PHY 712 Electrodynamics
10-10:50 AM MWF Olin 103

Notes for Lecture 36:

Review –

1. Motivation of final exam and how to optimize the experience; some comments on effective use of Maple, Mathematica
2. Some important mathematical tools and review of problem solving techniques
3. Particular electrodynamics topics

24	Mon: 03/24/2025	Chap. 9	Radiation from time harmonic sources	#20	03/26/2025
25	Wed: 03/26/2025	Chap. 9 & 10	Radiation from scattering	#21	03/28/2025
26	Fri: 03/28/2025	Chap. 11	Special Theory of Relativity	#22	03/31/2025
27	Mon: 03/31/2025	Chap. 11	Special Theory of Relativity	#23	04/02/2025
28	Wed: 04/02/2025	Chap. 11	Special Theory of Relativity	#24	04/04/2025
29	Fri: 04/04/2024	Chap. 14	Radiation from accelerating charged particles	#25	04/07/2025
30	Mon: 04/07/2025	Chap. 14	Analysis of synchroton radiation	#26	04/09/2025
31	Wed: 04/09/2025	Chap. 14	Synchrotron radiation and Compton scattering	#27	04/11/2025
32	Fri: 04/11/2025	Chap. 13 & 15	Other radiation -- Cherenkov & bremsstrahlung	#28	04/14/2025
33	Mon: 04/14/2025		Special topic:E & M aspects of superconductivity		
34	Wed: 04/16/2025		Special topic:Quantum effects in electrodynamics		
	Fri: 04/18/2025		Presentations I		
35	Mon: 04/21/2025		Special topic:More quantum effects in electrodynamics		
	Wed: 04/23/2025		Presentations II		
	Fri: 04/25/2025		Presentations III		
36	Mon: 04/28/2025		Review		

Important dates: Final exams available ~ May 2
Exams and outstanding HW due May 8

What will you do after May 8?

Relax a minute or two

Several of you will want to start preparing for the Qualifier Exams which will be administered (tentative dates):
Monday, June 9 to Thursday, June 12

Motivation for giving/taking final exam for PHY 712

1. Opportunity to review/solidify knowledge in the topic
2. Opportunity to practice problem solving techniques appropriate to the topic
3. Assessment of performance. Accordingly, the work you turn in must be your own (of course).
   - You are encouraged to consult with your instructor (**but no one else**!) if any questions arise about the exam questions
   - Extra credit awarded if you find errors/inconsistencies/ambiguities in the exam questions

Instructions on exam:

Note: This is a ``take-home'' exam which can be turned in any time before 5 PM Thursday, May 8, 2025. In addition to each worked problem, please attach ALL Maple (or Mathematica, Matlab, Wolfram, etc.), work sheets as well as a full list of resources used to complete these problems. It is assumed that all work on the exam is performed under the guidelines of the honor code. In particular, if you have any questions about the material, you may consult with the instructor ***but no one else***. For grading purposes, each question in multi-part problems are worth equal weight. Credit will be assigned on the basis of both the logical steps of the solution and on the correct answer.

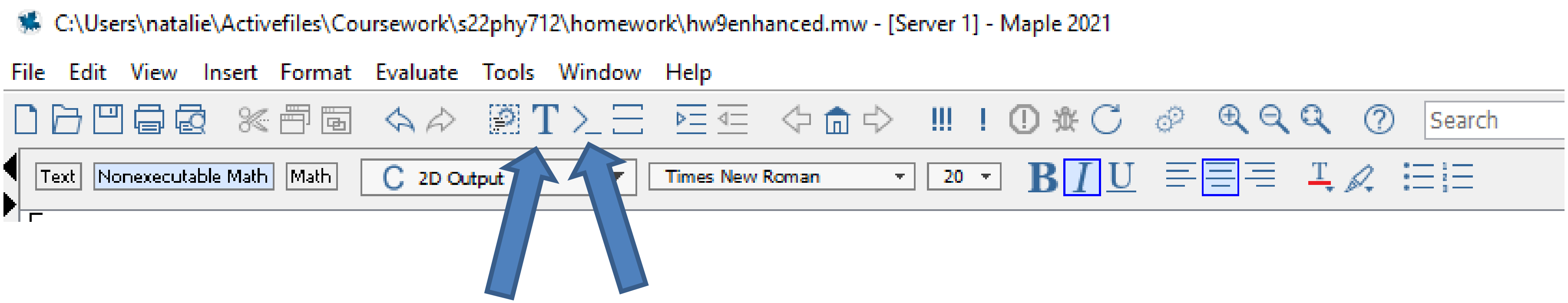
More advice about exam –

- It is important that the instructor is able to read your work and understand your reasoning.

- Since you will be using Maple or Mathematica or ?? to evaluate some of your results, consider integrating them into your exam paper or perhaps paste snips into your favorite word processor.

- Your exam paper does not need to be a work of art, but it does need to be readable. If you prefer to submit your exam paper electronically, that will be fine. (I may print it myself.)

Example solution using Maple --



Text mode

Math mode

More advice – accumulated trusted equations/mathematical relationships and know how to use them

Jackson
pg. 783

Table 4 Conversion Table for Given Amounts of a Physical Quantity

The table is arranged so that a given amount of some physical quantity, expressed as so many SI or Gaussian units of that quantity, can be expressed as an equivalent number of units in the other system. Thus the entries in each row stand for the same amount, expressed in different units. All factors of 3 (apart from exponents) should, for accurate work, be replaced by (2.997 924 58), arising from the numerical value of the velocity of light. For example, in the row for displacement (D), the entry ($12\pi \times 10^5$) is actually ($2.997\,924\,58 \times 4\pi \times 10^5$) and "9" is actually $10^{-16}\, c^2 = 8.987\,55\ldots$. Where a name for a unit has been agreed on or is in common usage, that name is given. Otherwise, one merely reads so many Gaussian units, or SI units.

Jackson pg. 783

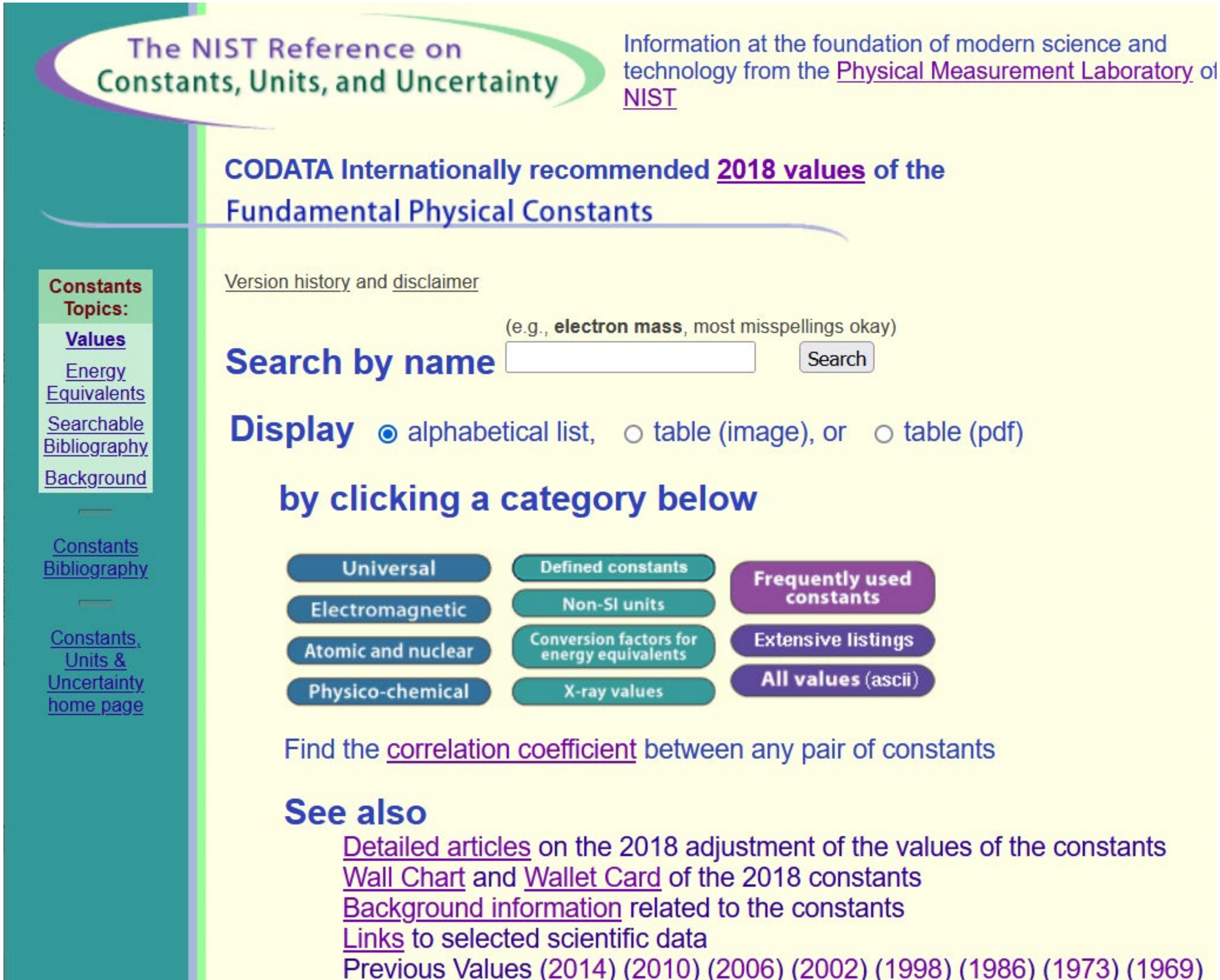
Note that some of the “fundamental” constants change slightly over the years……

Physical Quantity	Symbol	SI		Gaussian
Length	l	1 meter (m)	10^2	centimeters (cm)
Mass	m	1 kilogram (kg)	10^3	grams (g)
Time	t	1 second (s)	1	second (s)
Frequency	ν	1 hertz (Hz)	1	hertz (Hz)
Force	F	1 newton (N)	10^5	dynes
Work / Energy	W / U	1 joule (J)	10^7	ergs
Power	P	1 watt (W)	10^7	ergs s^{-1}
Charge	q	1 coulomb (C)	3×10^9	statcoulombs
Charge density	ρ	1 C m^{-3}	3×10^3	statcoul cm^{-3}
Current	I	1 ampere (A)	3×10^9	statamperes
Current density	J	1 A m^{-2}	3×10^5	statamp cm^{-2}
Electric field	E	1 volt m^{-1} (Vm^{-1})	$\frac{1}{3} \times 10^{-4}$	statvolt cm^{-1}
Potential	Φ, V	1 volt (V)	$\frac{1}{300}$	statvolt
Polarization	P	1 C m^{-2}	3×10^5	dipole moment cm^{-3}
Displacement	D	1 C m^{-2}	$12\pi \times 10^5$	statvolt cm^{-1} (statcoul cm^{-2})
Conductivity	σ	1 mho m^{-1}	9×10^9	s^{-1}
Resistance	R	1 ohm (Ω)	$\frac{1}{9} \times 10^{-11}$	s cm^{-1}
Capacitance	C	1 farad (F)	9×10^{11}	cm
Magnetic flux	ϕ, F	1 weber (Wb)	10^8	gauss cm^2 or maxwells
Magnetic induction	B	1 tesla (T)	10^4	gauss (G)
Magnetic field	H	1 A m^{-1}	$4\pi \times 10^{-3}$	oersted (Oe)
Magnetization	M	1 A m^{-1}	10^{-3}	magnetic moment cm^{-3}
Inductance*	L	1 henry (H)	$\frac{1}{9} \times 10^{-11}$	

*There is some confusion about the unit of inductance in Gaussian units. This stems from the use by some authors of a modified system of Gaussian units in which current is measured in electromagnetic units, so that the connection between charge and current is $I_m = (1/c)(dq/dt)$. Since inductance is defined through the induced voltage $V = L(dI/dt)$ or the energy $U = \frac{1}{2}LI^2$, the choice of current defined in Section 2 means that our Gaussian unit of inductance is equal in magnitude and dimensions (t^2l^{-1}) to the electrostatic unit of inductance. The electromagnetic current I_m is related to our Gaussian current I by the relation $I_m = (1/c)I$. From the energy definition of inductance, we see that the electromagnetic inductance L_m is related to our Gaussian inductance L through $L_m = c^2L$. Thus L_m has the dimensions of length. The modified Gaussian system generally uses the electromagnetic unit of inductance, as well as current. Then the voltage relation reads $V = (L_m/c)(dI_m/dt)$. The numerical connection between units of inductance is

$$1 \text{ henry} = \tfrac{1}{9} \times 10^{-11} \text{ Gaussian (es) unit} = 10^9 \text{ emu}$$

Source for standard measurements – https://physics.nist.gov/cuu/Constants/index.html



Vector relations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, and $f(r)$ is a well-behaved function of r, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$

$$\nabla\cdot[\mathbf{n}f(r)] = \frac{2}{r}f + \frac{\partial f}{\partial r} \qquad \nabla\times[\mathbf{n}f(r)] = 0$$

$$(\mathbf{a}\cdot\nabla)\mathbf{n}f(r) = \frac{f(r)}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] + \mathbf{n}(\mathbf{a}\cdot\mathbf{n})\frac{\partial f}{\partial r}$$

$$\nabla(\mathbf{x}\cdot\mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla\cdot\mathbf{a}) + i(\mathbf{L}\times\mathbf{a})$$

where $\mathbf{L} = \frac{1}{i}(\mathbf{x}\times\nabla)$ is the angular-momentum operator.

In the following ϕ, ψ, and $\mathbf{A}$ are well-behaved scalar or vector functions, V is a three-dimensional volume with volume element d^3x, S is a closed two-dimensional surface bounding V, with area element da and unit outward normal $\mathbf{n}$ at da.

$$\int_V \nabla \cdot \mathbf{A}\, d^3x = \int_S \mathbf{A} \cdot \mathbf{n}\, da \qquad \text{(Divergence theorem)}$$

$$\int_V \nabla\psi\, d^3x = \int_S \psi\mathbf{n}\, da$$

$$\int_V \nabla \times \mathbf{A}\, d^3x = \int_S \mathbf{n} \times \mathbf{A}\, da$$

$$\int_V (\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi)\, d^3x = \int_S \phi\mathbf{n} \cdot \nabla\psi\, da \qquad \text{(Green's first identity)}$$

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot \mathbf{n}\, da \qquad \text{(Green's theorem)}$$

In the following S is an open surface and C is the contour bounding it, with line element $d\mathbf{l}$. The normal $\mathbf{n}$ to S is defined by the right-hand-screw rule in relation to the sense of the line integral around C.

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da = \oint_C \mathbf{A} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}$$

$$\int_S \mathbf{n} \times \nabla\psi\, da = \oint_C \psi\, d\mathbf{l}$$

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$

Comment on cartesian unit vectors versus local (cylindrical or spherical) unit vectors

$$\hat{\mathbf{r}} = \sin\theta\ \cos\phi\ \hat{\mathbf{x}} + \sin\theta\ \sin\phi\ \hat{\mathbf{y}} + \cos\theta\ \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\ \cos\phi\ \hat{\mathbf{x}} + \cos\theta\ \sin\phi\ \hat{\mathbf{y}} - \sin\theta\ \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\varphi}} = \ -\sin\phi\ \hat{\mathbf{x}} + \cos\phi\ \hat{\mathbf{y}}$$

Note that Cartesian unit coordinates are fixed in space, but curvilinear unit coordinates change with angle.

Note that $\nabla^2 \mathbf{A} = \left(\nabla^2 A_x\right)\hat{\mathbf{x}} + \left(\nabla^2 A_y\right)\hat{\mathbf{y}} + \left(\nabla^2 A_z\right)\hat{\mathbf{z}}$

Also note that $\nabla^2 f(r) = \dfrac{\partial^2 f(r)}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial f(r)}{\partial r}$

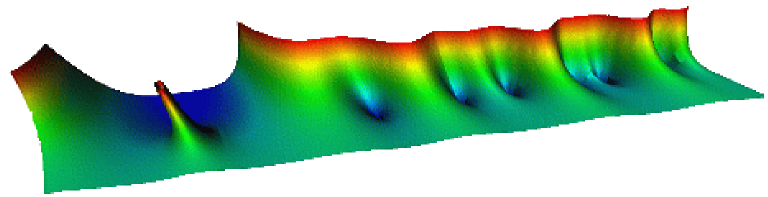
Special functions -- many are described in Jackson
Additional source [-- https://dlmf.nist.gov/](https://dlmf.nist.gov/)

Digital Library of Mathematical Functions

Index
Notations
Search
Help?
Citing
Customize

About the Project

NIST
National Institute of Standards and Technology
U.S. Department of Commerce



NIST Digital Library of Mathematical Functions

Project News

2022-03-15 *DLMF Update; Version 1.1.5*
2022-01-15 *DLMF Update; Version 1.1.4*
2021-09-15 *DLMF Update; Version 1.1.3*
2021-07-19 *Brian D. Sleeman, Associate Editor of the DLMF, dies at age 81*
More news

Foreword
Preface
Mathematical Introduction
1 Algebraic and Analytic Methods
2 Asymptotic Approximations
3 Numerical Methods
4 Elementary Functions
5 Gamma Function
6 Exponential, Logarithmic, Sine, and Cosine Integrals
7 Error Functions, Dawson's and Fresnel Integrals
8 Incomplete Gamma and Related Functions
9 Airy and Related Functions
10 Bessel Functions
20 Theta Functions
21 Multidimensional Theta Functions
22 Jacobian Elliptic Functions
23 Weierstrass Elliptic and Modular Functions
24 Bernoulli and Euler Polynomials
25 Zeta and Related Functions
26 Combinatorial Analysis
27 Functions of Number Theory
28 Mathieu Functions and Hill's Equation
29 Lamé Functions
30 Spheroidal Wave Functions
31 Heun Functions
32 Painlevé Transcendents
33 Coulomb Functions
34 $3j$, $6j$, $9j$ Symbols

Basic equations of electrodynamics

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

CGS (Gaussian)	SI
$\nabla \cdot \mathbf{D} = 4\pi\rho$	$\nabla \cdot \mathbf{D} = \rho$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
$u = \frac{1}{8\pi}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$	$u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$
$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{H})$	$\mathbf{S} = (\mathbf{E} \times \mathbf{H})$

More relationships

CGS (Gaussian)		MKS (SI)
$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}$		$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E}$
$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = \frac{1}{\mu}\mathbf{B}$		$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} = \frac{1}{\mu}\mathbf{B}$
$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$		$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$
$\mathbf{B} = \nabla \times \mathbf{A}$		$\mathbf{B} = \nabla \times \mathbf{A}$
ϵ	$\Leftrightarrow$	ϵ / ϵ_0
μ	$\Leftrightarrow$	μ / μ_0

More SI relationships:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \qquad \mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}) \qquad \mathbf{B} = \mu \mathbf{H} \qquad \mathbf{B} = F(\mathbf{H})$$

for ferromagnet

More Gaussian relationships:

$$\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P} \qquad \mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{H} = \mathbf{B} - 4\pi \mathbf{M}) \qquad \mathbf{B} = \mu \mathbf{H} \qquad \mathbf{B} = F(\mathbf{H})$$

for ferromagnet

elementary charge: $e = 1.6021766208 \times 10^{-19}$ C

$= 4.80320467299766 \times 10^{-10}$ statC

Energy and power (SI units)

Electromagnetic energy density: $u \equiv \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}\right)$

Poynting vector: $\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$

Equations for time harmonic fields:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^{*}(\mathbf{r},\omega)e^{i\omega t}\right)$$

$$\left\langle u\left(\mathbf{r},t\right)\right\rangle_{t\ \mathrm{avg}} = \frac{1}{4}\Re\left(\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\cdot\tilde{\mathbf{D}}^{*}(\mathbf{r},\omega)+\tilde{\mathbf{B}}(\mathbf{r},\omega)\cdot\tilde{\mathbf{H}}^{*}(\mathbf{r},\omega)\right)\right)$$

$$\left\langle \mathbf{S}\left(\mathbf{r},t\right)\right\rangle_{t\ \mathrm{avg}} = \frac{1}{2}\Re\left(\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{H}}^{*}(\mathbf{r},\omega)\right)\right) \quad \text{(omitting } e^{\pm 2i\omega t} \text{ terms)}$$

Solution of Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

Introduction of vector and scalar potentials:

$$\nabla \cdot \mathbf{B} = 0 \qquad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi \quad \text{or} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Scalar and vector potentials continued:

$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 :$

$$-\nabla^2 \Phi - \frac{\partial \left(\nabla \cdot \mathbf{A} \right)}{\partial t} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times \left(\nabla \times \mathbf{A} \right) + \frac{1}{c^2} \left(\frac{\partial \left(\nabla \Phi \right)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

Analysis of the scalar and vector potential equations:

$$-\nabla^2\Phi - \frac{\partial(\nabla\cdot\mathbf{A})}{\partial t} = \rho/\varepsilon_0$$

$$\nabla\times(\nabla\times\mathbf{A}) + \frac{1}{c^2}\left(\frac{\partial(\nabla\Phi)}{\partial t} + \frac{\partial^2\mathbf{A}}{\partial t^2}\right) = \mu_0\mathbf{J}$$

Lorenz gauge form -- require $\nabla\cdot\mathbf{A}_L + \frac{1}{c^2}\frac{\partial\Phi_L}{\partial t} = 0$

$$-\nabla^2\Phi_L + \frac{1}{c^2}\frac{\partial^2\Phi_L}{\partial t^2} = \rho/\varepsilon_0$$

$$-\nabla^2\mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} = \mu_0\mathbf{J}$$

Solution methods for scalar and vector potentials and their electrostatic and magnetostatic analogs:

$$-\nabla^2\Phi_L + \frac{1}{c^2}\frac{\partial^2\Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2\mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} = \mu_0\mathbf{J}$$

In your "bag" of tricks:

- ❑ **Direct (analytic or numerical) solution of differential equations**
- ❑ **Solution by expanding in appropriate orthogonal functions**
- ❑ **Green's function techniques**

How to choose most effective solution method --

- In general, Green's functions methods work well when source is contained in a finite region of space

Consider the electrostatic problem:

$$-\nabla^2\Phi_L = \rho / \varepsilon_0$$

Define: $\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$

$$\Phi_L(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r' \left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

For electrostatic problems where $\rho(\mathbf{r})$ is contained in a small region of space and $S \to \infty$, $\quad G(\mathbf{r},\mathbf{r}') = \dfrac{1}{|\mathbf{r}-\mathbf{r}'|}$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^{\,l}}{r_>^{\,l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^{\,*}(\theta',\varphi')$$

Example electrostatic problem with spherical polar geometry

PHY 712 -- Assignment #8

Assigned: 2/3/2025 Due: 2/5/2025

Complete reading Chapter 3 and start Chapter 4 in **Jackson** .

1. Consider the charge density of an electron bound to a proton in the ground state of a hydrogen atom -- $\rho(r) = (1/\pi a_0^3)\, e^{-2r/a_0}$, where a_0 denotes the Bohr radius. Find the electrostatic potential $\Phi(r)$ associated with $\rho(r)$. Compare your result to the electrostatic potential given in HW#2.

Note that the electron has a charge of -e (e indicating the elementary charge) so that we should really write

$$> \text{rho} := r \rightarrow -\frac{ee}{\text{Pi}\cdot a^3}\cdot \exp\left(-\frac{2\cdot r}{a}\right)$$

$$\rho := r \mapsto -\frac{ee\cdot e^{-\frac{2\cdot r}{a}}}{\pi\cdot a^3}$$

$$> assume(a > 0)$$

$$> check := 4\cdot \text{Pi}\cdot int(\text{rho}(x)\cdot x^2, x = 0..\text{infinity})$$

$$check := -ee$$

$$> \ \mathrm{Phi} := r \rightarrow \frac{4 \cdot \mathrm{Pi}}{4 \cdot \mathrm{Pi} \cdot epsilon0} \cdot \left(\frac{1}{r} \cdot int\left(\mathrm{rho}(x) \cdot x^2, x = 0 .. r\right) + int(\mathrm{rho}(x) \cdot x, x = r .. \mathrm{infinity}) \right)$$

$$\Phi := r \mapsto \frac{1 \cdot \pi \cdot \left(\frac{\int_0^r \rho(x) \cdot x^2 \, dx}{r} + \int_r^\infty \rho(x) \cdot x \, dx \right)}{\pi \cdot \epsilon 0}$$

$$> \ simplify(\mathrm{Phi}(r))$$

$$\frac{ee \left((a\sim + r) \, e^{-\frac{2r}{a\sim}} - a\sim \right)}{4 \, a\sim \, \epsilon 0 \, r \, \pi}$$

Using same notation:

$$> \ assume(a > 0)$$

$$> \ PhiHW2 := r \rightarrow \frac{ee}{4 \cdot \mathrm{Pi} \cdot epsilon0 \cdot r} \cdot \exp\left(-\frac{2 \cdot r}{a} \right) \cdot \left(1 + \frac{r}{a} \right)$$

$$PhiHW2 := r \mapsto \frac{ee \cdot e^{-\frac{2 \cdot r}{a}} \cdot \left(1 + \frac{r}{a} \right)}{4 \cdot \pi \cdot \epsilon 0 \cdot r}$$

$$> \ simplify(PhiHW2(r) - \mathrm{Phi}(r))$$

$$\frac{ee}{4 \, r \, \pi \, \epsilon 0}$$

Electromagnetic waves from time harmonic sources

Charge density: $\rho(\mathbf{r},t) = \Re\left(\tilde{\rho}(\mathbf{r},\omega)e^{-i\omega t}\right)$

Current density: $\mathbf{J}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{J}}(\mathbf{r},\omega)e^{-i\omega t}\right)$

Note that the continuity condition:

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t} + \nabla\cdot\mathbf{J}(\mathbf{r},t) = 0 \quad \Rightarrow -i\omega\tilde{\rho}(\mathbf{r},\omega) + \nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega) = 0$$

For dynamic problems where $\tilde{\rho}(\mathbf{r},\omega)$ and $\tilde{\mathbf{J}}(\mathbf{r},\omega)$ are contained in a small region of space and $S \to \infty$,

$$\tilde{G}(\mathbf{r},\mathbf{r}',\omega) = \frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$$

Electromagnetic waves from time harmonic sources – continued:

For scalar potential (Lorenz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

For vector potential (Lorenz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Electromagnetic waves from time harmonic sources – continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm} \tilde{\phi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

Electromagnetic waves from time harmonic sources – continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*{}_{lm}(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \sum_{lm} \tilde{\mathbf{a}}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega) = ik\mu_0 \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}',\omega) j_l(kr_<)h_l(kr_>)Y^*{}_{lm}(\hat{\mathbf{r}}')$$

Example from HW 20

PHY 712 -- Assignment #20

Assigned: 3/24/2025 Due: 3/26/2025

Continue reading Chapter 9 (Sec. 9.1-9.4) in **Jackson** .

1. Problem 9.10 in **Jackson** lists the harmonic frequency dependent charge and current densities of a radiating H atom. Instead of answering **Jackson's** questions, calculate the exact scalar $\Phi(\mathbf{r},\omega_0)$ field for $r>>a_0$ and compare your results with the scalar potential field calculated within the dipole approximation.

$$\rho(\mathbf{r},t) = \frac{2e}{\sqrt{24\pi a_0^4}} Y_{10}(\hat{\mathbf{r}}) r e^{-3r/(2a_0)-i\omega_0 t}$$

$$\mathbf{J}(\mathbf{r},t) = -i\frac{e^2}{4\pi\epsilon_0\hbar}\left(\frac{\hat{\mathbf{r}}}{2}+\frac{a_0}{z}\hat{\mathbf{z}}\right)\rho(\mathbf{r},t)$$

← Don't need to use this actually

$$\omega_0 = \frac{3}{4}\frac{e^2}{8\pi\epsilon_0\hbar a_0}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm} \tilde{\varphi}_{lm}(r,\omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\varphi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0} \int d^3r' \, \tilde{\rho}(\mathbf{r}',\omega) \, j_l(kr_<) \, h_l(kr_>) \, Y^*_{lm}(\hat{\mathbf{r}}')$$

In our case, $k = \omega_0 / c$

$$\Phi_{10}(r,t) = \frac{i\omega_0}{c\epsilon_0} \frac{2e}{\sqrt{24\pi a_0^4}} h_1(kr) e^{-i\omega_0 t} \int_0^\infty dr' \, r'^3 \, e^{-3r'/(2a_0)} \left(\frac{\sin(kr')}{(kr')^2} - \frac{\cos(kr')}{kr'} \right)$$

Results of exact integration compared with dipole approximation –

> $assume(a > 0); \; assume(k > 0);$

> $int\left(x^3 \cdot \exp\left(-\frac{3 \cdot x}{2 \cdot a} \right) \cdot \left(\frac{\sin(k \cdot x)}{k^2 \cdot x^2} - \frac{\cos(k \cdot x)}{k \cdot x} \right), x = 0 ..infinity \right);$ Exact

$$\frac{768 \, a\text{\textasciitilde}^5 \, k\text{\textasciitilde}}{\left(4 \, k\text{\textasciitilde}^2 \, a\text{\textasciitilde}^2 + 9\right)^3}$$

> $int\left(x^3 \cdot \exp\left(-\frac{3 \cdot x}{2 \cdot a} \right) \cdot \left(\frac{k \cdot x}{3} \right), x = 0 ..infinity \right);$ Dipole approximation

$$\frac{256 \, a\text{\textasciitilde}^5 \, k\text{\textasciitilde}}{243}$$

$$\Phi_{10}(r,t) = -\frac{i\omega_0}{c\epsilon_0}\frac{2e}{\sqrt{24\pi}}\left(\frac{e^{ikr}}{kr}\left(1+\frac{i}{kr}\right)\right)e^{-i\omega_0 t}\left(\frac{768ka_0}{\left(4k^2a_0^2+9\right)^3}\right) \text{ exact integral for } r >> a_0$$

$$\Phi_{10}(r,t) = -\frac{i\omega_0}{c\epsilon_0}\frac{2e}{\sqrt{24\pi}}\left(\frac{e^{ikr}}{kr}\left(1+\frac{i}{kr}\right)\right)e^{-i\omega_0 t}\left(\frac{256ka_0}{243}\right) \text{ dipole approximation}$$

Results are the same when $ka_0 << 1$

Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = -\nabla\tilde{\Phi}(\mathbf{r},\omega) + i\omega\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla \times \tilde{\mathbf{A}}(\mathbf{r},\omega)$$

Power radiated :

$$\frac{dP}{d\Omega} = r^2\hat{\mathbf{r}} \cdot \langle \mathbf{S} \rangle_{avg} = \frac{r^2\hat{\mathbf{r}}}{2\mu_0}\hat{\mathbf{r}} \cdot \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega) \times \tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right)$$

Example of time harmonic radiation source

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega) = \frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \left(ik\mu_0\right)\int_0^\infty r'^2\, dr'\, e^{-r'/R} h_0(kr_>) j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{J_0 k}{\varepsilon_0 \omega R}\cos\theta\int_0^\infty r'^2\, dr'\, e^{-r'/R} h_1(kr_>) j_1(kr_<)$$

Evaluation for $r >> R$:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r}\ \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = \frac{J_0 k}{\varepsilon_0 \omega}\cos\theta\,\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\ \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

Note that this result is valid for all *k*.

Example of time harmonic radiation source -- continued

Evaluation for $r >> R$:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = \frac{J_0 k}{\varepsilon_0 \omega} \cos\theta \frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right) \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

Relationship to pure dipole approximation (exact when $kR \rightarrow 0$)

$$\mathbf{p}(\omega) \equiv \int d^3r\, \mathbf{r}\tilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega}\int d^3r\, \tilde{\mathbf{J}}(\mathbf{r},\omega) = -\frac{8\pi R^3 J_0}{i\omega}\hat{\mathbf{z}}$$

Corresponding dipole fields: $\tilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{i}{4\pi\omega\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$

Electromagnetic waves from time harmonic sources – for dipole radiation --:

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = -\nabla\tilde{\Phi}(\mathbf{r},\omega) + i\omega\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left(k^2\left(\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right) + \left(\frac{3\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\mathbf{p}(\omega)\right)-\mathbf{p}(\omega)}{r^2}\right)(1-ikr)\right)$$

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla\times\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0 c^2}\frac{e^{ikr}}{r}k^2\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\left(1-\frac{1}{ikr}\right)$$

Power radiated for $kr >> 1$:

$$\frac{dP}{d\Omega} = r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg} = \frac{r^2\hat{\mathbf{r}}}{2\mu_0}\hat{\mathbf{r}}\cdot\Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right)$$

$$= \frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right|^2$$

Another useful approximation for analyzing radiation for time harmonic sources --

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{e^{ikr}}{r} e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \qquad k \equiv \frac{\omega}{c}$$

for $r >> r'$

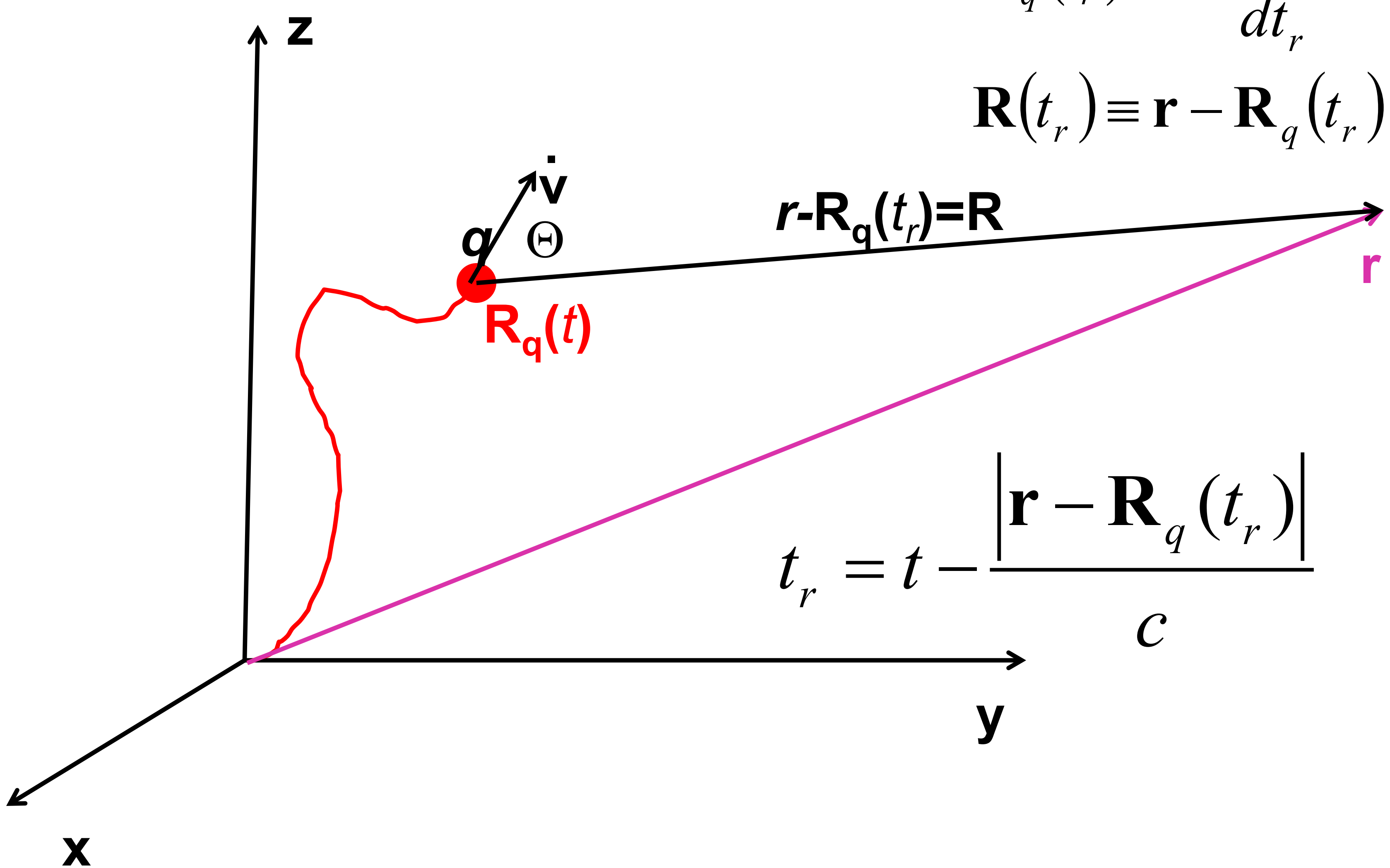
often used for antenna radiation

Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$



$$t_r = t - \frac{\left|\mathbf{r} - \mathbf{R}_q(t_r)\right|}{c}$$

Liènard-Wiechert potentials –(Gaussian units)

$$\dot{\mathbf{R}}_q\left(t_r\right) \equiv \frac{d\mathbf{R}_q\left(t_r\right)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}\left(t_r\right) \equiv \mathbf{r} - \mathbf{R}_q\left(t_r\right) \equiv \mathbf{R}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R}\times\left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}.$$

Electric and magnetic fields far from accelerating source:

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left\{\mathbf{R}\times\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}$$

Let $\hat{\mathbf{R}} \equiv \frac{\mathbf{R}}{R}$ $\quad \boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c}$ $\quad \dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{v}}}{c}$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

Poynting vector:

$$\mathbf{S}(\mathbf{r},t)=\frac{c}{4\pi}(\mathbf{E}\times\mathbf{B})$$

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{cR\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t)=\hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{S}(\mathbf{r},t)=\frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2=\frac{q^2}{4\pi cR^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

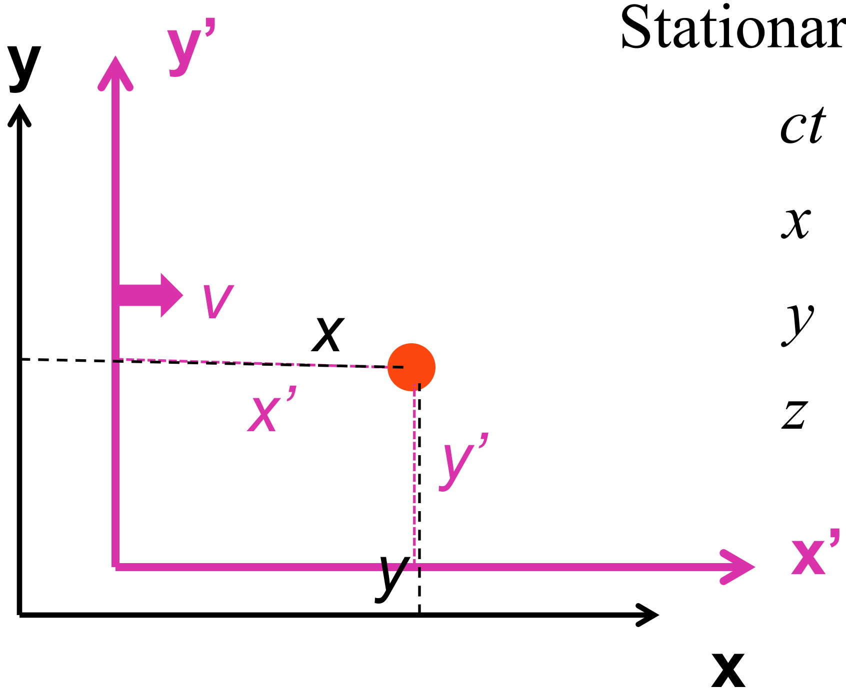
$$\frac{dP}{d\Omega}=\mathbf{S}\cdot\hat{\mathbf{R}}R^2=\frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

Lorentz transformations

Convenient notation :

$$\beta_v \equiv \frac{v}{c}$$

$$\gamma_v \equiv \frac{1}{\sqrt{1-\beta_v{}^2}}$$



Stationary frame		Moving frame
ct	$=$	$\gamma(ct'+\beta x')$
x	$=$	$\gamma(x'+\beta ct')$
y	$=$	y'
z	$=$	z'

Note, the concept of the Lorentz transformation is quite general, but the specific transformation form given in the following slides is special to the relative velocity along the x-axis.

Lorentz transformations -- continued

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \mathcal{L}_v \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \mathcal{L}_v^{-1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice:

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

Velocity relationships

Consider: $u_x = \dfrac{u'_x + v}{1 + vu'_x/c^2}$ $\quad u_y = \dfrac{u'_y}{\gamma_v\left(1 + vu'_x/c^2\right)}$ $\quad u_z = \dfrac{u'_z}{\gamma_v\left(1 + vu'_x/c^2\right)}.$

Note that $\gamma_u \equiv \dfrac{1}{\sqrt{1-(u/c)^2}} = \dfrac{1 + vu'_x/c^2}{\sqrt{1-(u'/c)^2}\sqrt{1-(v/c)^2}} = \gamma_v\gamma_{u'}\left(1 + vu'_x/c^2\right)$

$$\Rightarrow \gamma_u c = \gamma_v\left(\gamma_{u'}c + \beta_v\gamma_{u'}u'_x\right)$$

$$\Rightarrow \gamma_u u_x = \gamma_v\left(\gamma_{u'}u'_x + \gamma_{u'}v\right) = \gamma_v\left(\gamma_{u'}u'_x + \beta_v\gamma_{u'}c\right)$$

$$\Rightarrow \gamma_u u_y = \gamma_{u'}u'_y \qquad \gamma_u u_z = \gamma_{u'}u'_z$$

$$\rightarrow \quad \begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \mathcal{L}_v \begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} u'_x \\ \gamma_{u'} u'_y \\ \gamma_{u'} u'_z \end{pmatrix}$$

Field strength tensor $F^{\alpha\beta} \equiv \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)$

For stationary frame

$$F^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

For moving frame

$$F'^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -B'_z & B'_y \\ E'_y & B'_z & 0 & -B'_x \\ E'_z & -B'_y & B'_x & 0 \end{pmatrix}$$

➔ This analysis shows that the E and B fields must be treated as components of the field strength tensor and that in order to transform between inertial frames, we need to use the tensor transformation relationships:

Transformation of field strength tensor

$$F^{\alpha\beta} = \mathcal{L}_v{}^{\alpha\gamma} F'^{\gamma\delta} \mathcal{L}_v{}^{\delta\beta} \qquad \mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E'_x & -\gamma_v\left(E'_y+\beta_v B'_z\right) & -\gamma_v\left(E'_z-\beta_v B'_y\right) \\ E'_x & 0 & -\gamma_v\left(B'_z+\beta_v E'_y\right) & \gamma_v\left(B'_y-\beta_v E'_z\right) \\ \gamma_v\left(E'_y+\beta_v B'_z\right) & \gamma_v\left(B'_z+\beta_v E'_y\right) & 0 & -B'_x \\ \gamma_v\left(E'_z-\beta_v B'_y\right) & -\gamma_v\left(B'_y-\beta_v E'_z\right) & B'_x & 0 \end{pmatrix}$$

Inverse transformation of field strength tensor

$$F'^{\alpha\beta} = \mathcal{L}_v^{-1\alpha\gamma} F^{\gamma\delta} \mathcal{L}_v^{-1\delta\beta} \qquad \mathcal{L}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F'^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -\gamma_v\left(E_y - \beta_v B_z\right) & -\gamma_v\left(E_z + \beta_v B_y\right) \\ E_x & 0 & -\gamma_v\left(B_z - \beta_v E_y\right) & \gamma_v\left(B_y + \beta_v E_z\right) \\ \gamma_v\left(E_y - \beta_v B_z\right) & \gamma_v\left(B_z - \beta_v E_y\right) & 0 & -B_x \\ \gamma_v\left(E_z + \beta_v B_y\right) & -\gamma_v\left(B_y + \beta_v E_z\right) & B_x & 0 \end{pmatrix}$$

Summary of results:

$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y = \gamma_v\left(E_y - \beta_v B_z\right) \qquad B'_y = \gamma_v\left(B_y + \beta_v E_z\right)$$

$$E'_z = \gamma_v\left(E_z + \beta_v B_y\right) \qquad B'_z = \gamma_v\left(B_z - \beta_v E_y\right)$$