# PHY 712 Electrodynamics
# 10-10:50 AM MWF in Olin 103

## Plan for Lecture 7:

**Continue reading Chapters 2 & 3**

1. **Methods of images -- planes, spheres (JDJ Sec. 2.1-2.7)**
2. **Solution of Poisson equation in for other geometries – cylindrical (JDJ Sec. 3.7-3.8)**

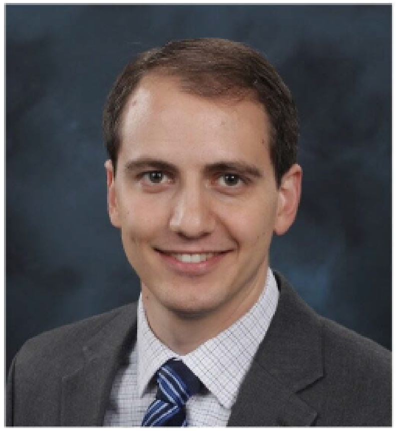
# Physics Colloquium

**– Thursday –**

**January 30, 2025**

### Neutron scattering: its history, philosophical implications, and its use to probe quantum materials

In the first half of this talk I give a history of neutron scattering, developed at Oak Ridge National Laboratory during the Manhattan project to build an atomic bomb. This technique gave some surprising insight into many-body quantum physics, such as spontaneous symmetry breaking and direct evidence of quasiparticles. Both philosophical concepts radically changed the way contemporary physicists view reality. In the second half I will discuss my work using neutron spectroscopy to solve current problems in quantum condensed matter: (i) using neutron scattering to search for and characterize quantum spin liquids (proposed states of matter exhibiting stable long-range quantum entanglement) and (ii) witnessing the presence of solid-state quantum entanglement between electron spins.



Dr. Allen Scheie
MPA-Q Division
Los Alamos National Laboratory

**4 PM in Olin 101**

Reception 3:30

# Course schedule for Spring 2025

(Preliminary schedule -- subject to frequent adjustment.)

| | Lecture date | JDJ Reading | Topic | HW | Due date |
|---|---|---|---|---|---|
| 1 | Mon: 01/13/2025 | Chap. 1 & Appen. | Introduction, units and Poisson equation | #1 | 01/15/2025 |
| 2 | Wed: 01/15/2025 | Chap. 1 | Electrostatic energy calculations | #2 | 01/17/2025 |
| 3 | Fri: 01/17/2025 | Chap. 1 | Electrostatic energy calculations | #3 | 01/22/2025 |
| | Mon: 01/20/2025 | No Class & 3 | Martin Luther King Jr. Holiday | | |
| 4 | Wed: 01/22/2025 | Chap. 1 | Electrostatic potentials and fields | #4 | 01/24/2025 |
| 5 | Fri: 01/24/2025 | Chap. 1 - 3 | Poisson's equation in multiple dimensions | | |
| 6 | Mon: 01/27/2025 | Chap. 1 - 3 | Brief introduction to numerical methods | #5 | 01/29/2025 |
| 7 | Wed: 01/29/2025 | Chap. 2 & 3 | Image charge constructions | #6 | 01/31/2025 |

# PHY 712 -- Assignment #6

Assigned: 1/27/2025 Due: 1/31/2025

Continue reading Chap. 2 in **Jackson**.

1. Eq. 2.5 on page 59 of **Jackson** was derived as the surface change density on a sphere of radius $a$ due to a charge $q$ placed at a radius $y > a$ outside the sphere. Determine the total surface charge on the sphere's outer surface.
2. Now consider the same system except assume $y < a$ representing the charge q being placed inside the sphere. What is the surface charge density and the total charge on the inner sphere surface in this case?

Survey of mathematical techniques for analyzing electrostatics – the Poisson equation

$$\nabla^2 \Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

1. Direct solution of differential equation
2. Solution by means of an integral equation; Green's function techniques
3. Orthogonal function expansions
4. Numerical methods (finite differences and finite element methods)
5. Method of images ← **today**

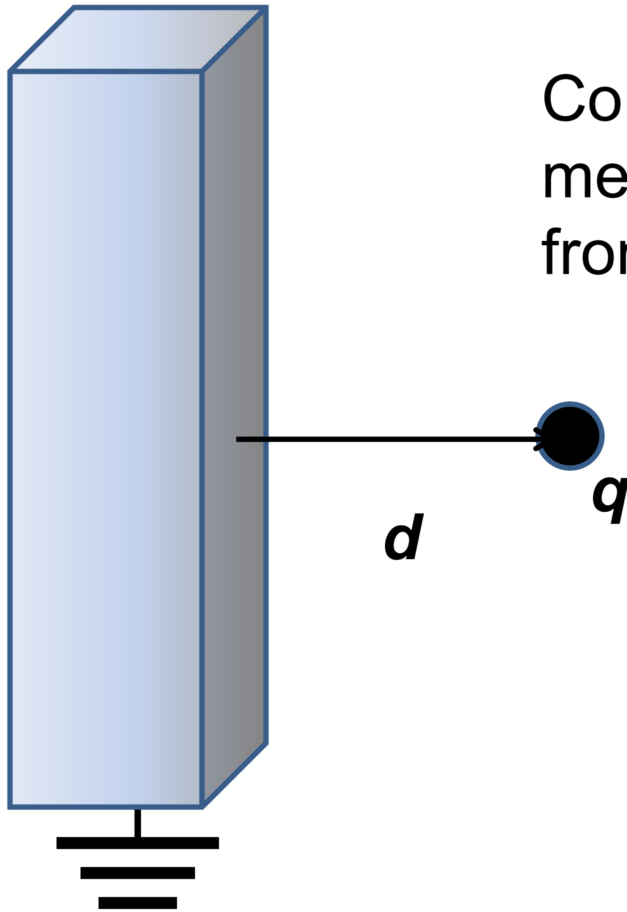
**Depends on geometry; Cartesian, spherical, and cylindrical cases considered in textbook**

Method of images

Clever trick for specialized geometries:

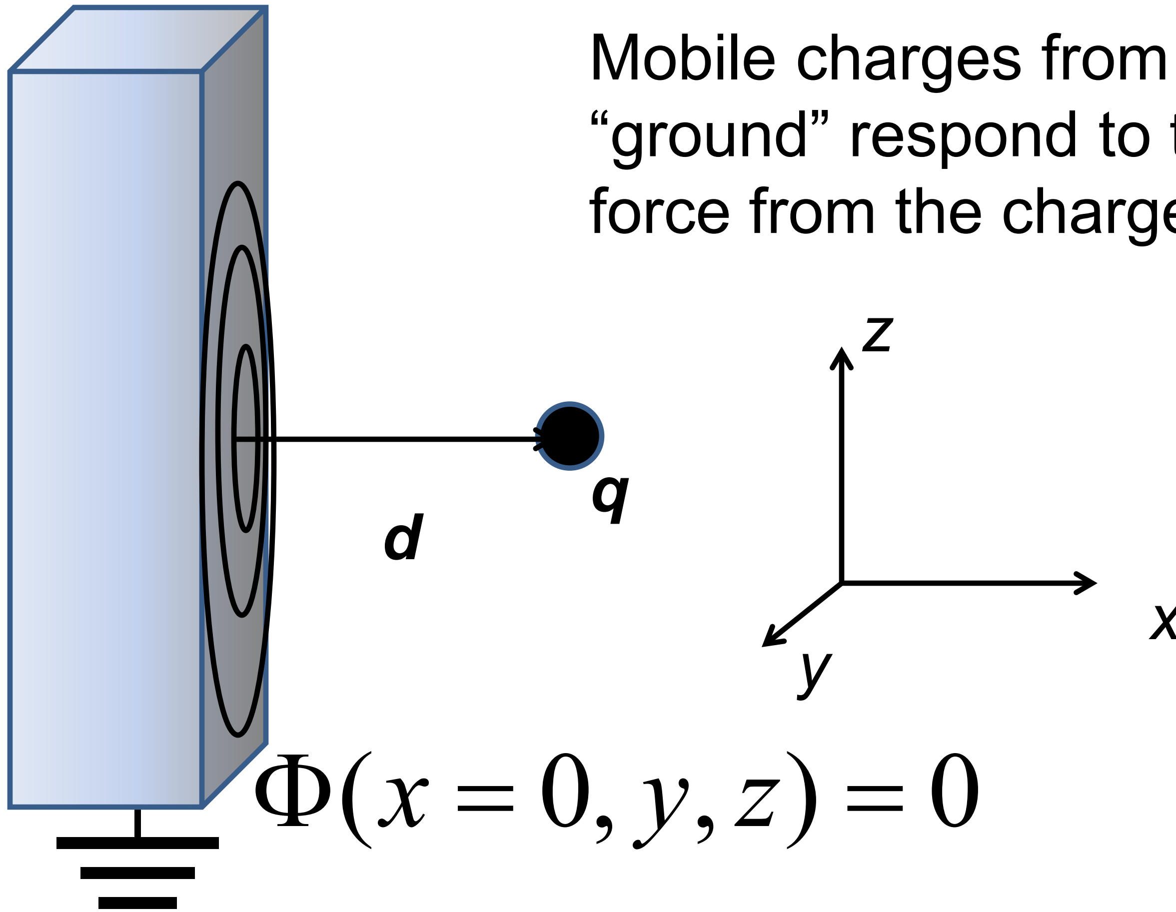
- Flat plane (surface)
- Sphere

Planar case:



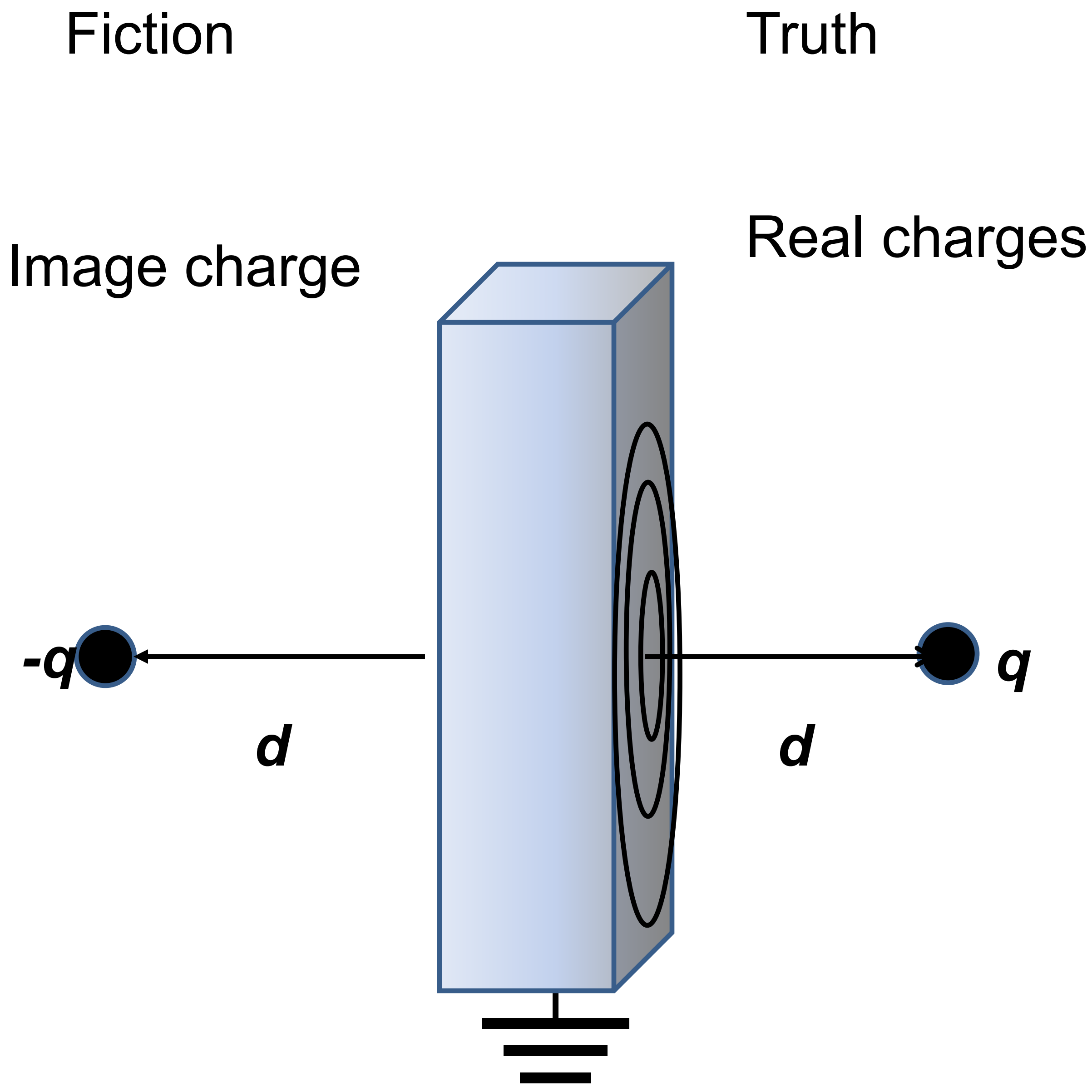

Consider a grounded metal sheet, a distance ***d*** from a point charge ***q***.

A grounded metal sheet, a distance ***d*** from a point charge ***q***.




Mobile charges from the "ground" respond to the force from the charge ***q***.

$$\Phi(x = 0, y, z) = 0$$


Fiction
Truth
Image charge
Real charges
-q
q
d
d

A grounded metal sheet, a distance ***d*** from a point charge ***q***.

$$\nabla^2\Phi = -\frac{q}{\varepsilon_0}\delta^3\left(\mathbf{r} - d\hat{\mathbf{x}}\right)$$

$$\Phi\left(x = 0, y, z\right) = 0$$

Trick for $x \geq 0$:

$$\Phi\left(x \geq 0, y, z\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\left|\mathbf{r} - d\hat{\mathbf{x}}\right|} - \frac{q}{\left|\mathbf{r} + d\hat{\mathbf{x}}\right|}\right)$$

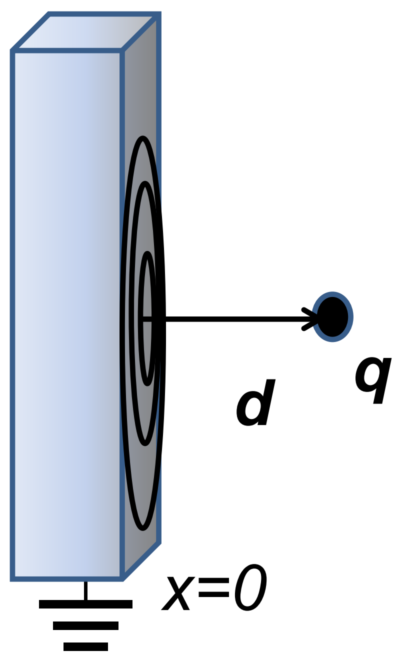
Surface charge density:

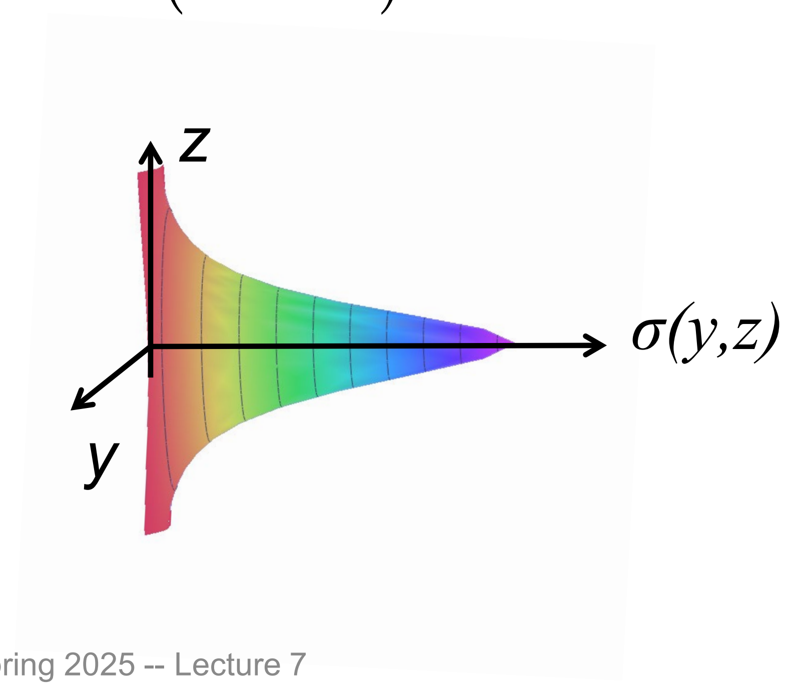
$$\sigma(y,z) = \varepsilon_0 E\left(0, y, z\right) = -\varepsilon_0 \frac{d\Phi\left(0, y, z\right)}{dx} = -\frac{q}{4\pi}\left(\frac{2d}{\left(d^2 + y^2 + z^2\right)^{3/2}}\right)$$

# A grounded metal sheet, a distance ***d*** from a point charge ***q***.

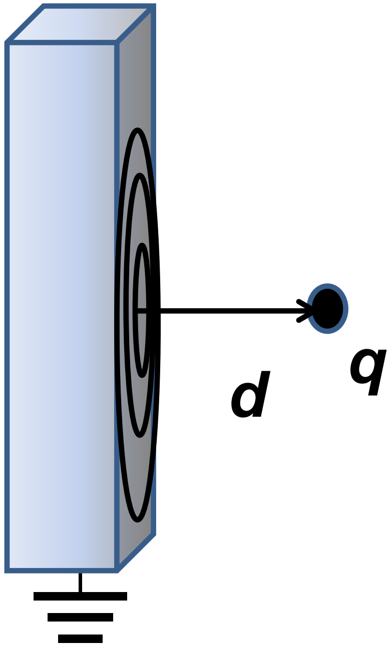
$$\text{Surface charge density}: \quad \sigma(y,z) = -\frac{q}{4\pi}\left(\frac{2d}{\left(d^2+y^2+z^2\right)^{3/2}}\right)$$

$$\text{Note}: \iint dydz \;\; \sigma(y,z) = -\frac{q2d}{4\pi}2\pi\int_0^\infty \frac{udu}{\left(d^2+u^2\right)^{3/2}} = -q$$



A grounded metal sheet, a distance ***d*** from a point charge ***q***.




Surface charge density :

$$\sigma(y,z) = -\frac{q}{4\pi}\left(\frac{2d}{\left(d^2+y^2+z^2\right)^{3/2}}\right)$$

Force between charge and sheet :

$$\mathbf{F} = \frac{-q^2\hat{\mathbf{x}}}{4\pi\varepsilon_0(2d)^2}$$

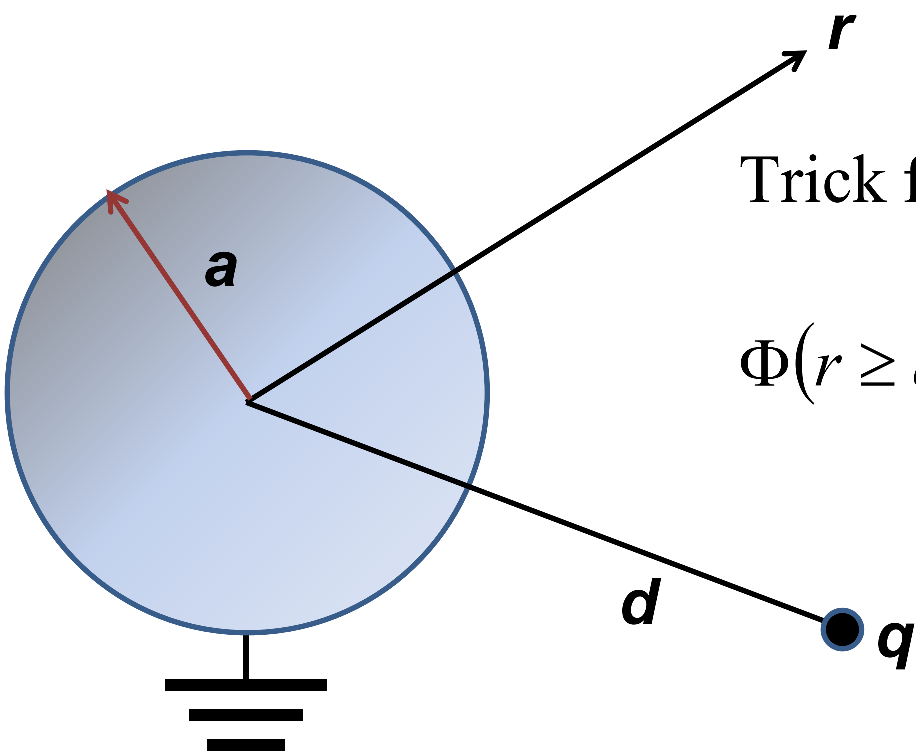
Image potential between charge and sheet at distance $x$ :

$$V(x) = \frac{-q^2}{4\pi\varepsilon_0(4x)}$$

**Note: this effect can be observed in photoemission experiments.**

Image charge methods can be used in some other geometries --

A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.




Trick for $r \geq a$ :

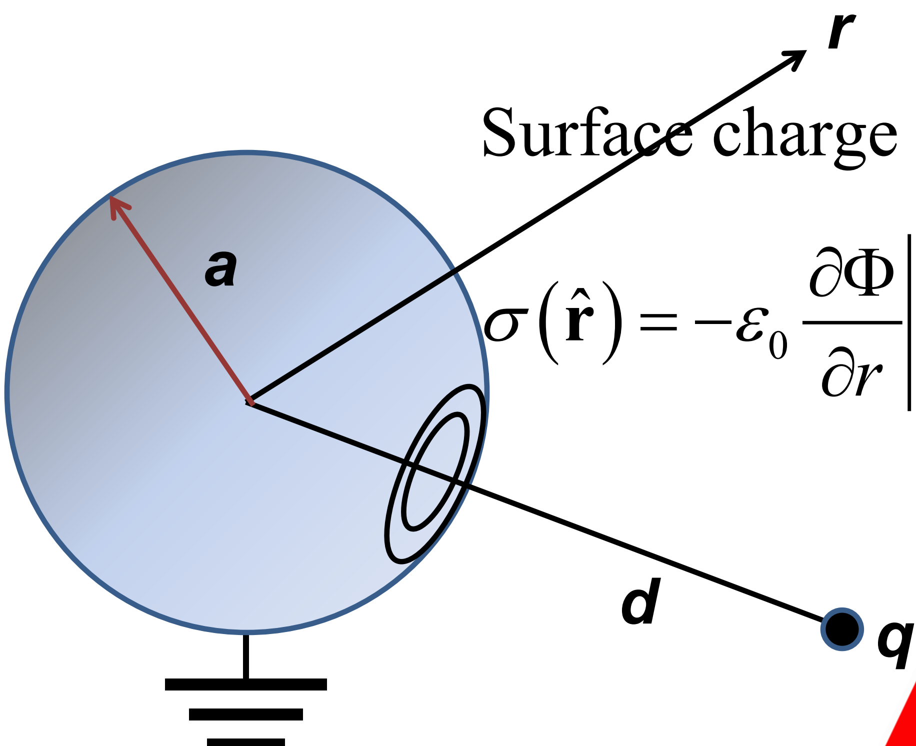
$$\Phi(r \geq a) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\mathbf{r}-\mathbf{d}|} - \frac{q}{\frac{d}{a}\left|\mathbf{r}-\mathbf{d}\frac{a^2}{d^2}\right|}\right)$$

Interpreted as

Image charge of $q' = -q\frac{a}{d}$

Located along $\hat{\mathbf{d}}$ at $\hat{\mathbf{d}}a\frac{a}{d}$

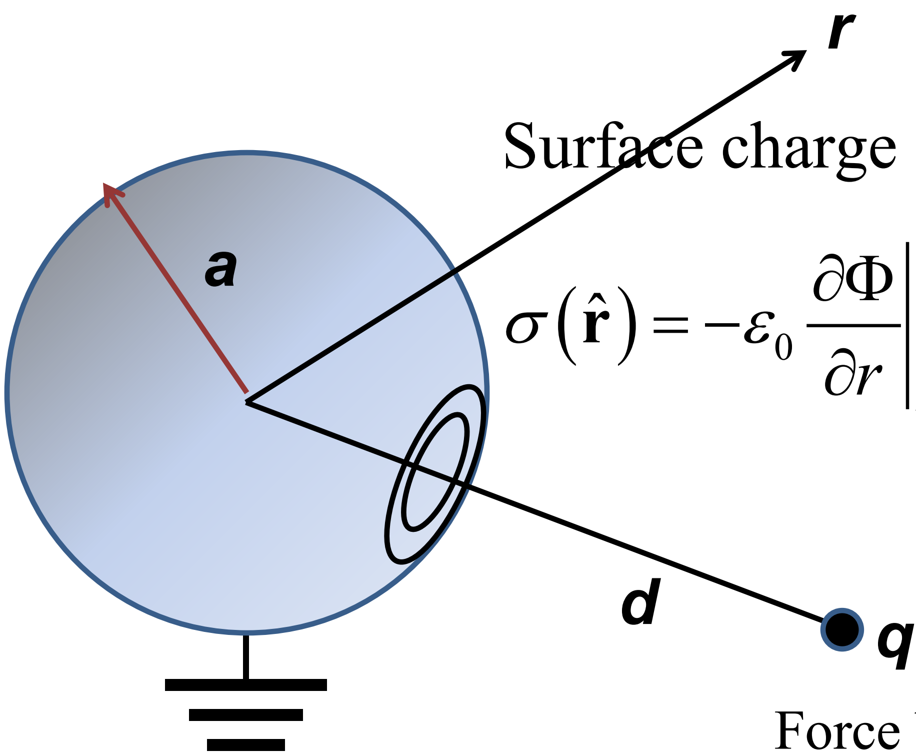
A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.




Surface charge density:

$$\sigma(\hat{\mathbf{r}}) = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = -\frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$

A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.



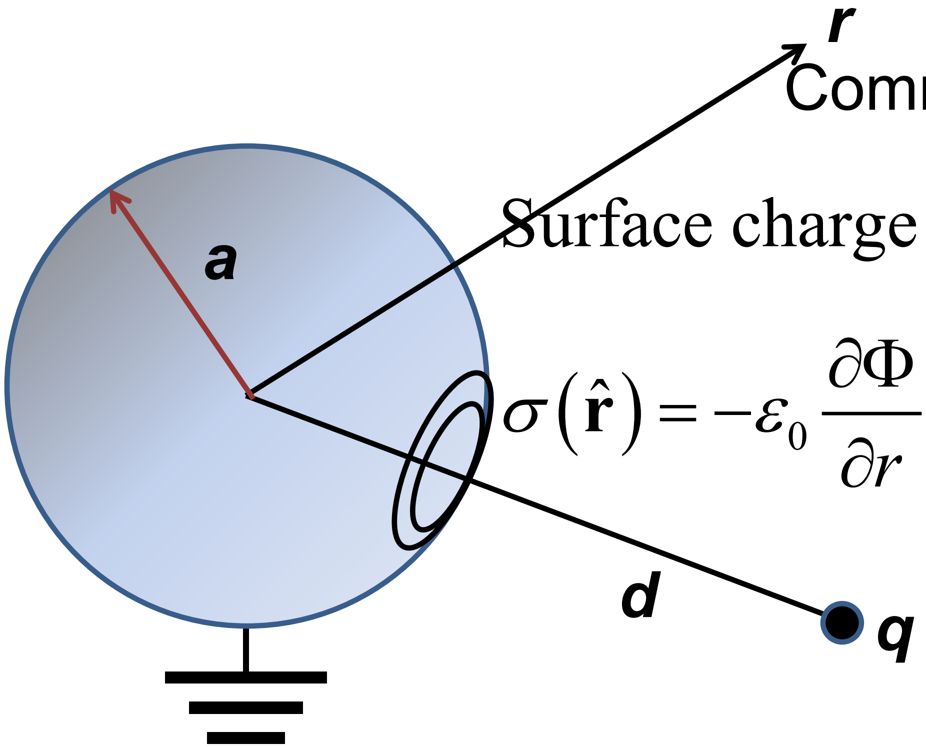

Surface charge density:

$$\sigma(\hat{\mathbf{r}}) = -\varepsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=a} = -\frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$

Force between $q$ and sphere

$$|\mathbf{F}| = \frac{1}{4\pi\epsilon_0}\frac{q^2(a/d)}{\left(d-a^2/d\right)^2} = \frac{q^2}{4\pi\epsilon_0}\frac{ad}{\left(d^2-a^2\right)^2}$$

# Comment on HW problem #6




Surface charge density:

$$\sigma(\hat{\mathbf{r}}) = -\varepsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=a} = -\frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$

For #1, integrate the charge induced on the outer surface of the sphere due to the point charge $q$ at the point $d > a$.

$$\int \sigma(\hat{\mathbf{r}})dS = -\int \frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}dS = -\frac{q}{4\pi a^2}\frac{a}{d}\left(1-\frac{a^2}{d^2}\right)2\pi a^2 \int \frac{d\cos\theta}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\cos\theta\right)^{3/2}}$$

For #2, the point charge $q$ is located at a point $d < a$ and a similar analysis follows. Integrate the charge induced on the inner surface of the sphere.

**(Answer to #2 should be different from that of #1.)**

# Use of image charge formalism to construct Green's function

Example:

Suppose we have a Dirichlet boundary value problem

on a sphere of radius $a$:

$$\nabla^2\Phi = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \qquad \Phi(r=a)=0$$

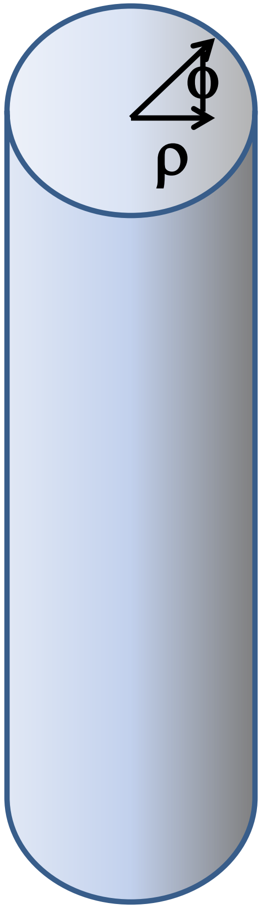
$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

$\Rightarrow$ For $r,r' > a$: $$G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{\frac{r'}{a}\left|\mathbf{r}-\frac{a^2}{r'^2}\mathbf{r}'\right|}$$

Analysis of Poisson/Laplace equation in various regular geometries

1. Rectangular geometries ➔ previous lectures
2. Cylindrical geometries ➔ introduce now, continue next time
3. Spherical geometries ➔ following lecture

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):




Corresponding orthogonal functions from solution of Laplace equation: $\nabla^2 \Phi = 0$

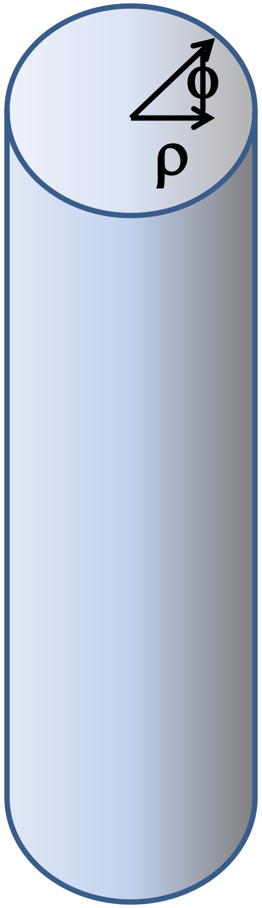
$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \varphi^2} = 0$$

$$\Phi(\rho,\varphi) = \Phi(\rho,\varphi + m2\pi)$$

$\Rightarrow$ General solution of the Laplace equation in these coordinates:

$$\Phi(\rho,\varphi) = \mathrm{A}_0 + \mathrm{B}_0 \ln(\rho) + \sum_{m=1}^{\infty}\left(A_m \rho^m + B_m \rho^{-m}\right)\sin(m\varphi + \alpha_m)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):



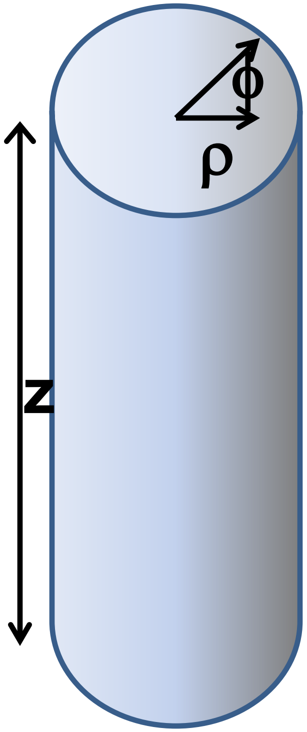

**Note that here ρ means radial coordinate**

Green's function appropriate for this geometry with boundary conditions at $\rho = 0$ and $\rho = \infty$:

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)G(\rho,\rho',\phi,\phi') \quad = \quad -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')$$

$$G(\rho,\rho',\phi,\phi') = -\ln\left(\rho_>{}^2\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos\left(m(\phi-\phi')\right)$$

# Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with *z*-dependence




Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

$$\Phi\left(\rho,\varphi,z\right)=\Phi\left(\rho,\varphi+m2\pi,z\right)$$

$$\Phi\left(\rho,\varphi,z\right)=R\left(\rho\right)Q\left(\varphi\right)Z\left(z\right)$$

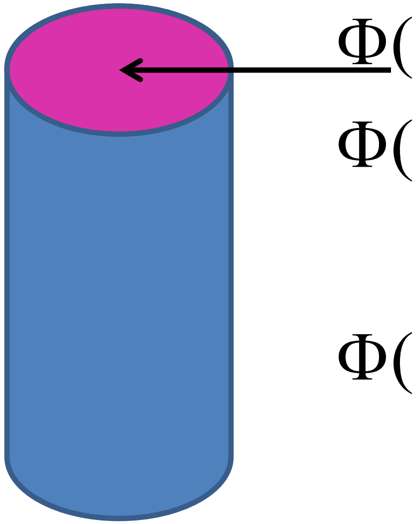
# Cylindrical geometry continued:

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \qquad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)R = 0 \qquad \Rightarrow J_m(k\rho), N_m(k\rho)$$

# Cylindrical geometry example:

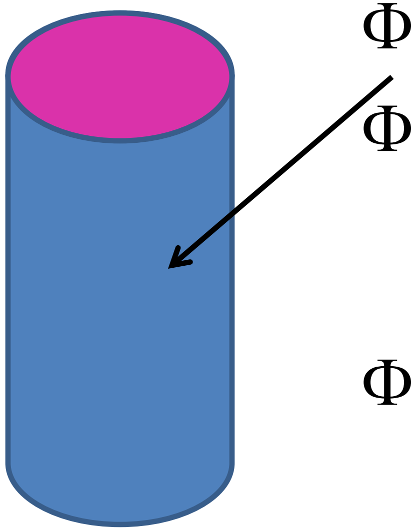


$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin\left(m\phi+\alpha_{mn}\right)$$

# Cylindrical geometry example:



$$\Phi(\rho = a, \phi, z) = V(\phi, z)$$

$$\Phi(\rho, \phi, z) = 0 \quad \text{on all other boundaries}$$

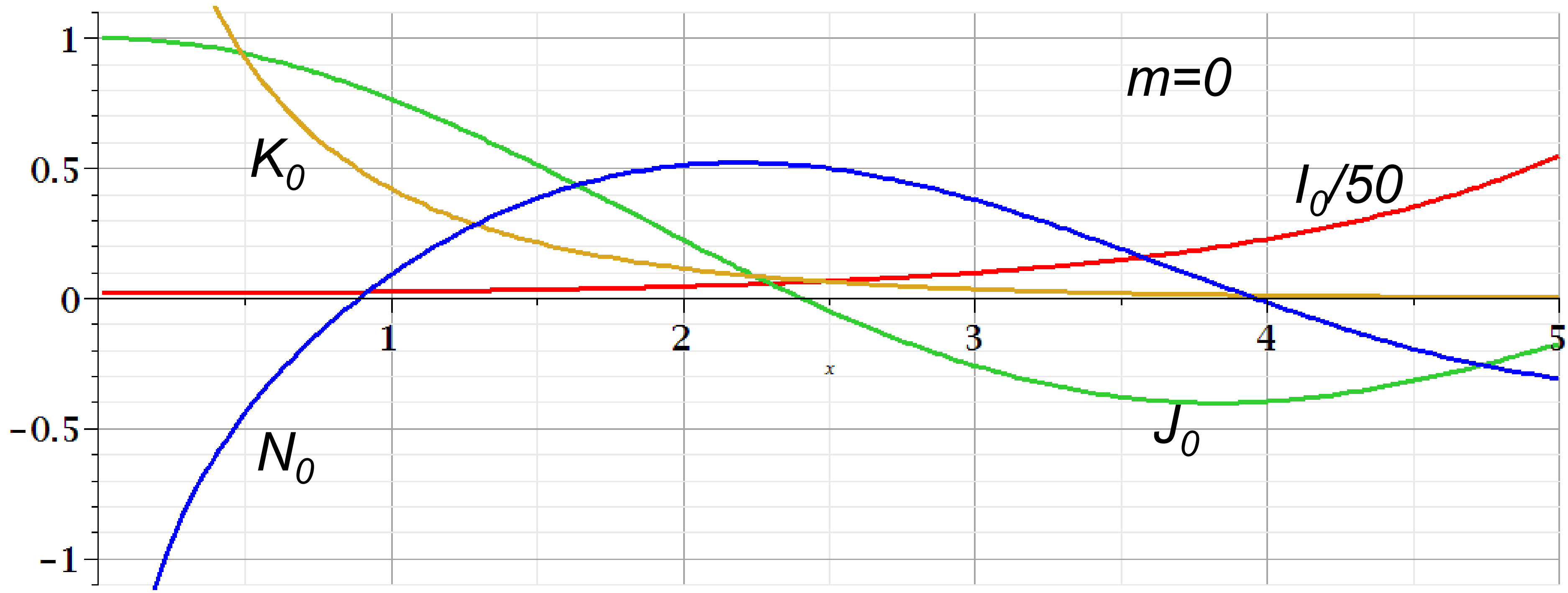
$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(m\phi + \alpha_{mn}\right)$$

# Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u),N_m(u),H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u),K_m(u)$$



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$$F_m^{+}(u)=J_m(u),N_m(u),H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u),K_m(u)$$