PHY 712 Electrodynamics
10-10:50 AM in Olin 103

Class notes for Lecture 4:

Reading: Chapter 1 in JDJ

1. **Review of electrostatics with one-dimensional examples**
2. **Poisson and Laplace Equations**
3. **Green's Theorem and its use in electrostatics**

Physics Colloquium

- Thursday - in Olin 101 at 4 PM

January 23, 2025

Forging a Multi-Messenger View of Supermassive Black Hole Binaries

Supermassive black hole binaries lurk deep within the cores of post-merger galaxies, and their identification represents the only key to unlock previously impossible probes of gravity, galaxy evolution, and the structure of the cosmos. While electromagnetic signatures probe plasma and gas in the environment around a binary, only pulsar timing arrays are currently sensitive to the low-frequency gravitational waves emitted directly by these slow-evolving giants. Pulsar timing array experiments have reached a critical turning point, and have recently announced that they have at last uncovered evidence of the stochastic gravitational wave background. The window to the gravitational-wave universe has been widened as we are now able to expand our view to include nanohertz gravitational wave frequencies, and will now turn our eye towards gravitational waves emitted directly by individual binary systems. Simultaneously, our electromagnetic capabilities to study the variable universe are on the brink of a new paradigm that will be opened by Rubin, Roman, and their dedicated surveys. In this talk, I will give an overview of electromagnetic, gravitational wave-, and multi-messenger studies of supermassive black hole

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Northwestern University

Reception 3:30

Course schedule for Spring 2025

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/13/2025	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/15/2025
2	Wed: 01/15/2025	Chap. 1	Electrostatic energy calculations	#2	01/17/2025
3	Fri: 01/17/2025	Chap. 1	Electrostatic energy calculations	#3	01/22/2025
	Mon: 01/20/2025	No Class	Martin Luther King Jr. Holiday		
4	Wed: 01/22/2025	Chap. 1	Electrostatic potentials and fields	#4	01/24/2025
5	Fri: 01/24/2025	Chap. 1 - 3	Poisson's equation in multiple dimensions	#5	01/29/2025

PHY 712 – Problem Set #4

Assigned: 01/22/2025 Due: 01/24/2025

Continue reading Chaper 1 in **Jackson**

1. Consider a one-dimensional charge distribution of the form:

$$\rho(x) = \begin{cases} 0 & \text{for} \quad x < -a \\ \rho_0 \sin(\pi x/a) & \text{for} \quad -a \leq x \leq a \\ 0 & \text{for} \quad x > a, \end{cases}$$

where ρ_0 and a are constants.

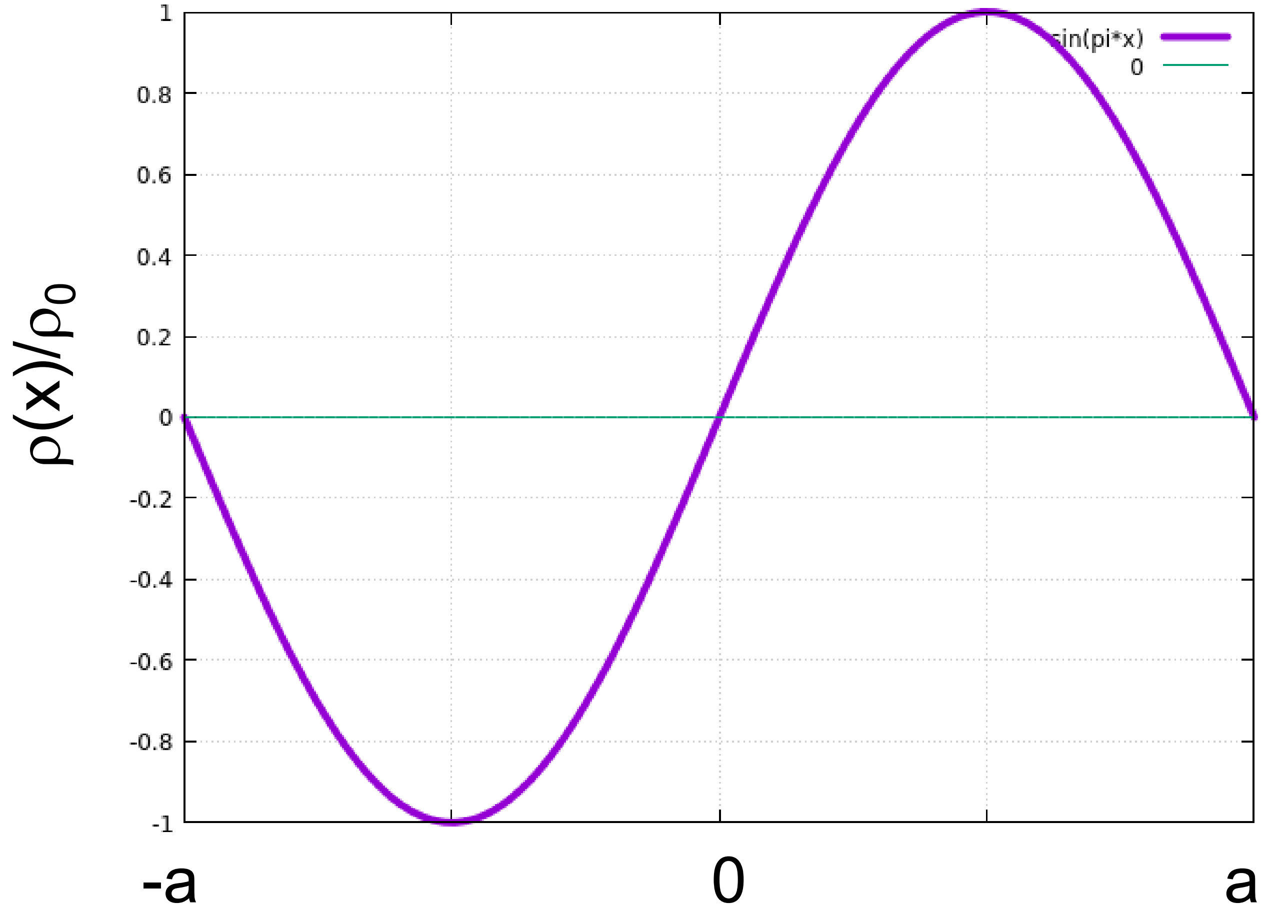
(a) Solve the Poisson equation for the electrostatic potential $\Phi(x)$ with the boundary conditions $\Phi(-a) = 0$ and $\frac{d\Phi}{dx}(a) = 0$.

(b) Find the corresponding electrostatic field $E(x)$.

(c) Plot $\Phi(x)$ and $E(x)$.

(d) Discuss your results in terms of possible electronic devices.

$$\frac{d^2\Phi}{dx^2} = -\frac{\rho(x)}{\varepsilon_0}$$



Poisson and Laplace Equations

We are concerned with finding solutions to the Poisson equation:

$$\nabla^2 \Phi_P(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

and the Laplace equation:

$$\nabla^2 \Phi_L(\mathbf{r}) = 0$$

The Laplace equation is the “homogeneous” version of the Poisson equation. The Green's theorem allows us to determine the electrostatic potential from volume and surface integrals:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

Poisson equation -- continued

Note that we have previously shown that the differential and integral forms of Coulomb's law is given by:

$$\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \quad \text{and} \quad \Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Generalization of analysis for non-trivial boundary conditions:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r' \left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla'G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

General comments on Green's theorem

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \rho(\mathbf{r}') G(\mathbf{r},\mathbf{r}') +$$
$$\frac{1}{4\pi}\int_S d^2r' \left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

This general form can be used in 1, 2, or 3 dimensions. In general, the Green's function must be constructed to satisfy the appropriate (Dirichlet or Neumann) boundary conditions. Alternatively, or in addition, boundary conditions can be adjusted using the fact that for any solution to the Poisson equation, $\Phi_P(\mathbf{r})$ other solutions may be generated by use of solutions of the Laplace equation

$$\Phi(\mathbf{r}) = \Phi_P(\mathbf{r}) + C\Phi_L(\mathbf{r}), \text{for any constant } C.$$

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The Green of Green Functions

In 1828, an English miller from Nottingham published a mathematical essay that generated little response. George Green's analysis, however, has since found applications in areas ranging from classical electrostatics to modern quantum field theory.

Lawrie Challis and Fred Sheard

Nottingham, an attractive and thriving town in the English Midlands, is famous for its association with Robin Hood, whose statue stands in the shadow of the castle wall. The Sheriff of Nottingham still has a special role in the city government although happily no longer strikes terror into the hearts of the good citizens.

Recently a new attraction, a windmill, has appeared on the Nottingham skyline (see figure 1). The sails turn on windy days and the adjoining mill shop sells packets of stone ground flour but also, more surprisingly, tracts on mathematical physics. The connection between the flour and the physics is part of the mill's unique character and is explained by a plaque once attached to the side of the mill tower that said,

HERE LIVED AND LABOURED
GEORGE GREEN
MATHEMATICIAN
B.1793–D.1841.

his family built a house next to the mill, Green spent most of his days and many of his nights working and indeed living in the mill. When he was 31, Jane Smith bore him a daughter. They had seven children in all but never married. It was said that Green's father felt that Jane was not a suitable wife for the son of a prosperous tradesman and landowner and threatened to disinherit him.

Little is known about Green's life from 1802 until 1823. In particular, it is not known whether he received any help in his mathematical development or if he was entirely self-taught. He may have received help from John Toplis, a fellow of Queens' College in the University of Cambridge and headmaster of the Nottingham Grammar School. Toplis's translation of Pierre-Simon Laplace's book *Mécanique Céleste*, published in Nottingham in 1814, seems a likely source of Green's interest in potential theory. The work was unusual in Britain at that time inasmuch as Toplis used Gottfried Leibniz's more convenient notation for differentials rather than Isaac Newton's. Because Green adapted the Leibniz notation, it seems plausible that Green was influenced by Toplis, but there is no evidence that Toplis acted in any way as his tutor.

In 1823, Green joined the Nottingham Subscription Library, the center of intellectual activity in the town. The library was situated in Bromley House (see figure 2). Library membership provided Green with encouragement

"Derivation" of Green's Theorem

Poisson equation: $$\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

Green's relation: $$\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}').$$

Divergence theorm: $$\int_V d^3r\ \nabla\cdot\mathbf{A} = \oint_S d^2r\ \mathbf{A}\cdot\hat{\mathbf{r}}$$

Let $$\mathbf{A} = f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})$$

$$\int_V d^3r\ \nabla\cdot\left(f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})\right) = \oint_S d^2r\ \left(f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})\right)\cdot\hat{\mathbf{r}}$$

$$\int_V d^3r\ \left(f(\mathbf{r})\nabla^2 g(\mathbf{r}) - g(\mathbf{r})\nabla^2 f(\mathbf{r})\right)$$

"Derivation" of Green's Theorem

Poisson equation: $\nabla^2\Phi(\mathbf{r}) = -\dfrac{\rho(\mathbf{r})}{\varepsilon_0}$

Green's relation: $\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$.

$$\int_V d^3r\, \left(f(\mathbf{r})\nabla^2 g(\mathbf{r}) - g(\mathbf{r})\nabla^2 f(\mathbf{r})\right) = \oint_S d^2r\, \left(f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})\right)\cdot\hat{\mathbf{r}}$$

$$f(\mathbf{r}) \leftrightarrow \Phi(\mathbf{r}) \qquad\qquad g(\mathbf{r}) = G(\mathbf{r},\mathbf{r}')$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

Example of charge density and potential varying in one dimension

Consider the following one dimensional charge distribution:

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$$

We want to find the electrostatic potential such that

$$\frac{d^2\Phi(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0},$$

with the boundary condition $\Phi(-\infty) = 0.$ and $\frac{d\Phi}{dx}(\infty) = 0$

Electrostatic field solution

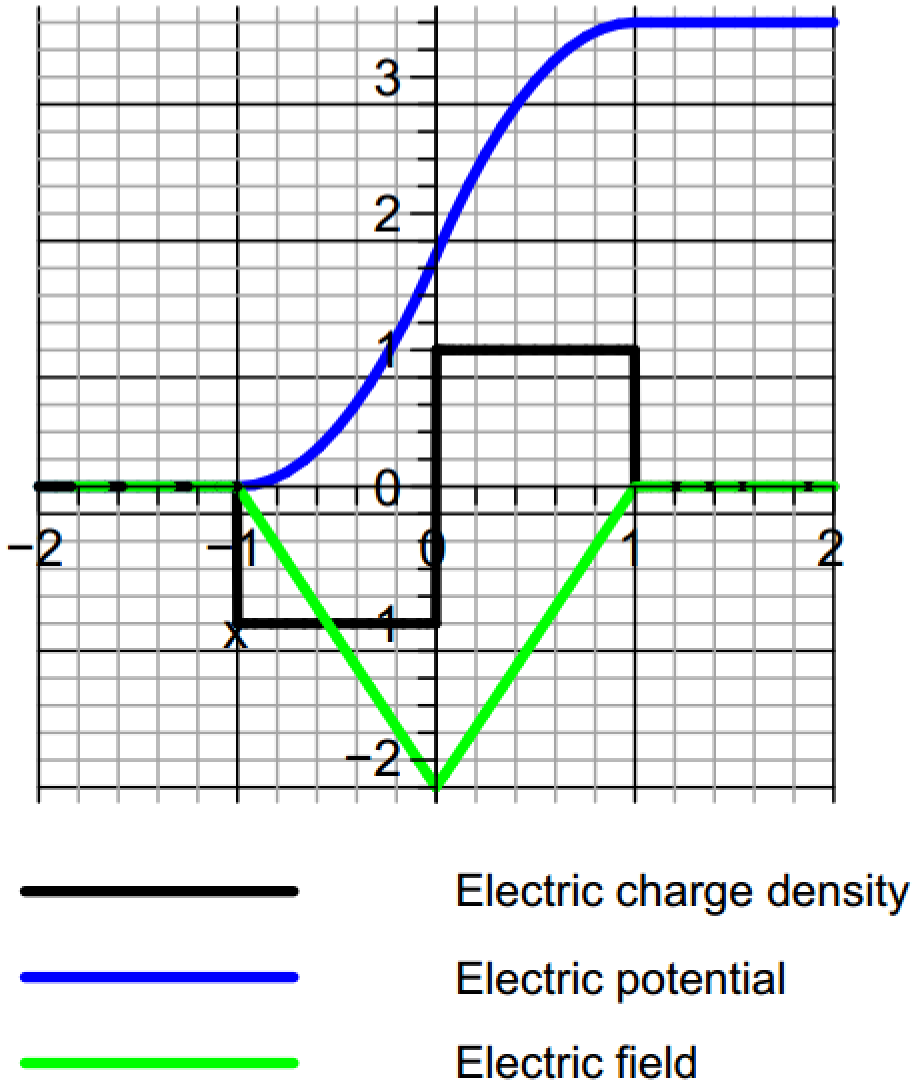
The solution to the Poisson equation is given by:

$$\Phi(x) = \begin{cases} 0 & \text{for } x < -a \\ \frac{\rho_0}{2\varepsilon_0}(x+a)^2 & \text{for } -a < x < 0 \\ -\frac{\rho_0}{2\varepsilon_0}(x-a)^2 + \frac{\rho_0 a^2}{\varepsilon_0} & \text{for } 0 < x < a \\ \frac{\rho_0}{\varepsilon_0}a^2 & \text{for } x > a \end{cases} .$$

Laplace
Poisson
Poisson
Laplace

The electrostatic field is given by:

$$E(x) = \begin{cases} 0 & \text{for } x < -a \\ -\frac{\rho_0}{\varepsilon_0}(x+a) & \text{for } -a < x < 0 \\ \frac{\rho_0}{\varepsilon_0}(x-a) & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases} .$$


3
2
1
0
−1
−2
−2
−1
0
1
2
x
Electric charge density
Electric potential
Electric field

Comment about the example and solution

This particular example is one that is used to model semiconductor junctions where the charge density is controlled by introducing charged impurities near the junction.

The solution of the Poisson equation for this case can be determined by piecewise solution within each of the four regions.   Alternatively, from Green's theorem in one-dimension, one can evaluate the expression --

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} G(x,x')\rho(x')dx' + \frac{1}{4\pi}\left[G(x,x')\frac{d\Phi(x')}{dx'} - \frac{dG(x,x')}{dx'}\Phi(x')\right]_{x'=-\infty}^{x'=\infty}$$

where $G(x,x') = 4\pi x_<$; $x_<$ should be taken as the smaller of x and x'.

Notes on the one-dimensional Green's function

The Green's function for the one-dimensional Poisson equation can be defined as a solution to the equation: $\nabla^2 G(x,x') \equiv \frac{\partial^2}{\partial x^2} G(x,x') = -4\pi\delta(x-x')$

Here the factor of 4π is not really necessary, but ensures consistency with your text's treatment of the 3-dimensional case. The meaning of this expression is that x' is held fixed while taking the derivative with respect to x.

Construction of a Green's function in one dimension

Consider two independent solutions to the homogeneous equation

$$\nabla^2 \phi_i(x) = 0$$

where $i = 1$ or 2. Let

$$G(x, x') = \frac{4\pi}{W} \phi_1(x_<)\phi_2(x_>).$$

This notation means that $x_<$ should be taken as the smaller of x and x' and $x_>$ should be taken as the larger.

W is defined as the "Wronskian":

$$W \equiv \frac{d\phi_1(x)}{dx}\phi_2(x) - \phi_1(x)\frac{d\phi_2(x)}{dx}.$$

Summary

$$\nabla^2 G(x,x') = -4\pi\delta(x-x')$$

$$G(x,x') = \frac{4\pi}{W}\phi_1(x_<)\phi_2(x_>)$$

$$W \equiv \frac{d\phi_1(x)}{dx}\phi_2(x) - \phi_1(x)\frac{d\phi_2(x)}{dx}$$

$$\lim_{\epsilon\to 0}\left(\left.\frac{dG(x,x')}{dx}\right|_{x=x'+\epsilon} - \left.\frac{dG(x,x')}{dx}\right|_{x=x'-\epsilon} \right) = -4\pi$$

One dimensional Green's function in practice

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} G(x,x')\rho(x')dx'$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{\int_{-\infty}^{x} G(x,x')\rho(x')dx' + \int_{x}^{\infty} G(x,x')\rho(x')dx'\right\}$$

For the one-dimensional Poisson equation, we can construct the Green's function by choosing $\phi_1(x) = x$ and $\phi_2(x) = 1; W = 1$:

$$\Phi(x) = \frac{1}{\varepsilon_0}\left\{\int_{-\infty}^{x} x'\rho(x')dx' + x\int_{x}^{\infty} \rho(x')dx'\right\}.$$

$$G(x,x') = 4\pi x_<$$

This expression gives the same result as previously obtained for the example $\rho(x)$ and more generally is appropriate for any neutral charge distribution.

Question -- How do we know which one of x and x' is the $x_<$ term?

$$G(x,x') = 4\pi x_<$$

$$\Phi(x) = \frac{1}{\varepsilon_0}\left\{\int_{-\infty}^{x} x'\rho(x')dx' + x\int_{x}^{\infty}\rho(x')dx'\right\}.$$

$x' < x$ $\qquad$ $x' > x$

Question – what happened to the boundary terms?

Full expression in this case --

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} G(x,x')\rho(x')dx' + \frac{1}{4\pi}\left[G(x,x')\frac{d\Phi(x')}{dx'} - \frac{dG(x,x')}{dx'}\Phi(x')\right]_{x'=-\infty}^{x'=\infty}$$

where $G(x,x') = 4\pi x_<$; $x_<$ should be taken as the smaller of x and x'.

We want to find the electrostatic potential such that

$$\frac{d^2\Phi(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0},$$

with the boundary condition $\Phi(-\infty) = 0$. and $\frac{d\Phi}{dx}(\infty) = 0$

$$\left[G(x,x')\frac{d\Phi(x')}{dx'} - \frac{dG(x,x')}{dx'}\Phi(x')\right]_{x'=-\infty}^{x'=\infty} = \left[-\frac{dG(x,x')}{dx'}\Phi(x')\right]_{x'=\infty} - \left[G(x,x')\frac{d\Phi(x')}{dx'}\right]_{x'=-\infty}$$

More clearly -- $\Phi(x) = \frac{1}{\varepsilon_0}\int_{-\infty}^{x} x'\rho(x')dx' + \frac{x}{\varepsilon_0}\int_{x}^{\infty}\rho(x')dx'$

$\Rightarrow G(x,x')$ is most conveniently constructed to incorporate boundary values.

Orthogonal function expansions and Green's functions

Suppose we have a "complete" set of orthogonal functions $\{u_n(x)\}$ defined in the interval $x_1 \leq x \leq x_2$ such that

$$\int_{x_1}^{x_2} u_n(x) u_m(x)\, dx = \delta_{nm}.$$

We can show that the completeness of this functions implies that

$$\sum_{n=1}^{\infty} u_n(x) u_n(x') = \delta(x - x').$$

This relation allows us to use these functions to represent a Green's function for our system. For the 1-dimensional Poisson equation, the Green's function satisfies

$$\frac{\partial^2}{\partial x^2} G(x, x') = -4\pi\delta(x - x').$$

Orthogonal function expansions –continued

Therefore, if

$$\frac{d^2}{dx^2}u_n(x) = -\alpha_n u_n(x),$$

where $\{u_n(x)\}$ also satisfy the appropriate boundary conditions, then we can write t

Green's functions as

$$G(x,x') = 4\pi \sum_n \frac{u_n(x)u_n(x')}{\alpha_n}.$$

Example

For example, consider the example discussed earlier in the interval $-a \leq x \leq a$ with

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases} \tag{24}$$

We want to solve the Poisson equation with boundary condition $d\Phi(-a)/dx = 0$ and $d\Phi(a)/dx = 0$. For this purpose, we may choose

$$u_n(x) = \sqrt{\frac{1}{a}} \sin\left(\frac{[2n+1]\pi x}{2a}\right). \tag{25}$$

The Green's function for this case as:

$$G(x,x') = \frac{4\pi}{a} \sum_{n=0}^{\infty} \frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right) \sin\left(\frac{[2n+1]\pi x'}{2a}\right)}{\left(\frac{[2n+1]\pi}{2a}\right)^2}. \tag{26}$$

Example – continued

$$\Phi(x) = \frac{\rho_0 a^2}{\epsilon_0}\left(16\sum_{n=0}^{\infty}\frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)}{([2n+1]\pi)^3} + \frac{1}{2}\right).$$

Constant shift to allow $\Phi(0) = 0$.