathbf{x}} \qquad R = \sqrt{v^2t_r^2 + b^2}$$

$$\mathbf{v} = v\hat{\mathbf{x}} \qquad t_r = t - \frac{R}{c}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right)\right]$$

Electric and magnetic fields produced →

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2}\right)\right]$$

Some details for vacuum case --

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2}\right)\right]$$

For our example:

$$\mathbf{R}_q(t_r) = vt_r\hat{\mathbf{x}} \qquad \mathbf{r} = b\hat{\mathbf{y}}$$

$$\mathbf{R} = b\hat{\mathbf{y}} - vt_r\hat{\mathbf{x}} \qquad R = \sqrt{v^2t_r^2 + b^2}$$

$$\mathbf{v} = v\hat{\mathbf{x}} \qquad t_r = t - \frac{R}{c}$$

t_r must be a solution to a quadratic equation: where $\frac{v}{c} \leq 1; \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

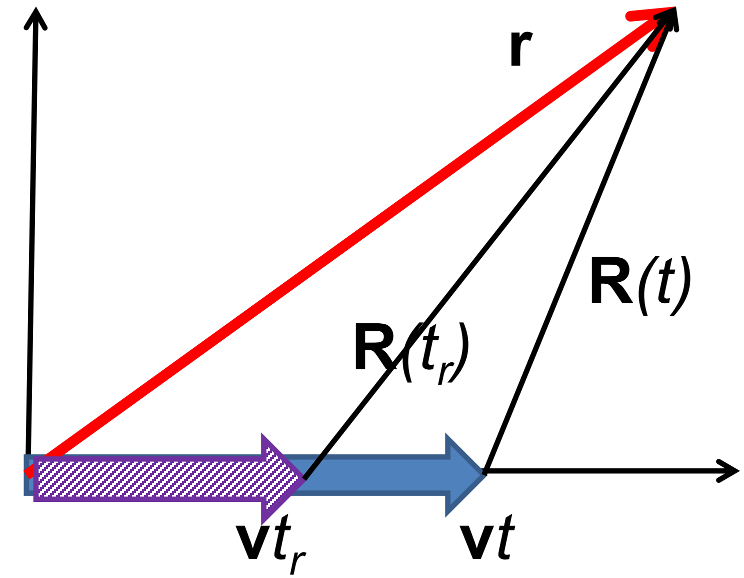
$$t_r - t = -\frac{R}{c} \quad \Rightarrow \quad t_r^2 - 2\gamma^2 t t_r + \gamma^2 t^2 - \gamma^2 b^2 / c^2 = 0$$

with the physical solution:

$$t_r = \gamma\left(\gamma t - \sqrt{(\gamma^2 - 1)t^2 + b^2 / c^2}\right) = \gamma\left(\gamma t - \frac{\sqrt{(v\gamma t)^2 + b^2}}{c}\right)$$

For Cherenkov case --

Consider a particle moving at constant velocity **v**; $v > c_n$



Some algebra

$$\mathbf{R}(t) = \mathbf{r} - \mathbf{v}t$$

$$\mathbf{R}(t_r) = \mathbf{r} - \mathbf{v}t_r = \mathbf{R}(t) + \mathbf{v}(t - t_r)$$

$$(t - t_r)c_n = R(t_r) = \left|\mathbf{R}(t) + \mathbf{v}(t - t_r)\right|$$

Quadratic equation for $(t - t_r)c_n$:

$$\left((t - t_r)c_n\right)^2 = R^2(t) + 2\mathbf{R}(t)\cdot\boldsymbol{\beta}_n(t - t_r)c_n + \beta_n^{\;2}\left((t - t_r)c_n\right)^2$$

$$\left(\beta_n^{\;2} - 1\right)\left((t - t_r)c_n\right)^2 + 2\mathbf{R}(t)\cdot\boldsymbol{\beta}_n(t - t_r)c_n + R^2(t) = 0$$

Quadratic equation for $(t-t_r)c_n \equiv R(t_r)$:

$$\left(\beta_n{}^2-1\right)\left((t-t_r)c_n\right)^2+2\mathbf{R}(t)\cdot\boldsymbol{\beta}_n(t-t_r)c_n+R^2(t)=0$$

For $\beta_n>1$, how can the equality be satisfied?

1. No problem
2. It cannot be satisfied.
3. It can only be satisfied for special conditions

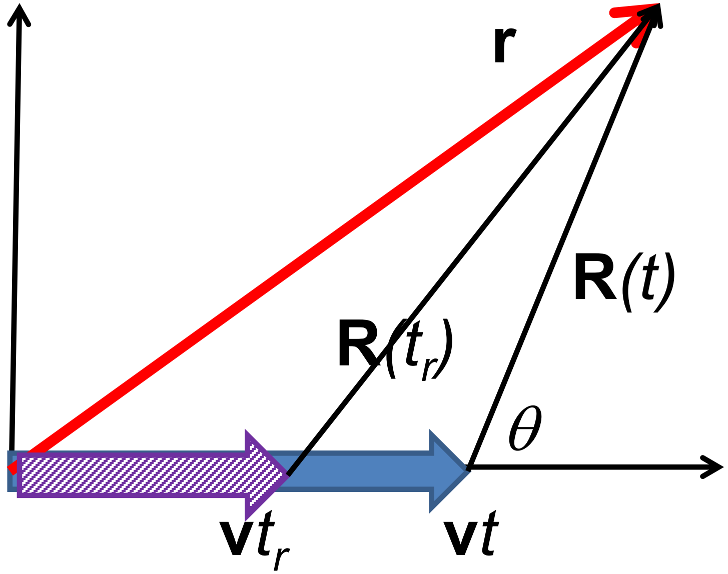
From solution of quadratic equation:

$$(t-t_r)c_n = R(t_r) = \frac{-\mathbf{R}(t)\cdot\boldsymbol{\beta}_n \pm \sqrt{\left(\mathbf{R}(t)\cdot\boldsymbol{\beta}_n\right)^2-\left(\beta_n{}^2-1\right)R^2(t)}}{\beta_n{}^2-1}$$

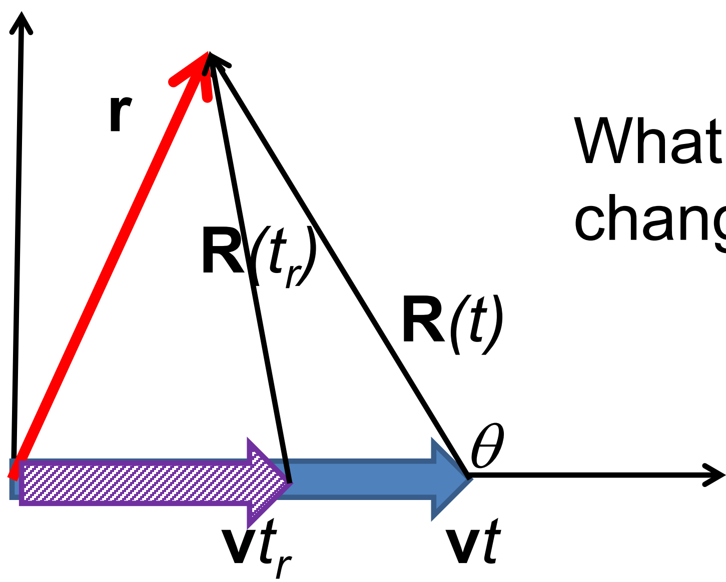
$\Rightarrow \mathbf{R}(t)\cdot\boldsymbol{\beta}_n<0$ (initial diagram is incorrect!)

Moreover, there are two retarded time solutions!

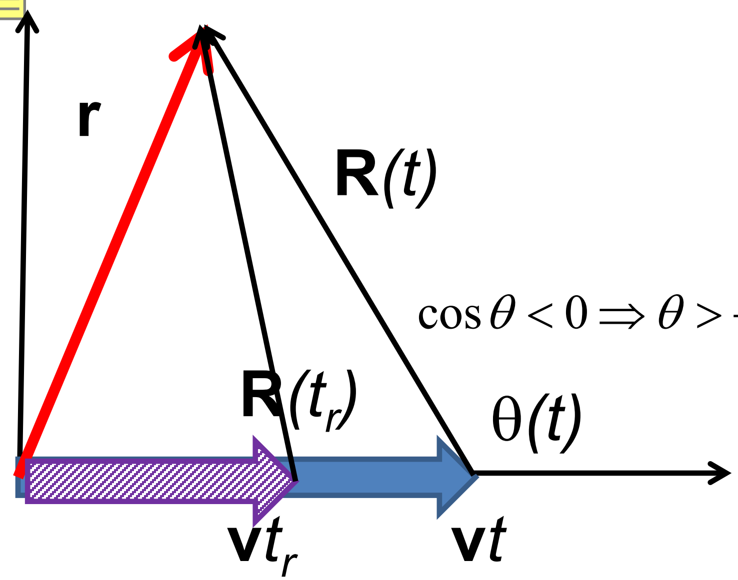
Original diagram:



New diagram:



What is the significance of changing the diagram?



$$\cos\theta < 0 \Rightarrow \theta > \frac{\pi}{2}$$

$$\cos\theta \equiv \hat{\mathbf{R}}(t)\cdot\hat{\mathbf{v}}$$

$$\mathbf{R}(t_r) = \mathbf{r} - \mathbf{v}t_r = \mathbf{R}(t) + \mathbf{v}(t - t_r)$$

$$(t - t_r)c_n = R(t_r)$$

$$R(t_r) - \mathbf{R}(t_r)\cdot\boldsymbol{\beta}_n =$$

$$(t - t_r)c_n\left(1 - \beta_n^2\right) - \mathbf{R}(t)\cdot\boldsymbol{\beta}_n$$

$$= R(t_r)\left(1 - \beta_n^2\right) - \mathbf{R}(t)\cdot\boldsymbol{\beta}_n$$

$$R(t_r) = \frac{-\mathbf{R}(t)\cdot\boldsymbol{\beta}_n \pm \sqrt{\left(\mathbf{R}(t)\cdot\boldsymbol{\beta}_n\right)^2 - \left(\beta_n^{\,2} - 1\right)R^2(t)}}{\beta_n^{\,2} - 1}$$

$$R(t_r) = \frac{R(t)}{\beta_n^{\,2} - 1}\left(-\beta_n\cos\theta \pm \sqrt{1 - \beta_n^{\,2}\sin^2\theta}\right) = (t - t_r)c_n$$

$$R(t_r) - \mathbf{R}(t_r)\cdot\boldsymbol{\beta}_n = \mp R(t)\sqrt{1 - \beta_n^{\,2}\sin^2\theta}$$

Recall the Liénard-Wiechert potential solutions:

$$\Phi(\mathbf{r},t) = \frac{q}{\varepsilon} \frac{1}{\left| R(t_r) - \boldsymbol{\beta}_n \cdot \mathbf{R}(t_r) \right|}$$

$$\mathbf{A}(\mathbf{r},t) = q\mu \frac{\boldsymbol{\beta}_n}{\left| R(t_r) - \boldsymbol{\beta}_n \cdot \mathbf{R}(t_r) \right|}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r)$$

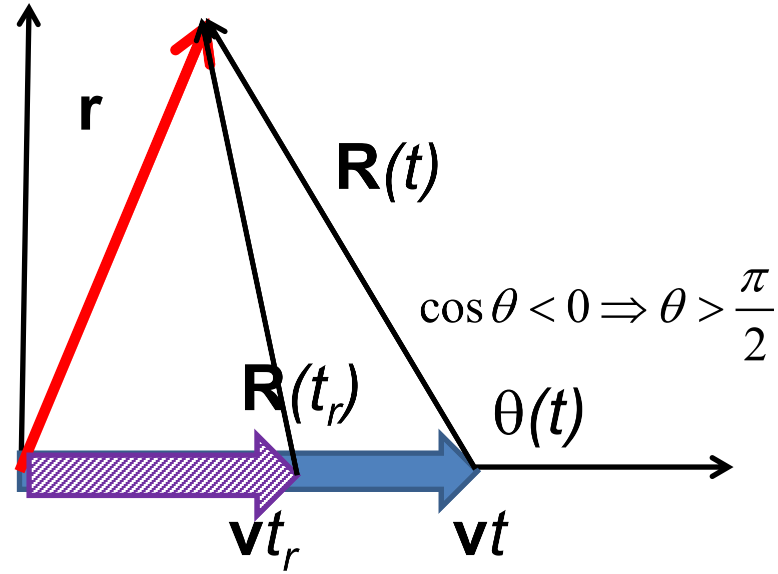
$$\boldsymbol{\beta}_n(t_r) \equiv \frac{\dot{\mathbf{R}}_q(t_r)}{c_n} \qquad c_n \equiv \frac{c}{\sqrt{\mu\varepsilon}} \equiv \frac{c}{n}$$

$$t_r = t - \frac{R(t_r)}{c_n}$$

Liénard-Wiechert potentials for two solutions:

$$\Phi(\mathbf{r},t) = \frac{q}{\varepsilon}\frac{1}{\left|\mp R(t)\sqrt{1-\beta_n{}^2\sin^2\theta}\right|}$$

$$\mathbf{A}(\mathbf{r},t) = q\mu\frac{\boldsymbol{\beta}_n}{\left|\mp R(t)\sqrt{1-\beta_n{}^2\sin^2\theta}\right|}$$



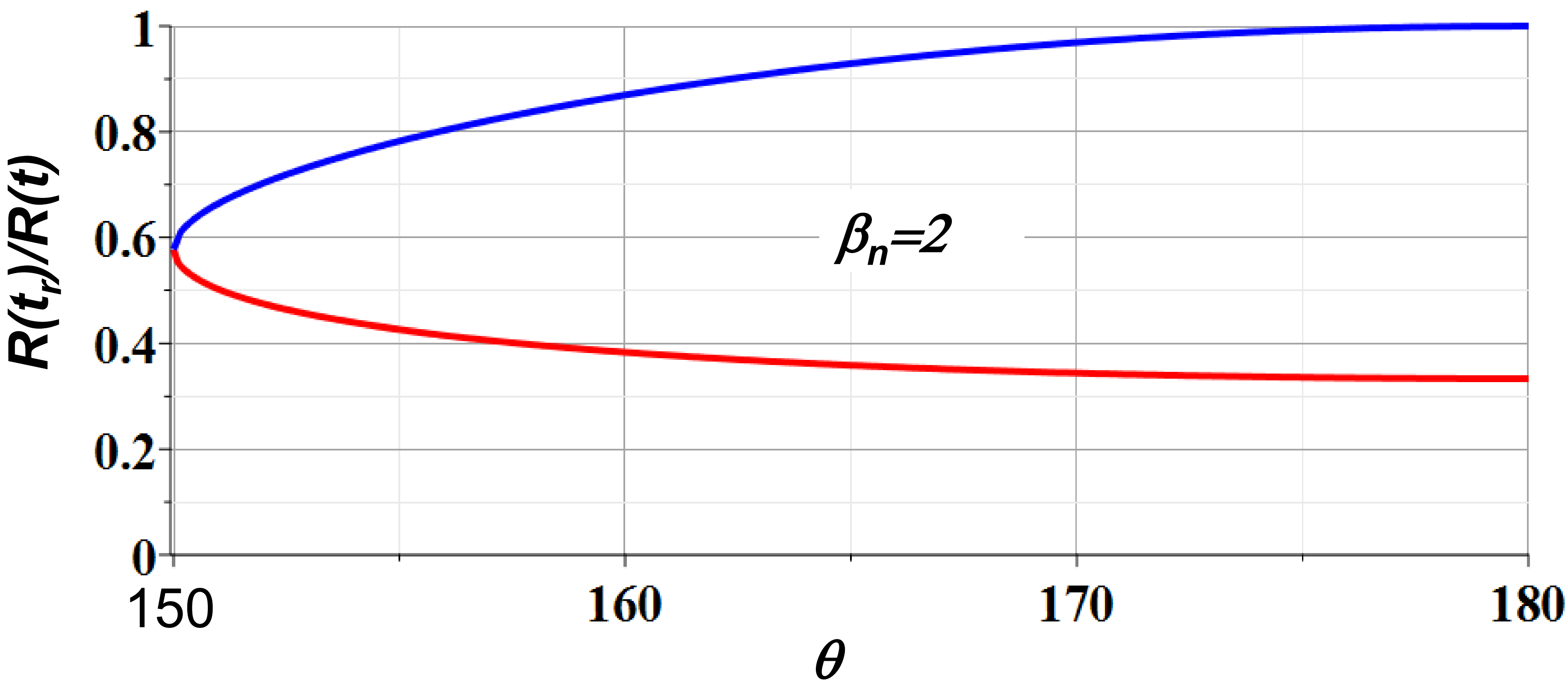
For $\beta_n > 1$, the range of θ is limited further:

$$R(t_r) = \frac{R(t)}{\beta_n{}^2 - 1}\left(-\beta_n\cos\theta \pm \sqrt{1-\beta_n{}^2\sin^2\theta}\right) \geq 0$$

$$\Rightarrow \left|\sin\theta\right| \leq \frac{1}{\beta_n} \equiv \left|\sin\theta_c\right| \quad \text{and} \quad \pi \geq \theta_c \geq \pi/2 \qquad \cos\ \theta_c = -\sqrt{1-\frac{1}{\beta_n^2}}$$

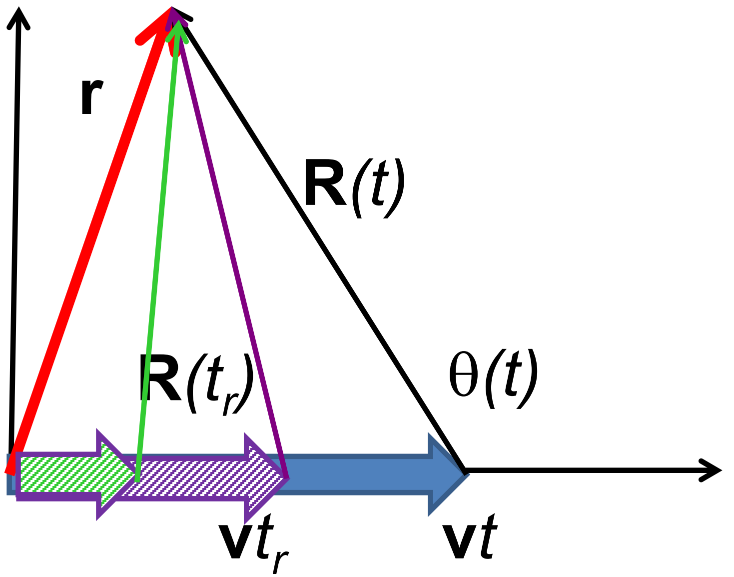
In this range, $\theta \geq \theta_c$

$$R(t_r) = \frac{R(t)}{\beta_n{}^2 - 1}\left(-\beta_n \cos\theta \pm \sqrt{1 - \beta_n{}^2 \sin^2\theta}\right)$$



$$\theta_c = 150^o \text{ for this case}$$

Physical fields for $\beta_n > 1$ -- two retarded solutions contribute



$$\theta \leq \sin^{-1}\left(\frac{1}{\beta_n}\right)$$

$$\text{Define} \quad \cos\theta_C \equiv -\sqrt{1-\frac{1}{\beta_n^{\ 2}}}$$

$$\Rightarrow \cos\theta \leq \cos\theta_C$$

Adding two solutions; in terms of Heaviside $\Theta(x)$:

$$\Phi(\mathbf{r},t) = \frac{2q}{\varepsilon}\frac{1}{R(t)\sqrt{1-\beta_n^{\ 2}\sin^2\theta}}\Theta(\cos\theta_C - \cos\theta(t))$$

$$\mathbf{A}(\mathbf{r},t) = 2q\mu\frac{\boldsymbol{\beta}_n}{R(t)\sqrt{1-\beta_n^{\ 2}\sin^2\theta}}\Theta(\cos\theta_C - \cos\theta(t))$$

Physical fields for $\beta_n > 1$

$$\Phi(\mathbf{r},t) = \frac{2q}{\varepsilon}\frac{1}{R(t)\sqrt{1-\beta_n{}^2\sin^2\theta}}\Theta(\cos\theta_C - \cos\theta(t))$$

$$\mathbf{A}(\mathbf{r},t) = 2q\mu\frac{\boldsymbol{\beta}_n}{R(t)\sqrt{1-\beta_n{}^2\sin^2\theta}}\Theta(\cos\theta_C - \cos\theta(t))$$

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi - \frac{1}{c_n}\frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{2q}{\varepsilon}\frac{\hat{\mathbf{R}}}{(R(t))^2\sqrt{1-\beta_n{}^2\sin^2\theta}} \times$$

$$\left(-\frac{\beta_n{}^2-1}{1-\beta_n{}^2\sin^2\theta}\Theta(\cos\theta_C - \cos\theta(t)) + \sqrt{\beta_n{}^2-1}\;\delta(\cos\theta_C - \cos\theta(t))\right)$$

$$\mathbf{B}(\mathbf{r},t) = -\beta_n\sin\theta\left(\hat{\theta}\times\mathbf{E}(\mathbf{r},t)\right)$$

From these results, we need to generate the power spectrum – following the approach in Sec. 23.7 in Zangwill's textbook.

When the dust clears, it can be shown the Cherenkov intensity per unit path length, per frequency is given by --

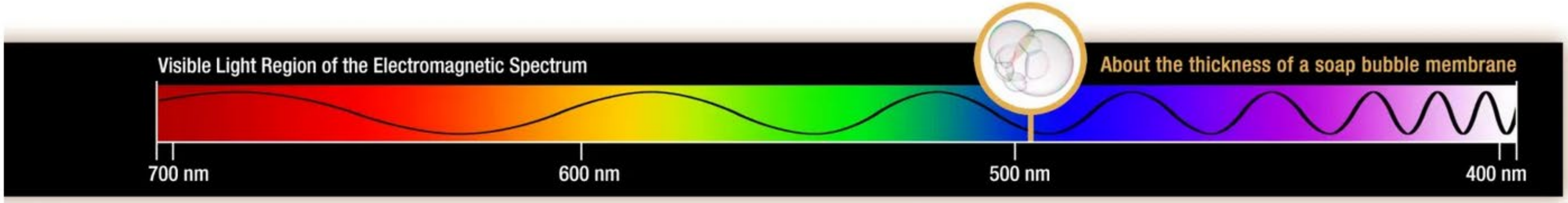
$$\frac{d^2 I}{d\ell d\omega} \propto \omega\left(\beta_n^2 - 1\right)$$

Noting that $c_n = \dfrac{c}{n(\omega)} = \dfrac{c}{\sqrt{\epsilon(\omega)}}$ $\qquad \beta_n = \dfrac{v}{c_n}$

$$\frac{d^2 I}{d\ell d\omega} \propto \omega\left(\epsilon(\omega)\frac{v^2}{c^2} - 1\right) = \frac{2\pi}{\lambda}\left(\epsilon(\omega)\frac{v^2}{c^2} - 1\right)$$

Visible Light Wavelengths --

700 600 500



$$\frac{d^2 I}{d\ell d\omega} \propto \frac{2\pi}{\lambda}\left(\epsilon(\omega)\frac{v^2}{c^2} - 1\right)$$

If $\varepsilon \approx 1.8$ for water, what is the slowest particle speed that can generate Cherenkov radiation?

a. $v = 0.9c$

b. $v = 0.8c$

c. $v = 0.7c$

d. $v = 0.6c$