PHY 712 Electrodynamics
10-10:50 AM MWF Olin 103

Notes for Lecture 35: Continued discussion about quantum effects in electrodynamics

1. Summary of pure of eigenstates of the quantum mechanical EM Hamiltonian and their properties
2. Various linear combinations of quantum mechanical EM eigenstates
    a. Black body radiation
    b. Coherent states
    c. Squeezed states

Note that we will conclude a few minutes early so that you can fill out the course evaluation.

24	Mon: 03/24/2025	Chap. 9	Radiation from time harmonic sources	#20	03/26/2025
25	Wed: 03/26/2025	Chap. 9 & 10	Radiation from scattering	#21	03/28/2025
26	Fri: 03/28/2025	Chap. 11	Special Theory of Relativity	#22	03/31/2025
27	Mon: 03/31/2025	Chap. 11	Special Theory of Relativity	#23	04/02/2025
28	Wed: 04/02/2025	Chap. 11	Special Theory of Relativity	#24	04/04/2025
29	Fri: 04/04/2024	Chap. 14	Radiation from accelerating charged particles	#25	04/07/2025
30	Mon: 04/07/2025	Chap. 14	Analysis of synchroton radiation	#26	04/09/2025
31	Wed: 04/09/2025	Chap. 14	Synchrotron radiation and Compton scattering	#27	04/11/2025
32	Fri: 04/11/2025	Chap. 13 & 15	Other radiation -- Cherenkov & bremsstrahlung	#28	04/14/2025
33	Mon: 04/14/2025		Special topic:E & M aspects of superconductivity		
34	Wed: 04/16/2025		Special topic:Quantum effects in electrodynamics		
	Fri: 04/18/2025		Presentations I		
35	Mon: 04/21/2025		Special topic:More quantum effects in electrodynamics		
	Wed: 04/23/2025		Presentations II		
	Fri: 04/25/2025		Presentations III		
36	Mon: 04/28/2025		Review		

Tue. Apr. 22, 2025 — Physics Honors Presentations

Thurs. Apr. 24, 2025 — Physics Honors and Awards Ceremonies

4 PM in Olin 101

Honors Presentations

- Tuesday -

April 22, 2025

The reaction of Catalase, Azide, and Peroxide produces Nitroxyl. New Insights on Oxidative Stress and Biomarker Preservation

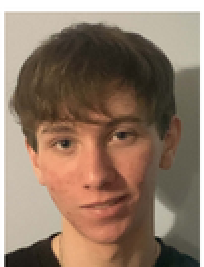
Jesse Gao
Advisor: Prof. D. Kim-Shapiro

An Investigation of Quantum Effects in the Exteriors and Interiors of Black Holes

Amanda Peake
Advisor: Prof. P. Anderson

Berry Curvature and Thermal Hall Transport in Bosonic Systems

Kate Choi
Advisor: Prof. S. Winter



Investigations of Fe-S Cluster Formations in DFT

Christopher Fivecoat
Advisor: Prof. T. Thonhauser

Wake Forest University

Class of 2025

Department of Physics

Olin Physical Laboratory
Room 101
4:00 PM

The Physics Department Presents
the *2025*

Honors and Awards Ceremony

Honors Thesis: Alex Evju

Awards Ceremony: For the extraordinary dedication of our department members

New $\Sigma\Pi\Sigma$ Majors

Graduating Seniors

April 24 Olin 101 4:00-5:00 PM

Wednesday 4/23/2025

	Presenter Name	Topic
10:00-10:24	Thomas Myers	Ising Model
10:25-10:50	Conall O'Leary	

Friday 4/25/2025

	Presenter Name	Topic
10:00-10:24	Julia Radtke	
10:25-10:50	Bhargava Jogi R	

References for today's lecture –

- Consultation with Professor Kandada
- Rodney Loudon, "The quantum theory of light" (1983)
- Leonard Mandel and Emil Wolf, "Optical Coherence and Quantum Optics" (2013)
- Yanhua Shih, "An Introduction to Quantum Optics" (2021)   (some typos, but generally informative)
- **Paul R Berman and Vladimir S. Malinovsky, "Principles of Laser Spectroscopy and Quantum Optics" (2011)**

Review of what we learned from Lecture 34

For a single mode plane wave with wave vector **k**, frequency $\omega_{\mathbf{k}}$ and polarization σ:

EM Field Hamiltonian acting on eigenstate $\left|n_{\mathbf{k}\sigma}\right\rangle$:

where $\mathbf{k}$ denotes wavevector and σ denotes polarization direction --

$$H_{\text{field}}^{\text{fixed}}\left|n_{\mathbf{k}\sigma}\right\rangle = \sum_{\mathbf{k}'\sigma'}\left(\hbar\omega_{\mathbf{k}'}a^{\dagger}_{\mathbf{k}'\sigma'}a_{\mathbf{k}'\sigma'}\right)\left|n_{\mathbf{k}\sigma}\right\rangle = \hbar\omega_{\mathbf{k}}n_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle$$

Here $n_{\mathbf{k}\sigma} = 0,1,2,3,4.......$

$$a_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle = \sqrt{n_{\mathbf{k}\sigma}}\left|n_{\mathbf{k}\sigma}-1\right\rangle$$

$$a^{\dagger}_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle = \sqrt{n_{\mathbf{k}\sigma}+1}\left|n_{\mathbf{k}\sigma}+1\right\rangle$$

Commutation relations:

$$\left[a_{\mathbf{k}\sigma}, a^{\dagger}_{\mathbf{k}'\sigma'}\right]=\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'} \quad \left[a_{\mathbf{k}\sigma}, a_{\mathbf{k}'\sigma'}\right]=0 \quad \left[a^{\dagger}_{\mathbf{k}\sigma}, a^{\dagger}_{\mathbf{k}'\sigma'}\right]=0$$

In terms of the same operators and with polarization unit vectors $\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}$ --

Vector potential:

$$\mathbf{A}(\mathbf{r},t)=\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}+a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

Electric field:

$$\mathbf{E}=-\frac{\partial\mathbf{A}}{\partial t}\quad\Rightarrow\mathbf{E}(\mathbf{r},t)=i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}-a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

Magnetic field:

$$\mathbf{B}=\nabla\times\mathbf{A}\quad\Rightarrow\mathbf{B}(\mathbf{r},t)=i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}-a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

While the photon eigenstates $\left|n_{\mathbf{k}'\sigma'}\right\rangle$ form a complete basis for describing quantum electromagnetic fields, they have some troublesome properties such as found in evaluating the field expectation values --

Vector potential:

$$\left\langle n_{\mathbf{k}'\sigma'}\right|\mathbf{A}(\mathbf{r},t)\left|n_{\mathbf{k}'\sigma'}\right\rangle=\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left\langle n_{\mathbf{k}'\sigma'}\right|\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}+a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)\left|n_{\mathbf{k}'\sigma'}\right\rangle=0$$

Electric field:

$$\left\langle n_{\mathbf{k}'\sigma'}\right|\mathbf{E}(\mathbf{r},t)\left|n_{\mathbf{k}'\sigma'}\right\rangle=i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left\langle n_{\mathbf{k}'\sigma'}\right|\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}-a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)\left|n_{\mathbf{k}'\sigma'}\right\rangle=0$$

Magnetic field:

$$\left\langle n_{\mathbf{k}'\sigma'}\right|\mathbf{B}(\mathbf{r},t)\left|n_{\mathbf{k}'\sigma'}\right\rangle=i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left\langle n_{\mathbf{k}'\sigma'}\right|\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}-a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)\left|n_{\mathbf{k}'\sigma'}\right\rangle=0$$

Quantum properties of electrodynamics and the properties of black body radiation (logical but not historically accurate development of the idea).

1. The quantum nature of the radiation Hamiltonian means that for each radiating frequency ω_i, the radiated energy is $\hbar\omega_i n_i$ for $n_i = 0, 1, 2, \ldots\infty$, where n_i denotes the number of photons.

2. If the radiating system is in thermal equilibrium at temperature T, the process follows Bose-Einstein statistics, since photons have spin 1.

3. After performing the summation over all n_i, the internal energy of the radiating system at temperature T, is given by

$$U = \sum_{i=1}^{\infty} \frac{\hbar\omega_i}{\exp(\hbar\omega_i / k_B T) - 1},$$ where k_B denotes the Boltmann constant.

continuing --

4. Evaluating the summation over i, with the understanding that $\omega_i = ck_i$ and for a system with volume V, k_i takes values

$$k_i = \frac{\pi}{V^{1/3}}\left(n_x^2 + n_y^2 + n_z^2\right)^{1/2} \text{ for all possible integers } n_x, n_y, n_z.$$

5. Converting the summation over i to an integral

$$U = \sum_{i=1}^{\infty} \frac{\hbar\omega_i}{\exp(\hbar\omega_i / k_B T) - 1} = \frac{V}{\pi^2 c^3}\int_0^{\infty} \omega^2 d\omega \frac{\hbar\omega}{\exp(\hbar\omega / k_B T) - 1}$$

Another aspect of blackbody radiation is the average photon number n each frequency ω at temperature T:

$$\langle n \rangle = \frac{1}{\exp(\hbar\omega / k_B T) - 1}$$

Returning to the ideal case of a single frequency radiation mode, it turns out that lasers can output radiation very similar to that so-called "coherent state" described by R. Glauber, Physical Review 131, 2766 (1963)

Gauber's coherent state: $\left|c_\alpha\right\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}} \left|n\right\rangle$

Here α represents a complex amplitude

It is possible to prove the following identies for the coherent states:

1. $\left\langle c_\alpha \middle| c_\alpha \right\rangle = 1$
2. $\left\langle c_\alpha \middle| a \middle| c_\alpha \right\rangle = \alpha$
3. $\left\langle c_\alpha \middle| a^\dagger \middle| c_\alpha \right\rangle = \alpha^*$
4. $\left|\left\langle c_\alpha \middle| c_\beta \right\rangle\right|^2 = e^{-|\alpha-\beta|^2}$

$$\left|c_\alpha\right\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}} \left|n\right\rangle$$ based on a single mode $n \to n_{\mathbf{k}\sigma}$

Electric field: $\left\langle c_\alpha \right| \mathbf{E}(\mathbf{r},t) \left| c_\alpha \right\rangle = i\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}} \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \left(\alpha_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} - \alpha^*_{\mathbf{k}\sigma} e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)} \right)$

Magnetic field: $\left\langle c_\alpha \right| \mathbf{B}(\mathbf{r},t) \left| c_\alpha \right\rangle = i\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}} \mathbf{k} \times \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \left(\alpha_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} - \alpha^*_{\mathbf{k}\sigma} e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)} \right)$

Let $\alpha = \Lambda e^{i\psi}$ where both Λ and Ψ are unitless real values.

$$\left\langle c_\alpha \right| \mathbf{E}(\mathbf{r},t) \left| c_\alpha \right\rangle = -2\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}} \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \Lambda \sin\left(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}}t + \psi\right)$$

$$\left\langle c_\alpha \right| \mathbf{B}(\mathbf{r},t) \left| c_\alpha \right\rangle = -2\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}} \mathbf{k} \times \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \Lambda \sin\left(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}}t + \psi\right)$$

Single mode coherent state continued

It can also be shown that

$$\left\langle c_\alpha \middle| \left|\mathbf{E}(\mathbf{r},t)\right|^2 \middle| c_\alpha \right\rangle = \frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}\left(4\Lambda^2 \sin^2\left(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t + \psi\right) + 1\right)$$

Therefore

$$\left\langle c_\alpha \middle| \left|\mathbf{E}(\mathbf{r},t)\right|^2 \middle| c_\alpha \right\rangle - \left|\left\langle c_\alpha \middle| \mathbf{E}(\mathbf{r},t) \middle| c_\alpha \right\rangle\right|^2 = \frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}$$

This means that variance of the E field for the coherent state is independent of the amplitude Λ. Therefore, for large Λ the variance is small in comparison.

Visualization of coherent state electric fields for various amplitudes

Source:
R. Loudon, "The Quantum Theory of Light"

The quantized radiation field 151

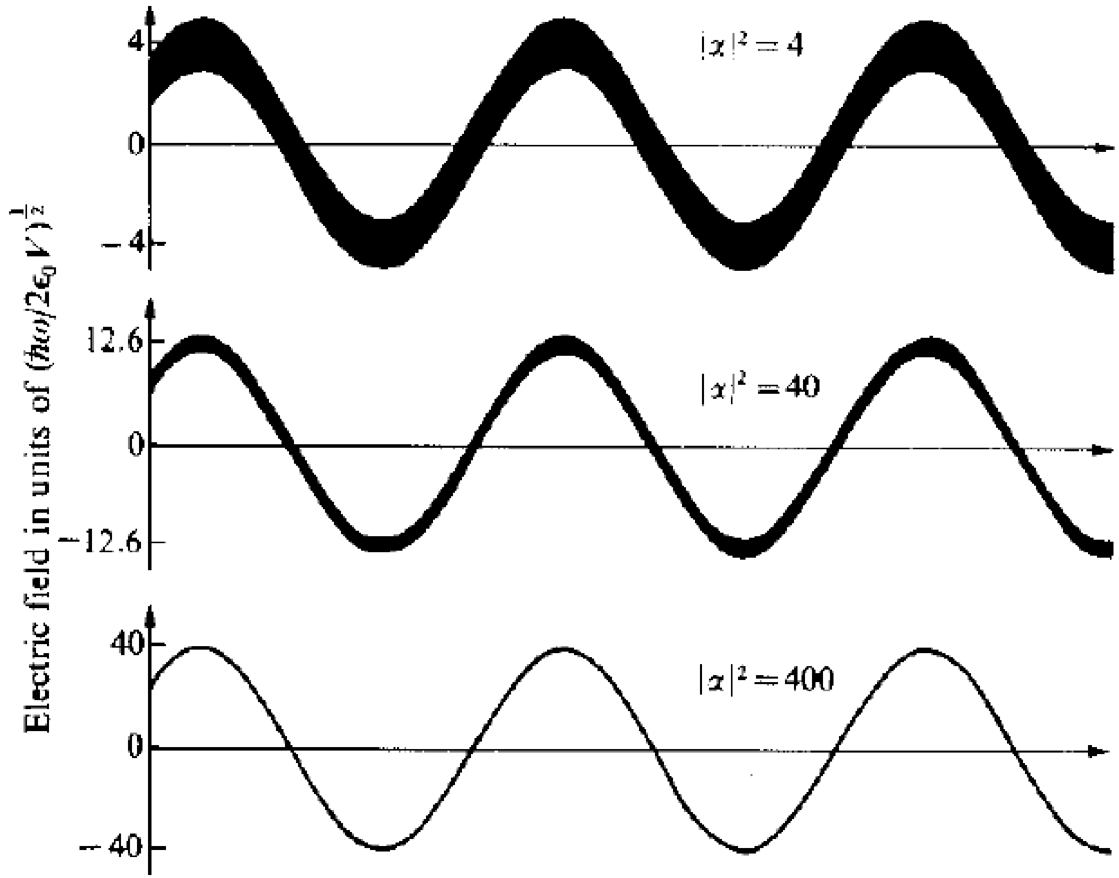


FIG. 4.3. Pictorial representation of the electric-field variation in a cavity mode excited to state $|\alpha\rangle$. Three different values of the mean photon number $|\alpha|^2$ are shown, the vertical scales being different for the three cases. The uncertainties in field values are indicated by the vertical widths $2\Delta E$ of the sine waves. These widths can also be regarded as combinations of the amplitude uncertainty associated with Δn and the phase uncertainty associated with $\Delta \cos\phi$.

Additional properties of single mode coherent state --

Consider the expectation values of the number operator and its square:

$$\mathbf{N}_{\mathbf{k}\sigma} \equiv a^{\dagger}_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}$$

$$\left\langle c_\alpha \left| \mathbf{N}_{\mathbf{k}\sigma} \right| c_\alpha \right\rangle = |\alpha|^2 \qquad\qquad \left\langle c_\alpha \left| \mathbf{N}_{\mathbf{k}\sigma}\mathbf{N}_{\mathbf{k}\sigma} \right| c_\alpha \right\rangle = |\alpha|^4 + |\alpha|^2$$

Square of the variance: $\left\langle c_\alpha \left| \mathbf{N}_{\mathbf{k}\sigma}\mathbf{N}_{\mathbf{k}\sigma} \right| c_\alpha \right\rangle - \left|\left\langle c_\alpha \left| \mathbf{N}_{\mathbf{k}\sigma} \right| c_\alpha \right\rangle\right|^2 = |\alpha|^2$

Fractional uncertainty in the number of photons for the coherent state:

$$\frac{\sqrt{\left\langle c_\alpha \left| \mathbf{N}_{\mathbf{k}\sigma}\mathbf{N}_{\mathbf{k}\sigma} \right| c_\alpha \right\rangle - \left|\left\langle c_\alpha \left| \mathbf{N}_{\mathbf{k}\sigma} \right| c_\alpha \right\rangle\right|^2}}{\left\langle c_\alpha \left| \mathbf{N}_{\mathbf{k}\sigma} \right| c_\alpha \right\rangle} = \frac{\sqrt{|\alpha|^4 + |\alpha|^2 - |\alpha|^4}}{|\alpha|^2} = \frac{1}{|\alpha|} = \frac{1}{\Lambda}$$

when $\alpha = \Lambda \mathrm{e}^{i\psi}$

Interpretation of a single mode coherent state

$$\left|c_\alpha\right\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}} \left|n\right\rangle$$ based on a single mode $n \rightarrow n_{\mathbf{k}\sigma}$

The probability of finding n photons in this state is given by:

$$\left|\left\langle n \middle| c_\alpha \right\rangle\right|^2 = \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}$$

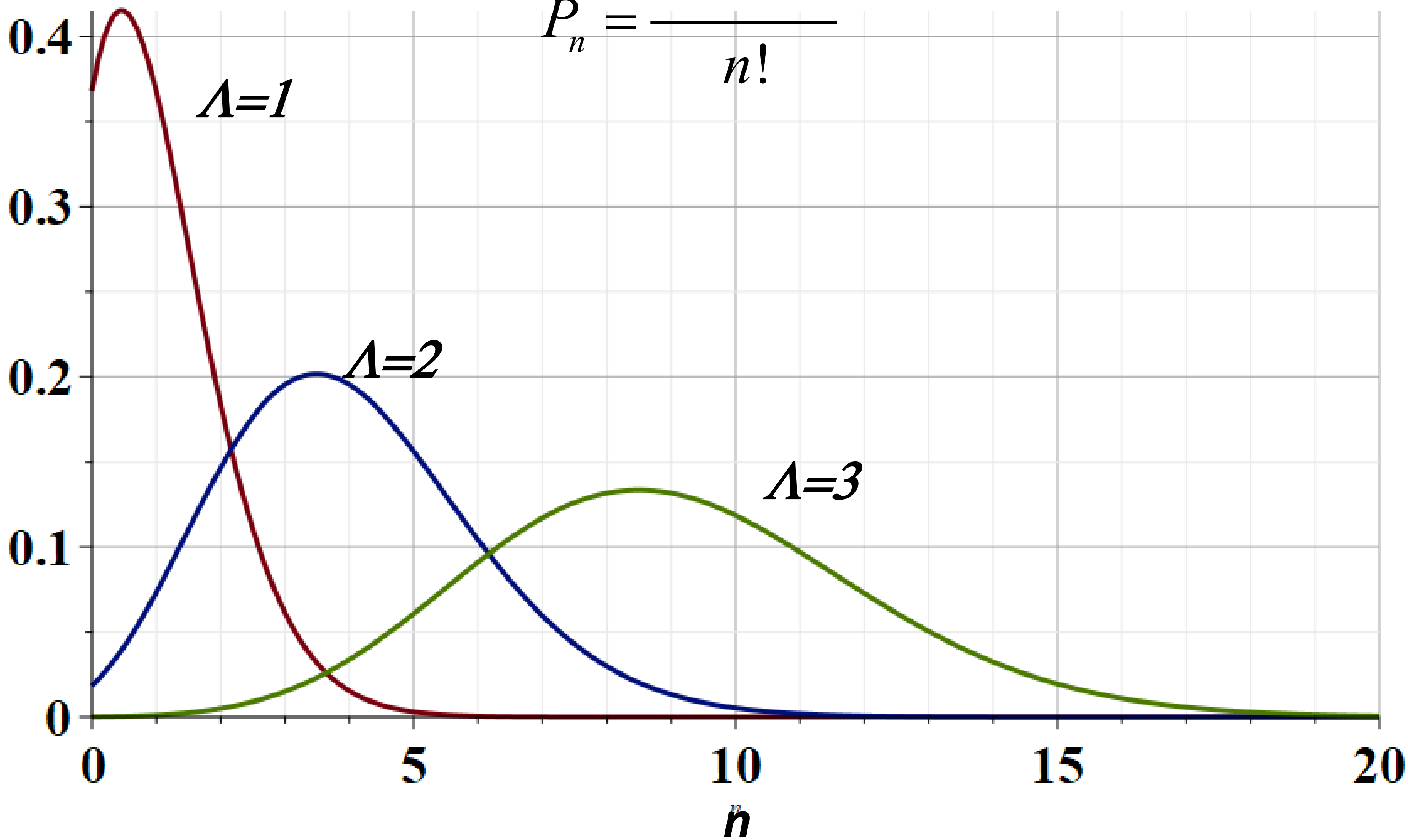
This is the form of a Poisson distribution for a mean value of $|\alpha|^2$.

For $\alpha = \Lambda e^{i\psi}$, the probability of finding the eigenstate with eigenstate $\left|n\right\rangle$ is given by

$$P_n = \left|\left\langle n \middle| c_\alpha \right\rangle\right|^2 = \frac{|\Lambda|^{2n} e^{-|\Lambda|^2}}{n!}$$

Poisson distributions

$$P_n = \frac{\Lambda^{2n} e^{-\Lambda^2}}{n!}$$



Focusing on a particular pure EM mode with wavenumber $\mathbf{k}$ and frequency $\omega_{\mathbf{k}}$:

For a coherent state c_α with $\alpha = \Lambda e^{i\Psi}$, the probability of finding the eigenstate with photon number $|n\rangle$ is given by

$$P_n^{\text{Coherent}} = \left| \langle n | c_\alpha \rangle \right|^2 = \frac{|\Lambda|^{2n} e^{-|\Lambda|^2}}{n!}$$

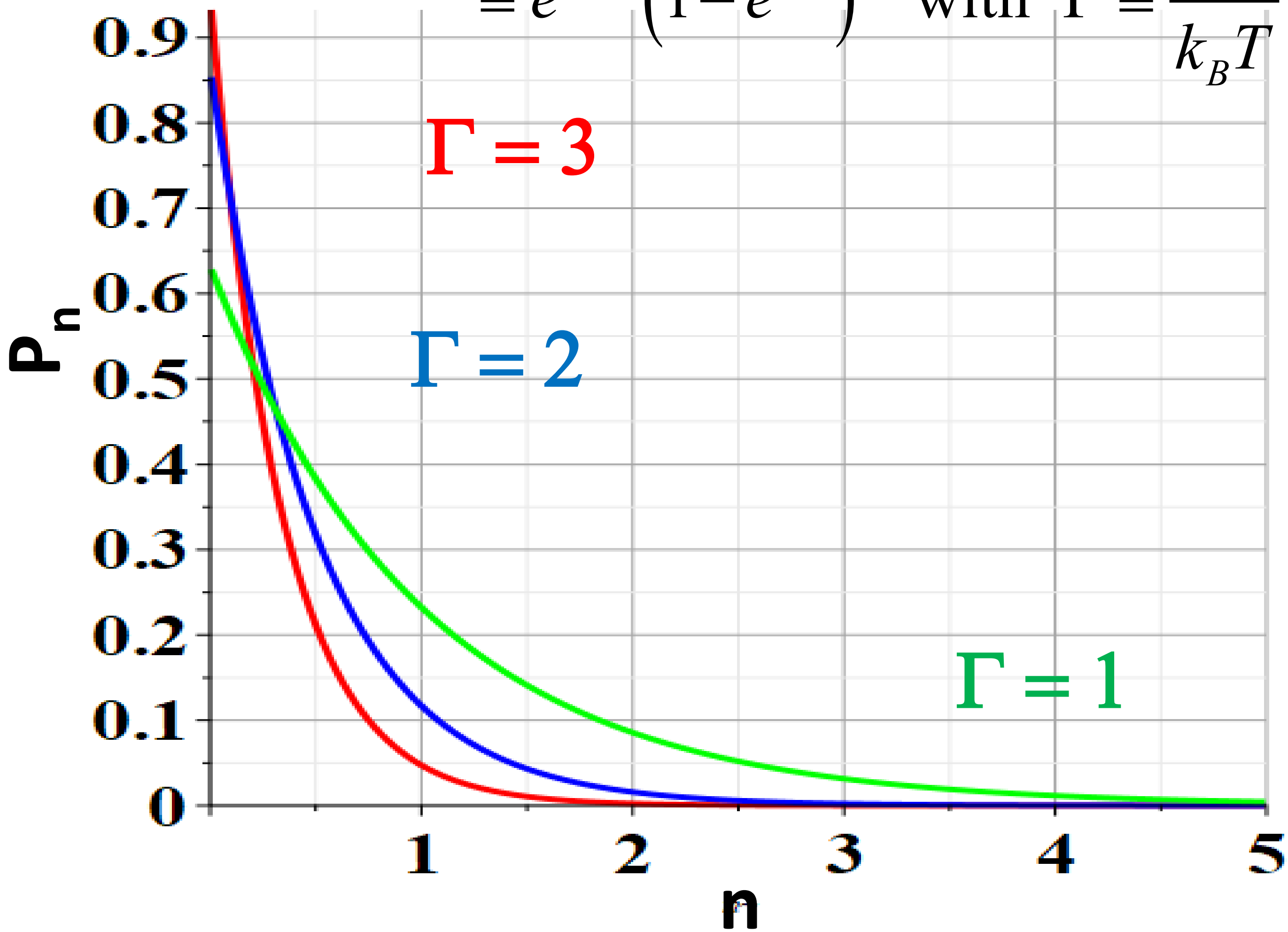
For "a black body system" at temperature T, the probability of finding the eigenstate with photon number $|n\rangle$ is given by

$$P_n^{\text{Thermal}}(T) = e^{-n\hbar\omega/k_B T}\left(1 - e^{-\hbar\omega/k_B T}\right)$$

Thermal distributions:

$$P_n^{\text{Thermal}}(T) = e^{-n\hbar\omega/k_BT}\left(1-e^{-\hbar\omega/k_BT}\right)$$

$$\equiv e^{-n\Gamma}\left(1-e^{-\Gamma}\right) \quad \text{with } \Gamma \equiv \frac{\hbar\omega}{k_BT}$$



Beyond the coherent state – There are problematic issues with the coherent state basis stemming from the fact that it is mathematically "over complete".

Following Berman and Malinovsky we explore the notion of "quadrature operators" for the radiation Hamiltonian.

Recall that for a given radiation more $\mathbf{k}\sigma$, the fields are expressed

$$\mathbf{A}(\mathbf{r},t)=\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}+a^{\dagger}_{\mathbf{k}\sigma}e^{-\left(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t\right)}\right)$$

$$\mathbf{E}(\mathbf{r},t)=i\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}-a^{\dagger}_{\mathbf{k}\sigma}e^{-\left(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t\right)}\right)$$

$$\mathbf{B}(\mathbf{r},t)=i\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}-a^{\dagger}_{\mathbf{k}\sigma}e^{-\left(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t\right)}\right)$$

In the following, we will drop the indication of the mode parameters $\mathbf{k}\sigma$

Define the Hermitian operators

$$a_1 \equiv \frac{1}{2}\left(a + a^\dagger\right) \qquad\qquad a_2 \equiv \frac{1}{2i}\left(a - a^\dagger\right)$$

Commutation relations: $\left[a_1, a_2\right] = \frac{i}{2}$

Variance relationship: $\left(\Delta a_1\right)\left(\Delta a_2\right) \geq \frac{1}{4}$

Recall the general relationship: $[A, B] = iC$ implies $\Delta A \Delta B \geq \frac{1}{2}\left|\left\langle C\right\rangle\right|$

For pure radiation eigenstates $|n\rangle$ --

Expectation values and variances of a_1 and a_2

$$\langle n|a_1|n\rangle = 0 = \langle n|a_2|n\rangle$$

$$\langle n|a_1^2|n\rangle = \frac{1}{2}\left(n+\frac{1}{2}\right) = \langle n|a_2^2|n\rangle$$

$$(\Delta a_1)(\Delta a_2) = \frac{1}{2}\left(n+\frac{1}{2}\right)$$

$\Rightarrow$ Pure radiation eigenstate is not a minimum uncertainty state unless $n = 0$.

For coherent states with $\left|c_\alpha\right\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}}\left|n\right\rangle$

Expectation values and variances of a_1 and a_2

$$\left\langle c_\alpha\right|a_1\left|c_\alpha\right\rangle = \frac{\alpha+\alpha^*}{2} \qquad \left\langle c_\alpha\right|a_2\left|c_\alpha\right\rangle = \frac{\alpha-\alpha^*}{2i}$$

$$\left\langle c_\alpha\right|a_1^2\left|c_\alpha\right\rangle - \left\langle c_\alpha\right|a_1\left|c_\alpha\right\rangle^2 = \frac{1}{4} = \left\langle n\right|a_2^2\left|n\right\rangle - \left\langle c_\alpha\right|a_2\left|c_\alpha\right\rangle^2$$

$\Rightarrow$ Coherent state is a minimum uncertainty state

Is it possible to do better than the coherent state?

Change of notation --

$$\hat{Q} \equiv 2a_1 \qquad \hat{P} \equiv 2a_2$$

$$\lambda \equiv \alpha$$

Review of equations related to quantized EM fields --

Recall that we can write the EM Hamiltonian for a single mode $\omega_{\mathbf{k}} \equiv \omega --$

$$H_{rad} = \frac{1}{2}\hbar\omega\left(a^\dagger a + aa^\dagger\right) = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \quad \text{where } \left[a, a^\dagger\right] = 1$$

Field eigenstates $\quad a^\dagger a\left|n\right\rangle = n\left|n\right\rangle$

For further analysis, it is convenient to define "Quadrature operators" which are unitless and Hermitian:

$$\hat{Q} \equiv \left(a^\dagger + a\right) \quad \text{and} \quad \hat{P} \equiv i\left(a^\dagger - a\right) \quad \Rightarrow \left[\hat{Q}, \hat{P}\right] = 2i$$

Note that some texts define Q and P with a prefactor of ½.

Note that: $\quad H = \frac{\hbar\omega}{2}\left(\hat{Q}^2 + \hat{P}^2\right)$

From the Heisenberg uncertainty ideas applied to the standard deviations:

$$\Delta\hat{Q}\Delta\hat{P} \geq 1$$

Also note that $\left\langle n\left|\hat{Q}\right|n\right\rangle = 0 = \left\langle n\left|\hat{P}\right|n\right\rangle$

For the coherent state:

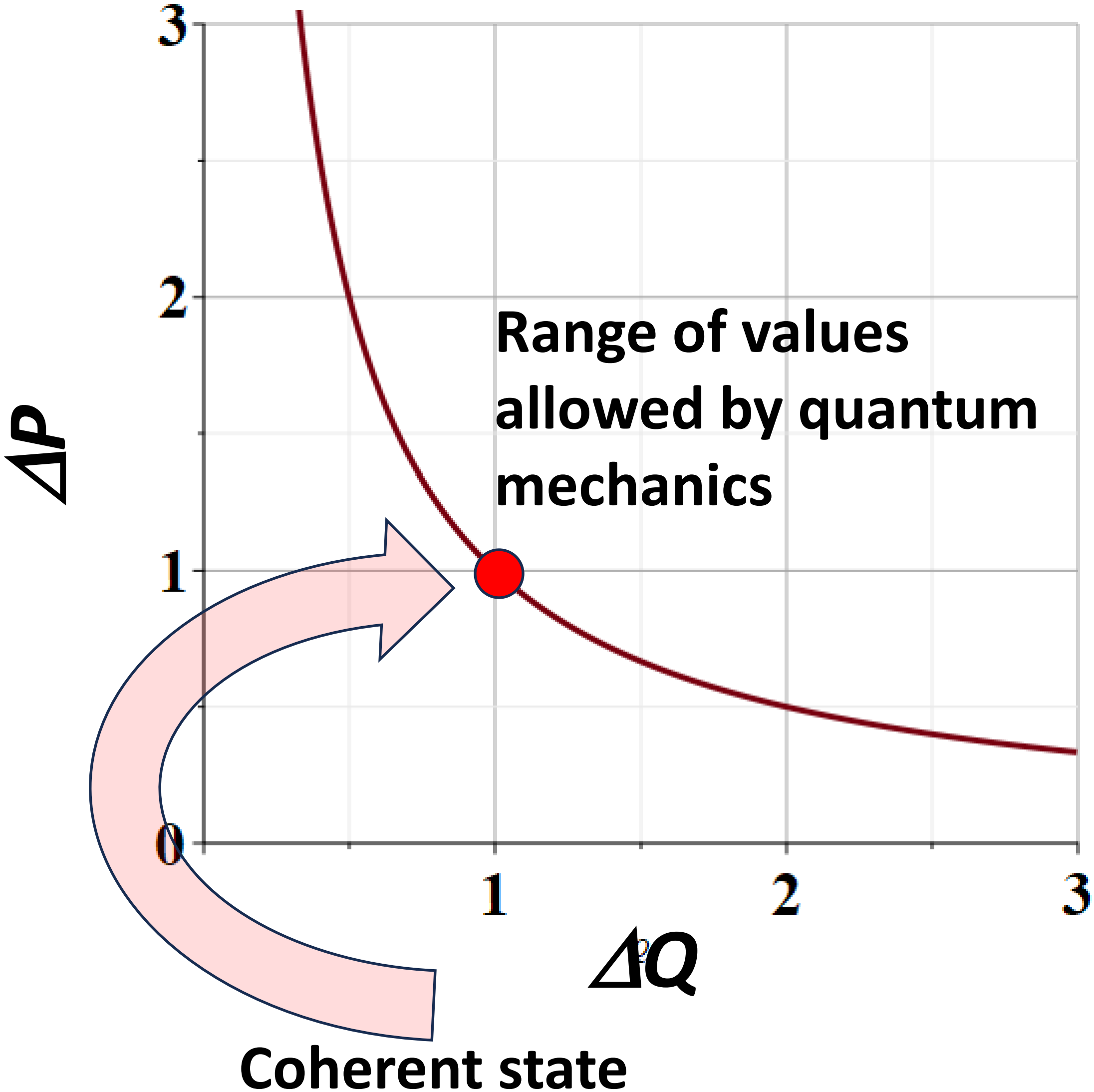
$$\left|\lambda\right\rangle = e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\left|n\right\rangle$$

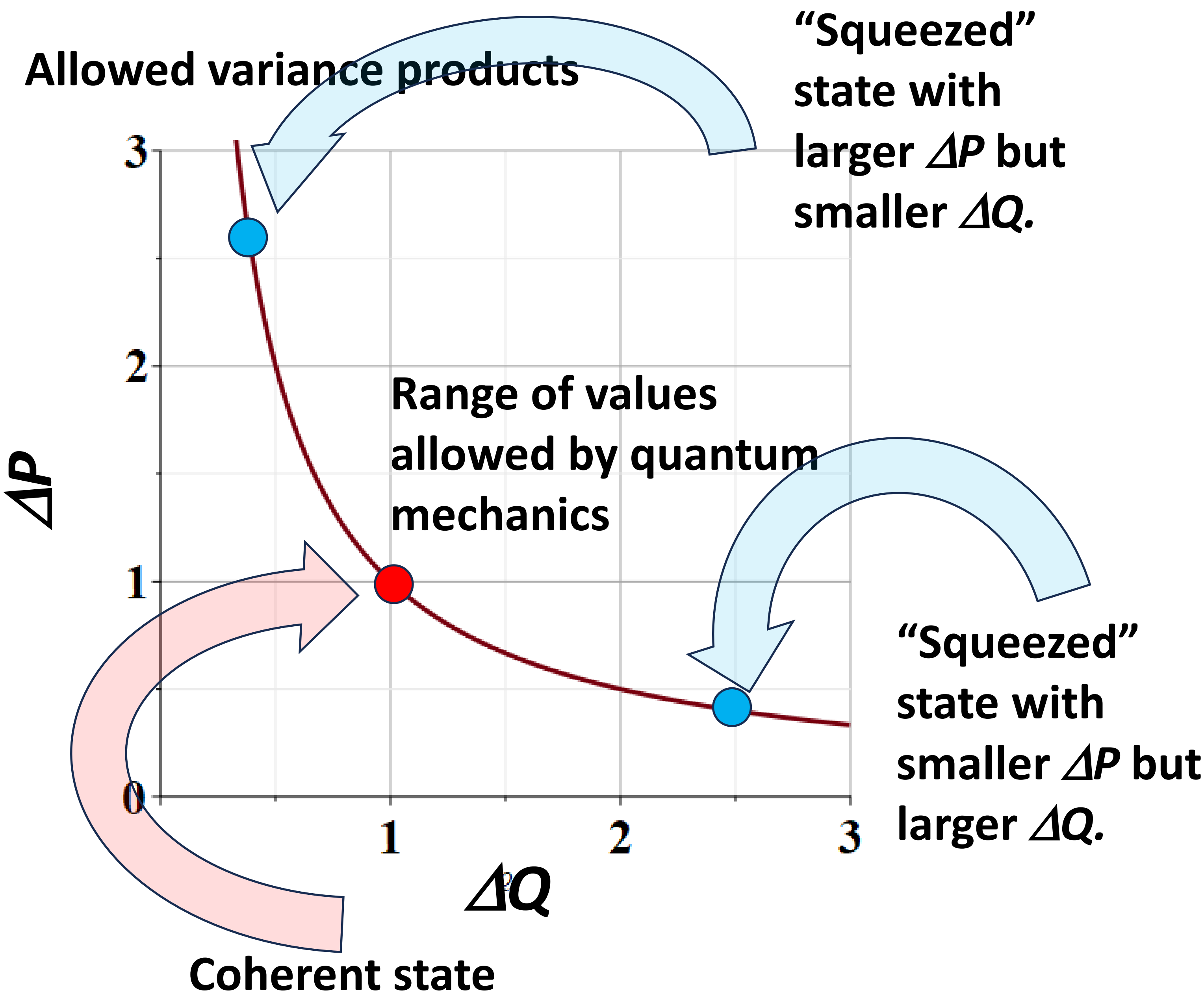
$$\Delta\hat{Q}_\lambda = \sqrt{\left\langle\lambda\left|\hat{Q}^2\right|\lambda\right\rangle - \left|\left\langle\lambda\left|\hat{Q}\right|\lambda\right\rangle\right|^2} = 1 = \Delta\hat{P}_\lambda$$

$$\Rightarrow \Delta\hat{Q}_\lambda\Delta\hat{P}_\lambda = 1$$

In this sense, the coherent state represents the minimum uncertainty process.

Allowed variance products




Allowed variance products
"Squeezed" state with larger ΔP but smaller ΔQ.
3
2
1
0
ΔP
Range of values allowed by quantum mechanics
"Squeezed" state with smaller ΔP but larger ΔQ.
1
2
3
ΔQ
Coherent state

In terms of the eigenstates of the EM Hamiltonian:

$$H_{rad}\left|n\right\rangle = \hbar\omega\left(n+\frac{1}{2}\right)\left|n\right\rangle$$

$$\Delta\hat{Q}_n = \sqrt{\left\langle n\left|\hat{Q}^2\right|n\right\rangle - \left|\left\langle n\left|\hat{Q}\right|n\right\rangle\right|^2} = \sqrt{2n+1} = \Delta\hat{P}_n$$

$$\Rightarrow \Delta\hat{Q}_n\Delta\hat{P}_n = 2n+1 \geq 1$$

In terms of coherent states: --

For the coherent state:

$$\left|\lambda\right\rangle = e^{-|\lambda|^2/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \left|n\right\rangle$$

$$\Delta\hat{Q}_\lambda = \sqrt{\left\langle\lambda\left|\hat{Q}^2\right|\lambda\right\rangle - \left|\left\langle\lambda\left|\hat{Q}\right|\lambda\right\rangle\right|^2} = 1 = \Delta\hat{P}_\lambda$$

$$\Rightarrow \Delta\hat{Q}_\lambda \Delta\hat{P}_\lambda = 1$$

In this sense, the coherent state represents the minimum uncertainty process.

How can we transform the quadrature functions to reduce the variances of ΔQ or ΔP?

Following Mandel and Wolf, we introduce the squeeze operator

$$\hat{S}(z) \equiv \exp\left(\frac{1}{2}\left(z^*\hat{a}^2 - z\hat{a}^{\dagger 2}\right)\right)$$

$$= 1 + \frac{1}{2}\left(z^*\hat{a}^2 - z\hat{a}^{\dagger 2}\right) + \frac{1}{2}\left(\frac{1}{2}\left(z^*\hat{a}^2 - z\hat{a}^{\dagger 2}\right)\right)^2 + \frac{1}{3!}\left(\frac{1}{2}\left(z^*\hat{a}^2 - z\hat{a}^{\dagger 2}\right)\right)^3 \ldots\ldots$$

Note that $\hat{S}(z)$ is a unitary operator $\quad \hat{S}(z)\left(\hat{S}(z)\right)^\dagger = 1$

Let $z = re^{i\theta}$

Squeeze operator with $z = re^{i\theta}$

$$\hat{S}(z) \equiv \exp\left(\frac{1}{2}\left(z^*\hat{a}^2 - z\hat{a}^{\dagger 2}\right)\right) \qquad z = re^{i\theta}$$

$$\hat{A}(z) \equiv \hat{S}(z)\hat{a}\left(\hat{S}(z)\right)^\dagger \qquad \text{and} \qquad \hat{A}^\dagger(z) \equiv \hat{S}(z)\hat{a}^\dagger\left(\hat{S}(z)\right)^\dagger$$

$$\hat{A}(z) = \hat{a} + z\hat{a}^\dagger + \frac{|z|^2\,\hat{a}}{2!} + \frac{z|z|^2\,\hat{a}^\dagger}{3!} + ... \qquad \text{(not totally trivial...)}$$

$$\Rightarrow \hat{A}(z) = \hat{a}\ \cosh r + \hat{a}^\dagger e^{i\theta}\sinh r$$

$$\hat{A}^\dagger(z) = \hat{a}\ e^{-i\theta}\sinh r + \hat{a}^\dagger\cosh r$$

Inverting these relations --

$$\hat{a} = \hat{A}(z)\cosh r - \hat{A}^\dagger(z)e^{i\theta}\sinh r$$

$$\hat{a}^\dagger = \hat{A}^\dagger(z)\cosh r - \hat{A}(z)e^{-i\theta}\sinh r$$

Now recall the "Quadrature operators"

$$\hat{Q} \equiv \left(a^\dagger + a\right) \quad \text{and} \quad \hat{P} \equiv i\left(a^\dagger - a\right) \quad \Rightarrow \left[\hat{Q}, \hat{P}\right] = 2i$$

From the Heisenberg uncertainty ideas -- $\Delta\hat{Q}\Delta\hat{P} \geq 1$

More generally, we can use the altered operators --

$$\hat{Q}_\beta \equiv \left(a^\dagger e^{i\beta} + a e^{-i\beta}\right) \quad \text{and} \quad \hat{P}_\beta \equiv i\left(a^\dagger e^{i\beta} - a e^{-i\beta}\right)$$

Note that $[\hat{Q}_\beta,\ \hat{P}_\beta] = 2i$ which implies $\Delta\hat{Q}_\beta\Delta\hat{P}_\beta \geq 1$

We are seeking a "squeezed" states for which $\Delta\hat{Q}_\beta < 1$

Consider a "squeezed" coherent state: $\left|z,\lambda\right\rangle \equiv S(z)\left|\lambda\right\rangle$

Evaluating the variance $\Delta\hat{Q}_\beta$ for this squeezed coherent state --

Evaluating the variance --

$$\left\langle z,\lambda\left|\hat{Q}_\beta\right|z,\lambda\right\rangle = \left\langle z,\lambda\left|\hat{a}^\dagger e^{i\beta} + \hat{a}e^{-i\beta}\right|z,\lambda\right\rangle$$

$$\hat{a} = \hat{A}(z)\cosh r - \hat{A}^\dagger(z)e^{i\theta}\sinh r$$

$$\hat{a}^\dagger = \hat{A}^\dagger(z)\cosh r - \hat{A}(z)e^{-i\theta}\sinh r$$

When the dust clears -- (Details in Mandel and Wolf and other references)

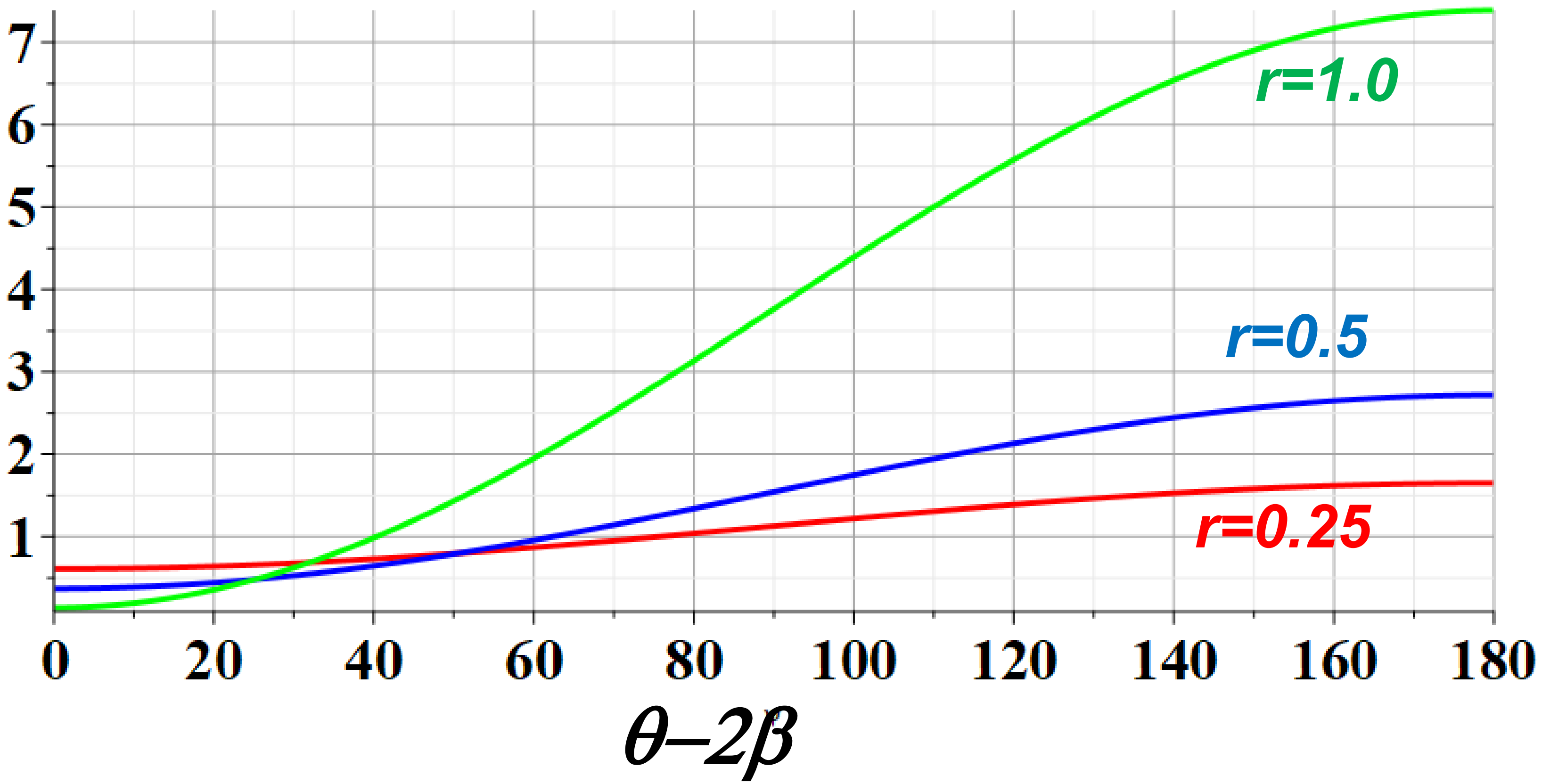
$$\left\langle z,\lambda\left|\hat{Q}_\beta\right|z,\lambda\right\rangle = \left(\lambda^*\cosh r - \lambda e^{-i\theta}\sinh r\right)e^{i\beta} + \left(\lambda\cosh r - \lambda^* e^{i\theta}\sinh r\right)e^{-i\beta}$$

After more dust --

$$\left\langle z,\lambda\left|\left(\Delta\hat{Q}_\beta\right)^2\right|z,\lambda\right\rangle = \left\langle z,\lambda\left|\left(\hat{Q}_\beta\right)^2\right|z,\lambda\right\rangle - \left|\left\langle z,\lambda\left|\hat{Q}_\beta\right|z,\lambda\right\rangle\right|^2$$

$$= \cosh(2r) - \sinh(2r)\cos\left(\theta - 2\beta\right)$$

$$\left\langle z,\lambda\right|\left(\Delta\hat{Q}_\beta\right)^2\left|z,\lambda\right\rangle = \left\langle z,\lambda\right|\left(\hat{Q}_\beta\right)^2\left|z,\lambda\right\rangle - \left|\left\langle z,\lambda\right|\hat{Q}_\beta\left|z,\lambda\right\rangle\right|^2$$
$$= \cosh(2r) - \sinh(2r)\cos\left(\theta - 2\beta\right)$$



Searching for the best squeeze parameters

$$\langle z,\lambda|\left(\Delta\hat{Q}_\beta\right)^2|z,\lambda\rangle = \langle z,\lambda|\left(\hat{Q}_\beta\right)^2|z,\lambda\rangle - \left|\langle z,\lambda|\hat{Q}_\beta|z,\lambda\rangle\right|^2$$
$$= \cosh(2r) - \sinh(2r)\cos\left(\theta - 2\beta\right)$$

For each r, the smallest result is obtained when $\beta=\theta/2$

$$\langle z,\lambda|\left(\Delta\hat{Q}_\beta\right)^2|z,\lambda\rangle = \cosh(2r) - \sinh(2r) = \exp(-2r) \leq 1$$

It can also be shown that for the same choice of parameters

$$\langle z,\lambda|\left(\Delta\hat{P}_\beta\right)^2|z,\lambda\rangle = \cosh(2r) + \sinh(2r) = \exp(2r) \geq 1$$

➔Despite the constraints of the uncertainty principle, it is possible to improve the measurement of one of the two non-commuting processes.

Experimental evidence

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Generation of Squeezed States by Parametric Down Conversion

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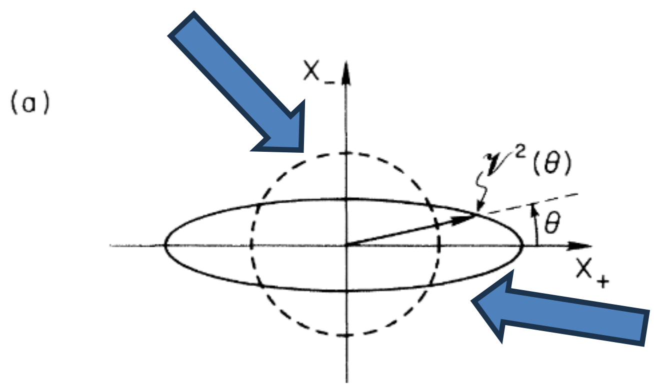
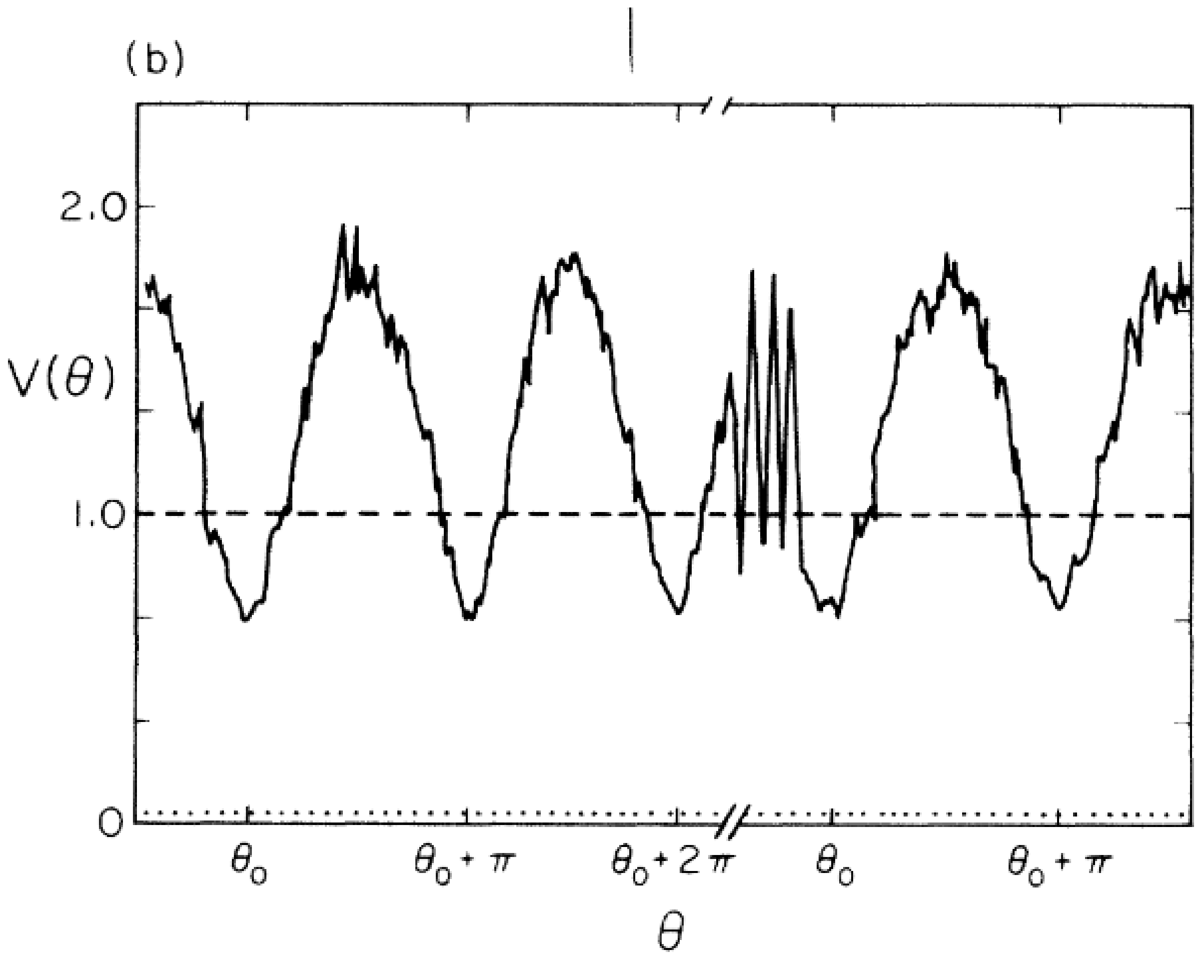
Squeezed states of the electromagnetic field are generated by degenerate parametric down conversion in an optical cavity. Noise reductions greater than 50% relative to the vacuum noise level are observed in a balanced homodyne detector. A quantitative comparison with theory suggests that the observed squeezing results from a field that in the absence of linear attenuation would be squeezed by greater then tenfold.

Coherent state



Squeezed state


(b)
2.0
1.0
0
V(θ)
θ₀
θ₀ + π
θ₀ + 2π
θ₀
θ₀ + π
θ

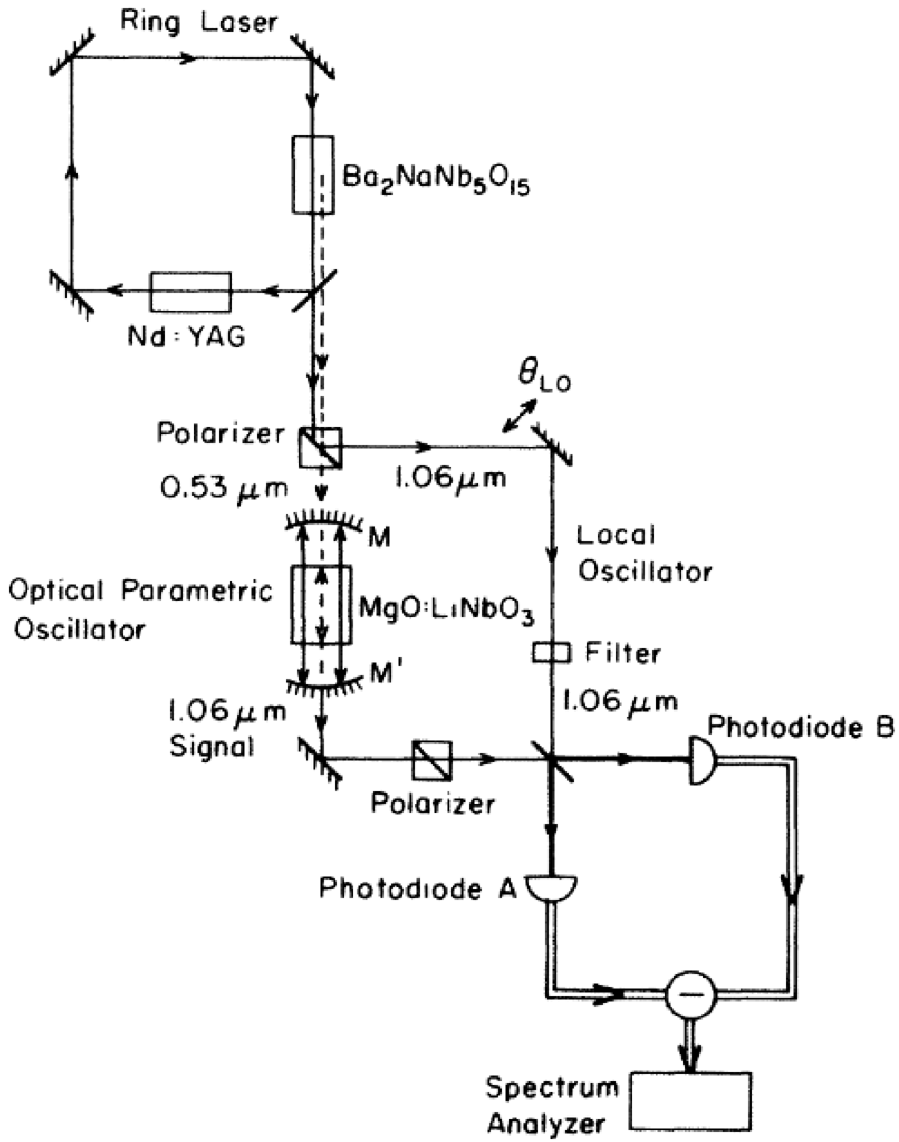


FIG. 2. Diagram of the principal elements of the apparatus for squeezed-state generation by degenerate parametric down conversion.