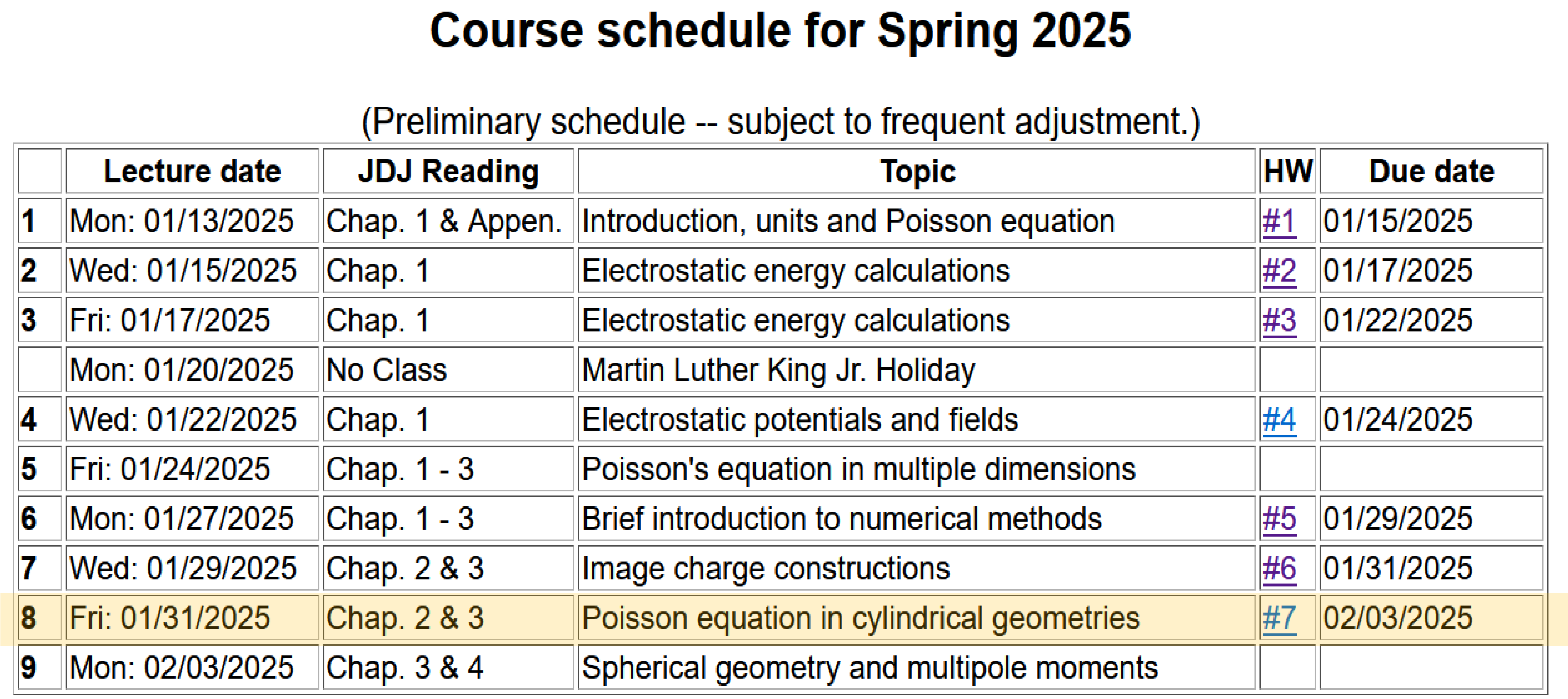
PHY 712 Electrodynamics
10-10:50 AM MWF in Olin 103

Class notes for Lecture 8:

Solution of Poisson/Laplace equation for special geometries –

- Cylindrical – (Sec. 2.11, 3.7, 3.8, 3.11 in JDJ)
- Spherical -- (Sec. 3.1-3.6 in JDJ); next lecture



Course schedule for Spring 2025

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/13/2025	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/15/2025
2	Wed: 01/15/2025	Chap. 1	Electrostatic energy calculations	#2	01/17/2025
3	Fri: 01/17/2025	Chap. 1	Electrostatic energy calculations	#3	01/22/2025
	Mon: 01/20/2025	No Class	Martin Luther King Jr. Holiday		
4	Wed: 01/22/2025	Chap. 1	Electrostatic potentials and fields	#4	01/24/2025
5	Fri: 01/24/2025	Chap. 1 - 3	Poisson's equation in multiple dimensions		
6	Mon: 01/27/2025	Chap. 1 - 3	Brief introduction to numerical methods	#5	01/29/2025
7	Wed: 01/29/2025	Chap. 2 & 3	Image charge constructions	#6	01/31/2025
8	Fri: 01/31/2025	Chap. 2 & 3	Poisson equation in cylindrical geometries	#7	02/03/2025
9	Mon: 02/03/2025	Chap. 3 & 4	Spherical geometry and multipole moments		

PHY 712 – Problem Set #7

Assigned: 01/31/2025 Due: 02/03/2025

Continue reading Chapters 1-3 in **Jackson**

1. Consider a long uniform cylindrical shell similar the example discussed in class, which can be described as a function of cylinder radius r in terms of an inner radius a and an outer radius b, with $b > a$ and constant density scale parameter ρ_0:

$$\rho(r) = \begin{cases} 0 & \text{for} \quad r < a \\ \rho_0(a-r)(b-r) & \text{for} \quad a \le r \le b \\ 0 & \text{for} \quad r > b, \end{cases}$$

   (a) Solve the Poisson equation for the electrostatic potential $\Phi(r)$ with the boundary condition that $\Phi(r)$ is well-behaved as $r \to 0$ and $\frac{d\Phi}{dr}(r)$ tends to zero as $r \to \infty$.

   (b) Plot the shape of $\Phi(r)$ for the case that $a = 1$ and $b = 2$ in length units.

Electrostatic equations for scalar potential function $\Phi(\mathbf{r})$

$$\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

Green's function solution:

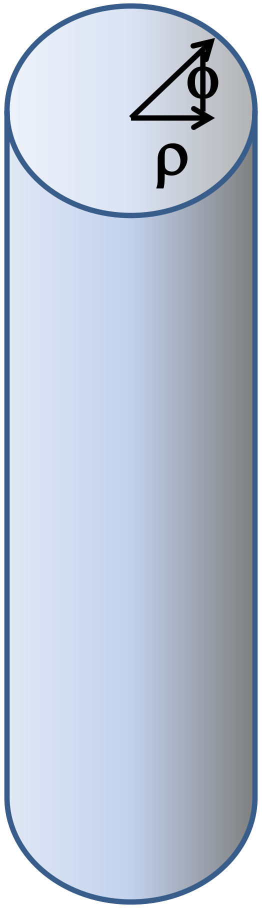
$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla'G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

Comment on surface boundary conditions – As discussed in JDJ (Sec. 1.9), there are two named boundary conditions – named for famous mathematicians.

- Dirichlet boundary condition is named for specifying the potential function Φ(**r**) on the boundary.
- Neumann boundary condition is named for specifying the electric field function **E**(**r**) on the boundary.

Solution of the Poisson/Laplace equation in various geometries ➔cylindrical geometry with no *z*-dependence (infinitely long wire, for example):



Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

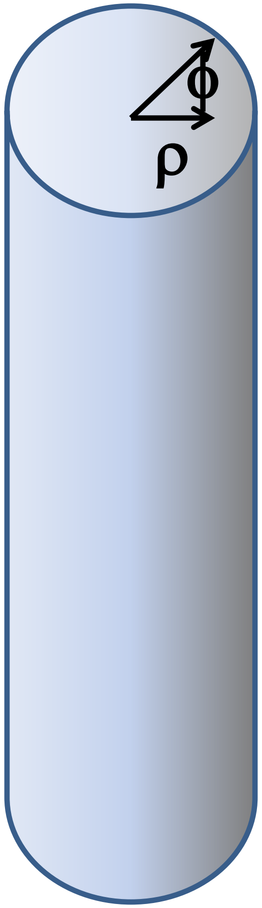
$$\Phi(\rho,\varphi) = \Phi(\rho,\varphi + m2\pi)$$

Assume: $\Phi(\rho,\phi) = f(\rho)g(\phi)$

Suppose $\frac{d^2g(\phi)}{d\phi^2} = -m^2g(\phi)$

$g(\phi) = \cos(m\phi + \alpha_m) \quad \Rightarrow m = \text{integer}, \alpha_m = \text{phase}$

Solution of the Poisson/Laplace equation in various geometries ➔cylindrical geometry with no *z*-dependence (infinitely long wire, for example):



Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2 \Phi = 0$

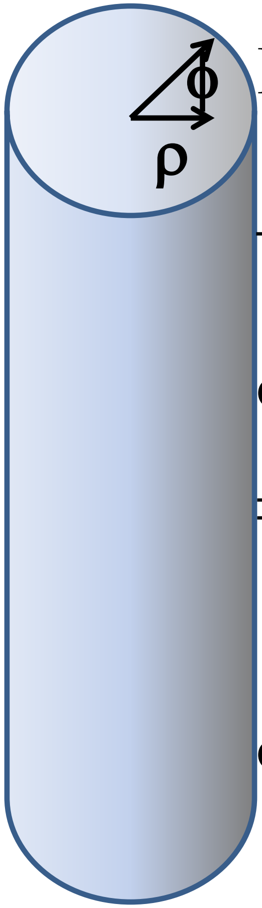
$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \varphi^2} = 0$$

Assume: $\Phi(\rho,\phi) = f(\rho)g(\phi)$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{df_m(\rho)}{d\rho}\right) - \frac{m^2}{\rho^2} f_m(\rho) = 0$$

$$f_0(\rho) = \begin{cases} 1 \\ \ln \rho \end{cases} \qquad f_{m>0} = \rho^{\pm m}$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):



Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

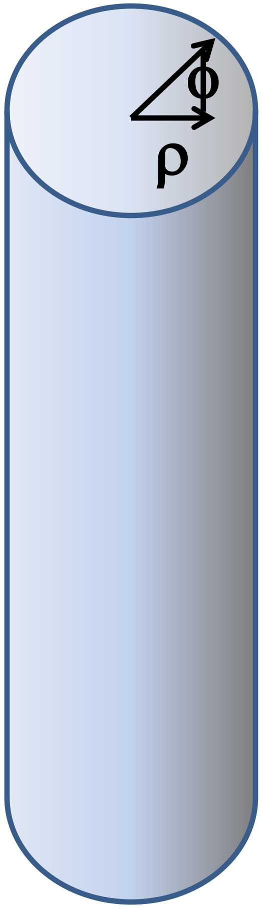
$$\Phi(\rho,\varphi) = \Phi(\rho,\varphi + m2\pi)$$ ➔*m*=integer

$\Rightarrow$ General solution of the Laplace equation

in these coordinates:

$$\Phi(\rho,\varphi) = \mathrm{A}_0 + \mathrm{B}_0\ln(\rho) + \sum_{m=1}^{\infty}\left(A_m\rho^m + B_m\rho^{-m}\right)\cos(m\varphi + \alpha_m)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):



Green's function appropriate for this geometry with boundary conditions at $\rho = 0$ and $\rho = \infty$:

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right)G(\rho,\rho',\varphi,\varphi') = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\varphi-\varphi')$$

It can be shown that the following form can be used:

$$G(\rho,\rho',\varphi,\varphi') = -\ln\left(\rho_>{}^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos\left(m(\varphi-\varphi')\right)$$

Note that the previous example is similar to the construction for the 2-d cartesian case --

For the 2-d cartesian case, for example, we can assume that the Green's function can be written in the form:

$$G(x,x',y,y') = \sum_n u_n(x) u_n(x') g_n(y,y') \quad \text{where} \quad \frac{d^2}{dx^2} u_n(x) = -\alpha_n u_n(x)$$

The y dependence of this equation will have the required behavior, if we choose: $\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right] g_n(y,y') = -4\pi\delta(y-y'),$

which in turn can be expressed in terms of the two independent solutions $v_{n_1}(y)$ and $v_{n_2}(y)$ of the homogeneous equation:

$$\left[\frac{d^2}{dy^2} - \alpha_n\right] v_{n_i}(y) = 0,$$

and the Wronskian constant: $K_n \equiv \frac{dv_{n_1}}{dy} v_{n_2} - v_{n_1} \frac{dv_{n_2}}{dy}$

Cartesian example continued --

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right] g_n(y, y') = -4\pi\delta(y - y'),$$

$$g_n(y, y') = \frac{4\pi}{K_n} v_{n_1}(y_<) v_{n_2}(y_>)$$

where: $$\left[\frac{d^2}{dy^2} - \alpha_n\right] v_{n_i}(y) = 0,$$

and $$K_n \equiv \frac{dv_{n_1}}{dy} v_{n_2} - v_{n_1} \frac{dv_{n_2}}{dy}$$

For example, choose $v_{n_1}(y) = \sinh(\sqrt{\alpha_n}\, y)$ and $v_{n_2}(y) = \sinh(\sqrt{\alpha_n}(b - y))$

where $K_n = \sqrt{\alpha_n} \sinh(\sqrt{\alpha_n}\, b)$

using the identity: $\cosh(r)\sinh(s) + \sinh(r)\cosh(s) = \sinh(r + s)$

$$G(x, x', y, y') = \sum_n u_n(x) u_n(x') \frac{4\pi}{K_n} v_{n_1}(y_<) v_{n_2}(y_>).$$

In the cylindrical geometry case,

$$u_n(x) \rightarrow \{\sin(m\varphi), \cos(m\varphi)\}$$

$$v_{n_{1,2}} \rightarrow \{1, \ln(\rho), \rho^m, \rho^{-m}\}$$

$$G(\rho, \rho', \varphi, \varphi') = -\ln\left(\rho_>{}^2\right) + 2\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos\left(m\left(\varphi - \varphi'\right)\right)$$

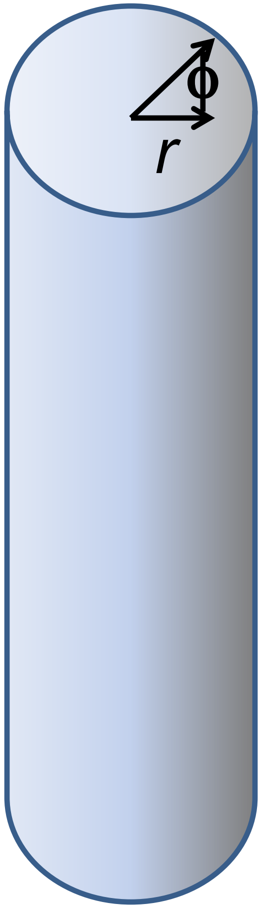
Note that, because we are using curvilinear coordinates, the Wronskian and the form of the delta function is modified.

More details given in **Jackson** Sec. 3.7 - 3.11.

Comments and details

Change notation

$\rho \Rightarrow r$

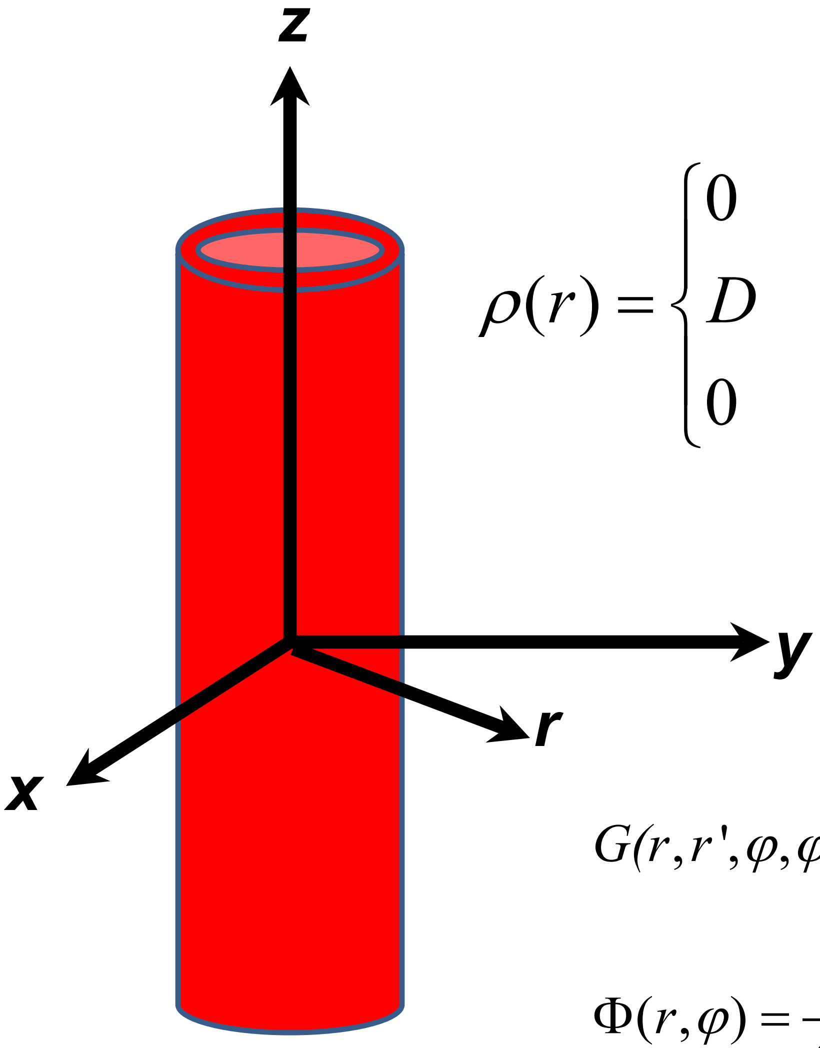


$$G(r,r',\varphi,\varphi') = -\ln\left(r_>{}^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m(\varphi-\varphi')\right)$$

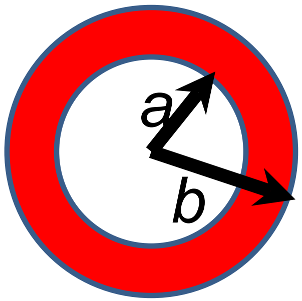
$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

Note that in this case, we have assumed that the surface integral contributions are trivial.

Example – uniform cylindrical shell:



Top view:



$$\rho(r)=\begin{cases}0 & r<a\\ D & a\le r\le b\\ 0 & r>b\end{cases}$$

$$\nabla^2\Phi=-\frac{\rho}{\epsilon_0}$$

$$\nabla^2\Phi(r,\phi)=\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\Phi(r,\phi)}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi(r,\phi)}{\partial^2\phi}$$

$$G(r,r',\varphi,\varphi')=-\ln\left(r_>{}^2\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m\cos\left(m\left(\varphi-\varphi'\right)\right)$$

$$\Phi(r,\varphi)=\frac{1}{4\pi\epsilon_0}\int_0^{2\pi}d\varphi'\int_0^{\infty}r'dr'G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

Boundary condition: $\lim_{r\to\infty}\left(\frac{\partial\Phi(r,\varphi)}{\partial r}\right)=0$

Question – Why only m=0 for this case?

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^{\,2}\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m\left(\varphi-\varphi'\right)\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

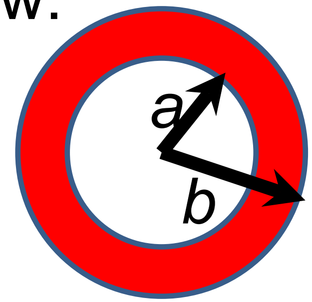
Note that $\int_0^{2\pi} d\varphi' \cos(m(\varphi-\varphi')) = 0$ for $m > 0$

So that $\Phi(r,\varphi) = \frac{2\pi}{4\pi\epsilon_0}\int_0^{\infty} r'dr'\left(-\ln\left(r_>^{\,2}\right)\right)\rho(r')$

$$= \frac{2\pi}{4\pi\epsilon_0}\left(\left(-\ln(r^2)\right)\int_0^{r} r'dr'\rho(r') - \int_r^{\infty} r'dr'\left(\ln\left(r'^2\right)\right)\rho(r')\right)$$

Some details

Top view:



$$\rho(r)=\begin{cases}0 & r<a\\ D & a\le r\le b\\ 0 & r>b\end{cases}$$

In our case: $\Phi(r,\varphi)=\frac{2\pi}{4\pi\epsilon_0}\int_0^\infty r'dr'\left(-\ln\left(r_>^{\,2}\right)\right)\rho(r')$

$$=\frac{2\pi}{4\pi\epsilon_0}\left(\left(-\ln(r^2)\right)\int_0^r r'dr'\rho(r')\;-\int_r^\infty r'dr'\left(\ln\left(r'^2\right)\right)\rho(r')\;\right)$$

$$\Phi(r,\varphi)=\frac{2\pi D}{4\pi\epsilon_0}\int_a^b r'dr'\left(-\ln(r_>^2)\right)=\frac{D}{\epsilon_0}\int_a^b r'dr'\left(-\ln(r_>)\right)$$

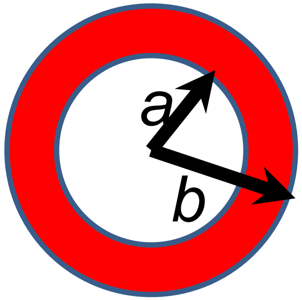
For $0\le r<a$: $\Phi(r,\varphi)=\frac{D}{\epsilon_0}\int_a^b r'dr'\left(-\ln(r')\right)$

For $a\le r<b$: $\Phi(r,\varphi)=\frac{D}{\epsilon_0}\left(\left(-\ln(r)\right)\int_a^r r'dr'+\int_r^b r'dr'\left(-\ln(r')\right)\right)$

For $r>b$: $\Phi(r,\varphi)=\frac{D}{\epsilon_0}\left(-\ln(r)\right)\int_a^b r'dr'$

Example continued -- *m=0* only --

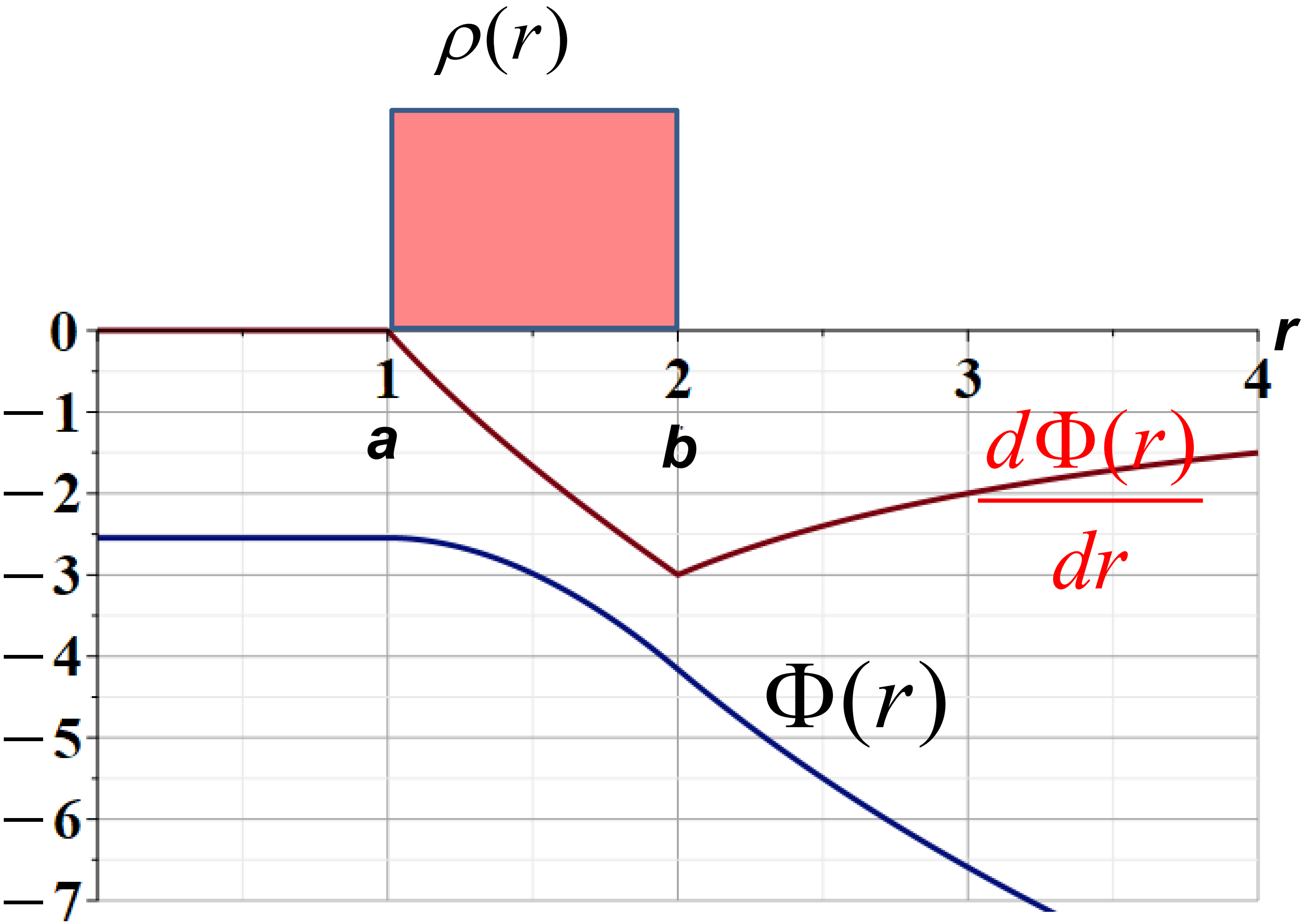
Top view:



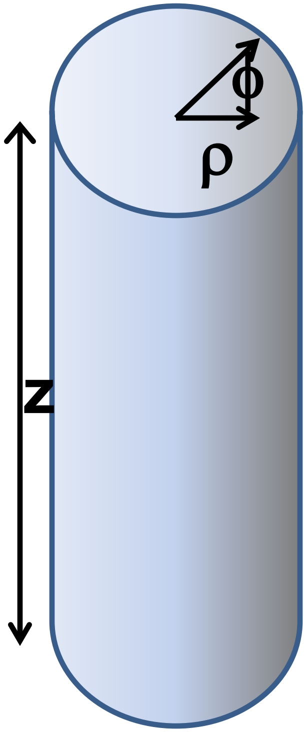
$$\rho(r) = \begin{cases} 0 & 0 < r < a \\ D & a \le r \le b \\ 0 & r > b \end{cases}$$

$$\Phi(r) = \begin{cases} \frac{D}{4\epsilon_0}\left(b^2 - a^2 - b^2\ln(b^2) + a^2\ln(a^2)\right) & 0 < r < a \\ \frac{D}{4\epsilon_0}\left(b^2 - r^2 - b^2\ln(b^2) + a^2\ln(r^2)\right) & a \le r \le b \\ \frac{D}{4\epsilon_0}\left(a^2 - b^2\right)\ln(r^2) & r > b \end{cases}$$

Example continued --



Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with *z*-dependence

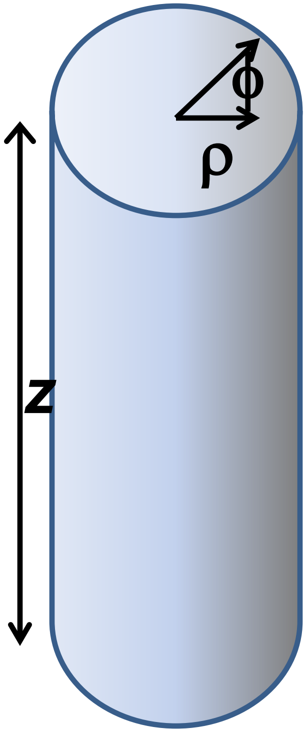


$$\text{Laplace equation:} \qquad \nabla^2\Phi = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

$$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$$

Cylindrical geometry continued:



$$\text{Laplace equation}: \qquad \nabla^2\Phi = 0$$

$$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$$

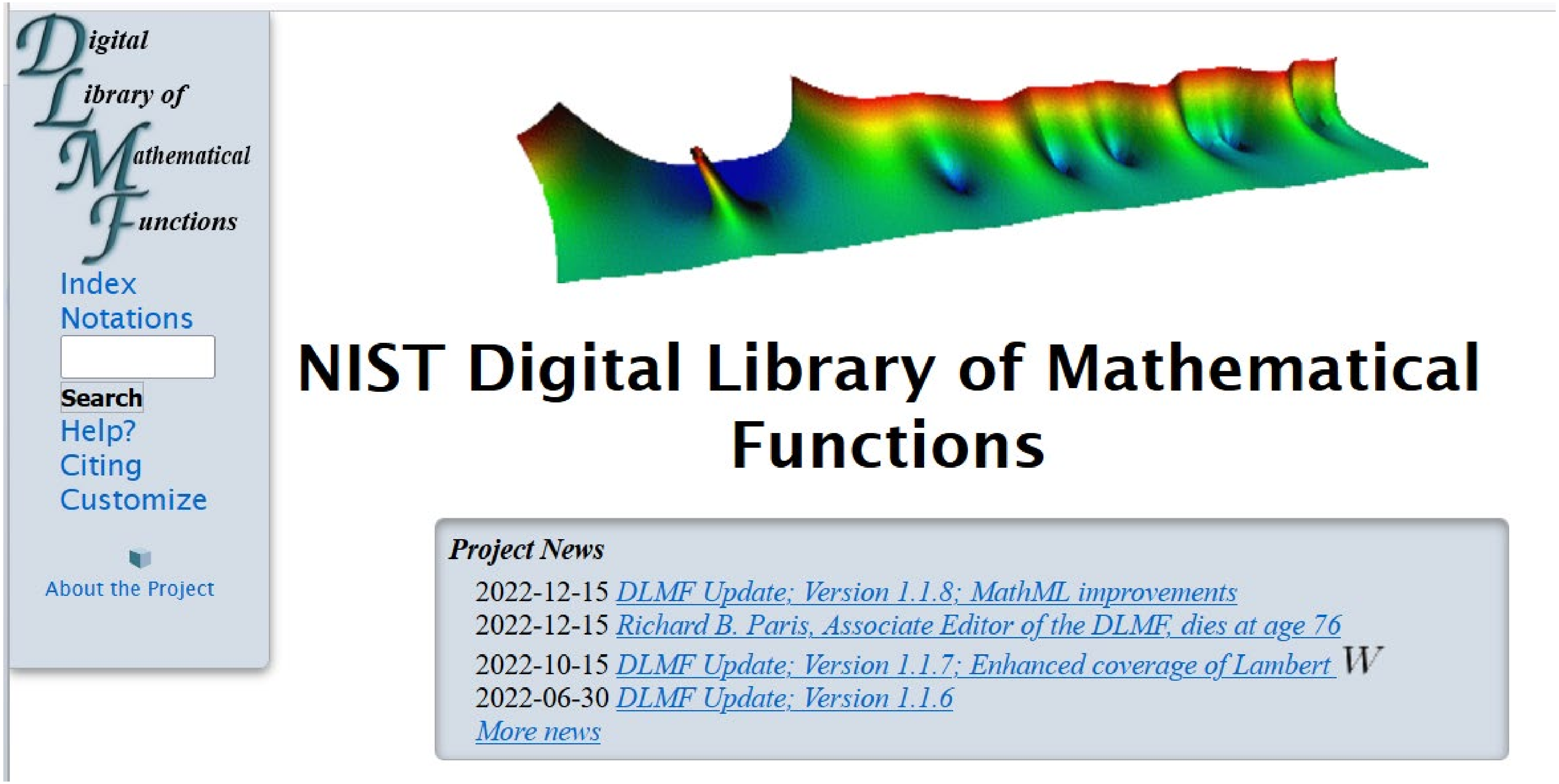
One possibility:

$$\frac{d^2Z}{dz^2} - k^2Z = 0 \qquad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2Q}{d\phi^2} + m^2Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)R = 0 \qquad \Rightarrow J_m(k\rho), N_m(k\rho)$$

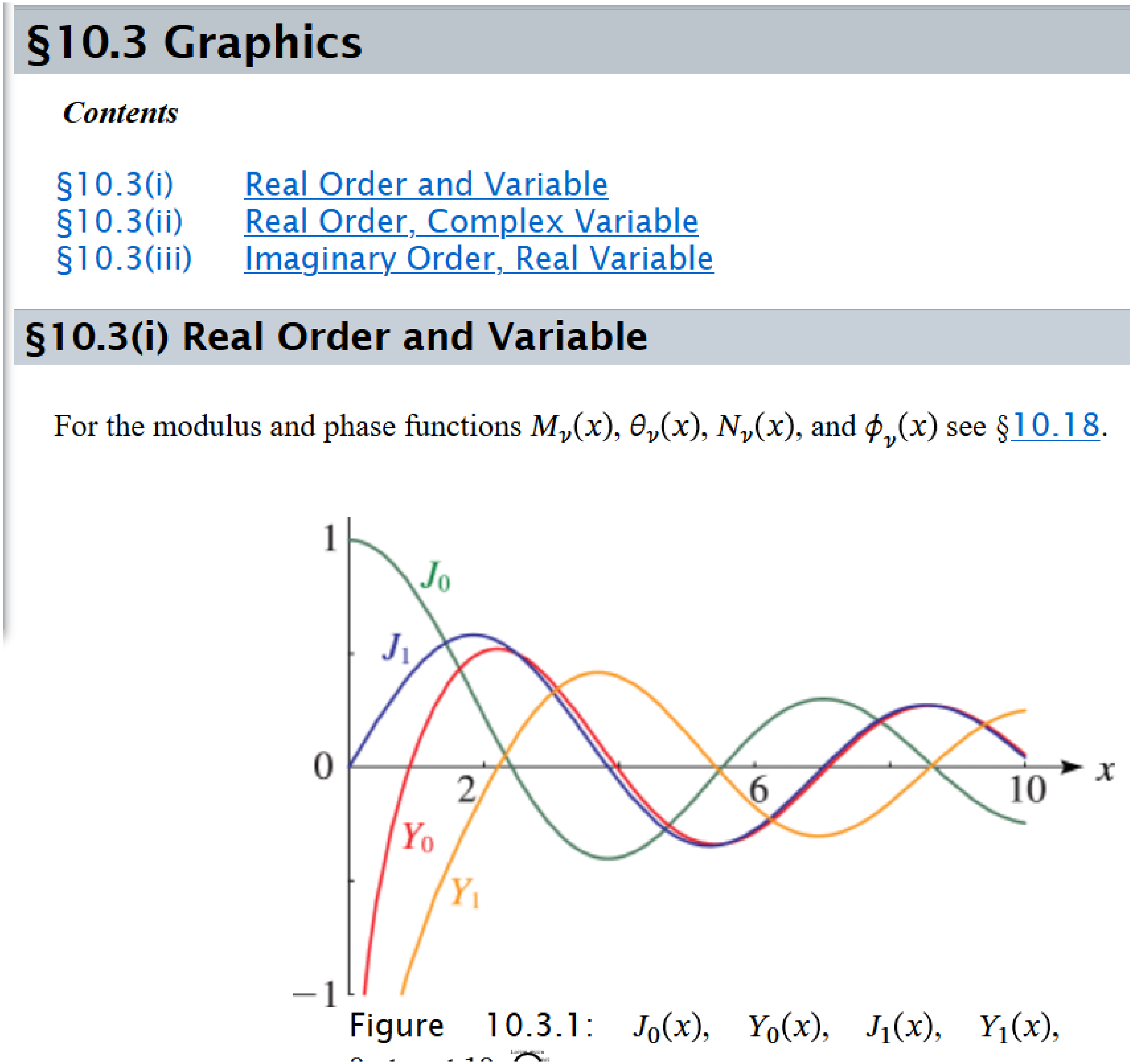
https://dlmf.nist.gov/



NIST Digital Library of Mathematical Functions

Project News

2022-12-15 *DLMF Update; Version 1.1.8; MathML improvements*
2022-12-15 *Richard B. Paris, Associate Editor of the DLMF, dies at age 76*
2022-10-15 *DLMF Update; Version 1.1.7; Enhanced coverage of Lambert* W
2022-06-30 *DLMF Update; Version 1.1.6*
More news



§10.3 Graphics

Contents

§10.3(i) Real Order and Variable

For the modulus and phase functions $M_\nu(x)$, $\theta_\nu(x)$, $N_\nu(x)$, and $\phi_\nu(x)$ see §10.18.

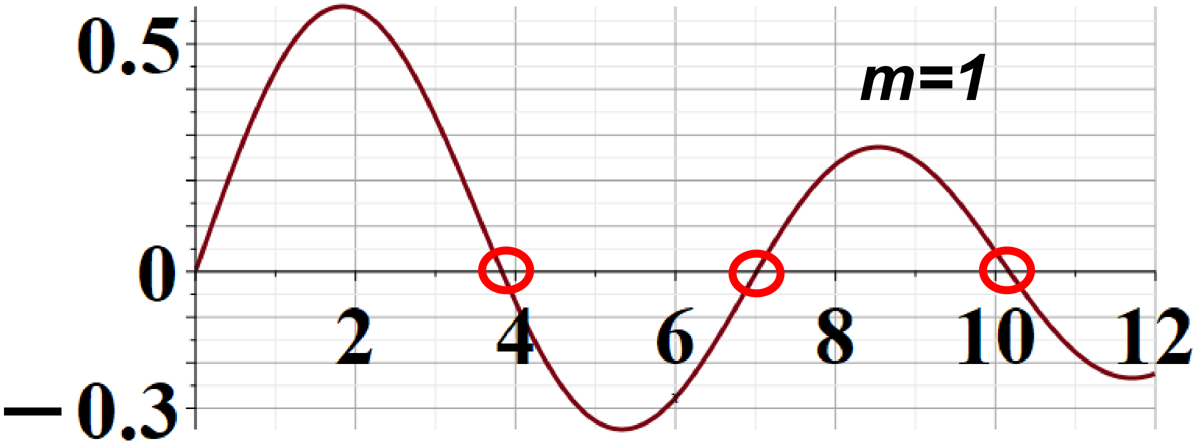
Figure 10.3.1: $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$,

Some useful identities involving cylindrical Bessel functions from **Jackson** Sec. 3.7

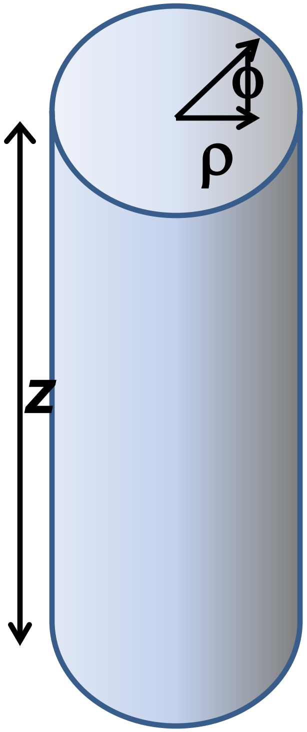
$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(1-\frac{m^2}{u^2}\right)\right)J_m(u)=0 \qquad \text{for integer } m$$

Properties of Bessel functions in terms of zeros: x_{mn}; $\quad J_m(x_{mn})=0$

$$\int_0^a \rho d\rho J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn'}\rho}{a}\right)=\frac{a^2}{2}\left(J_{m+1}(x_{mn})\right)^2\delta_{nn'}$$



Cylindrical geometry continued:



Laplace equation: $\nabla^2\Phi = 0$

$$\Phi(\rho,\varphi,z) = R(\rho)Q(\varphi)Z(z)$$

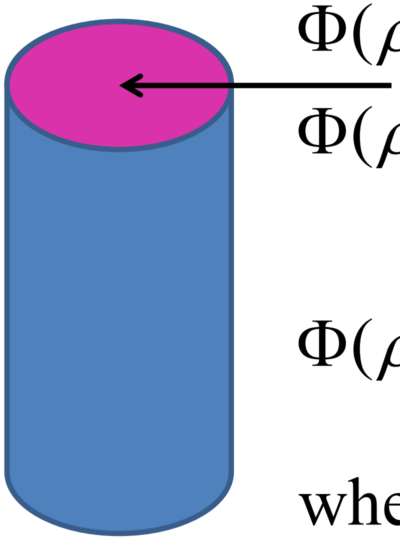
Another possibility:

$$\frac{d^2Z}{dz^2} + k^2Z = 0 \qquad \Rightarrow Z(z) = \sin(kz), \cos(kz), e^{\pm ikz}$$

$$\frac{d^2Q}{d\phi^2} + m^2Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(-k^2 - \frac{m^2}{\rho^2}\right)R = 0 \qquad \Rightarrow I_m(k\rho), K_m(k\rho)$$

Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at *z=L*



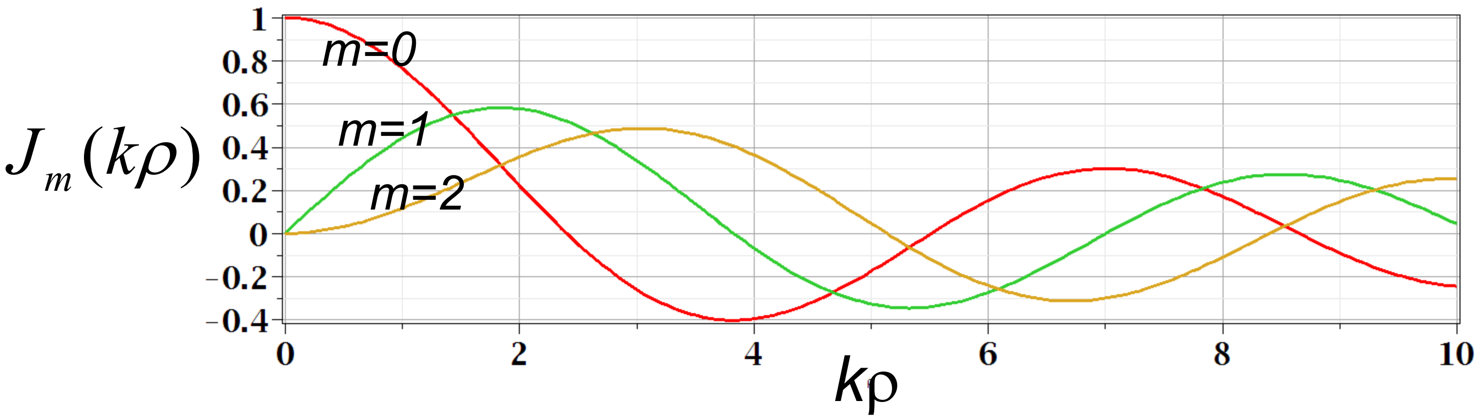
$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

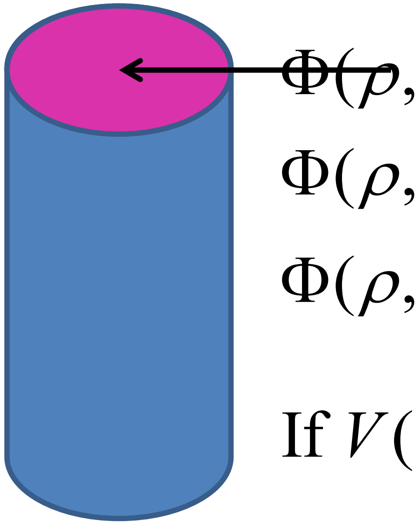
Well behaved at *ρ=0*

$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin\left(m\phi+\alpha_{mn}\right)$$

$$\text{where} \quad J_m(k_{mn}a)=0$$



Solutions of Laplace equation inside cylindrical shape Example with non-trivial boundary value at *z=L*



$$\Phi(\rho,\varphi,z=L)=V(\rho,\varphi)$$

$$\Phi(\rho,\varphi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\varphi,z)=\sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin\left(m\varphi+\alpha_{mn}\right)$$

If $V(\rho,\varphi)$ is an even function of φ so that $\alpha_{mn}=\pi/2$:

$$A_{mn}=\frac{\int_0^{2\pi} d\varphi\cos\left(m\varphi\right)\int_0^a \rho d\rho J_m(k_{mn}\rho)V(\rho,\varphi)}{\sinh(k_{mn}L)\int_0^{2\pi} d\varphi\cos^2\left(m\varphi\right)\int_0^a \rho d\rho J_m{}^2(k_{mn}\rho)}$$

§10.26 Graphics

Contents

§10.26(i) Real Order and Variable

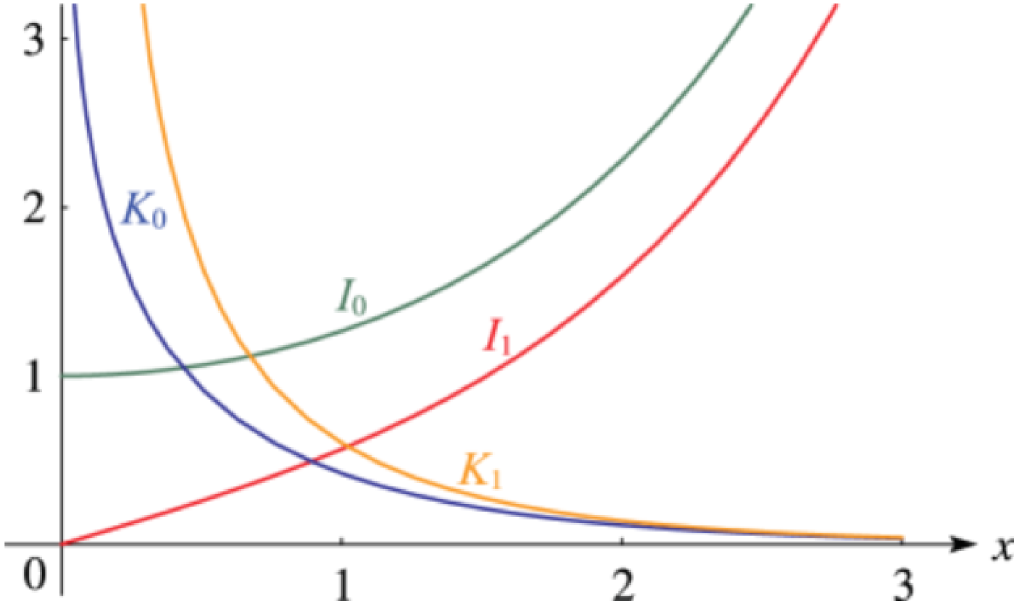
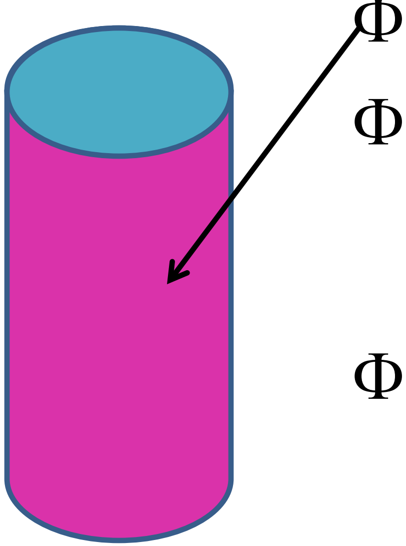


Figure 10.26.1: $I_0(x)$, $I_1(x)$, $K_0(x)$, $K_1(x)$,

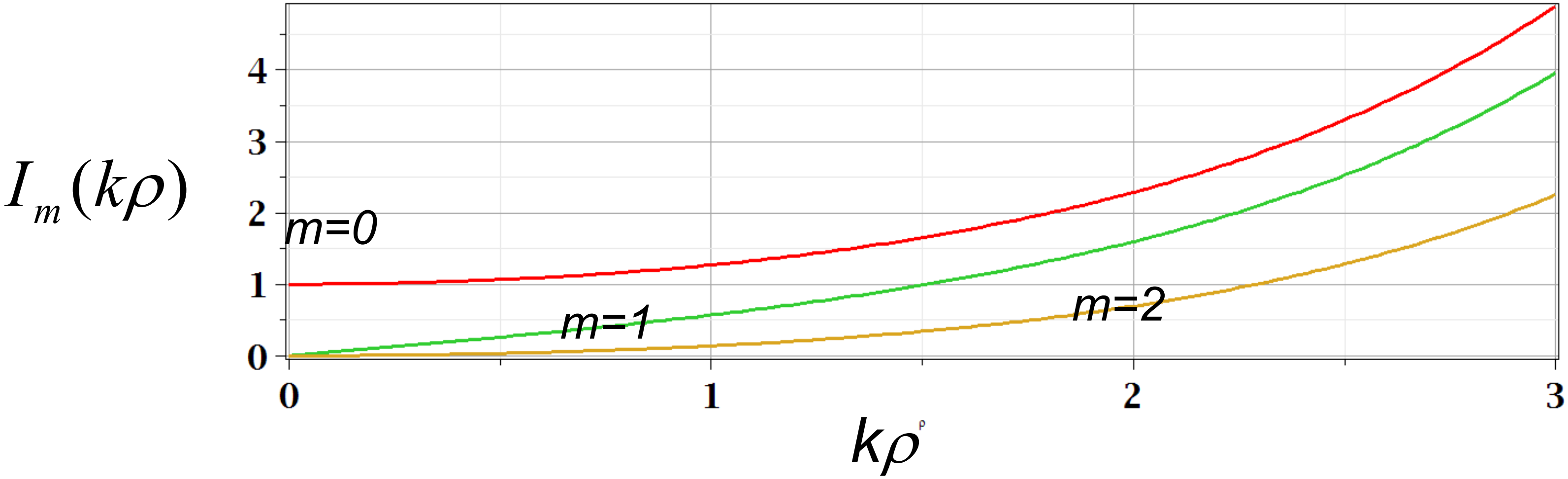
Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at $\rho=a$



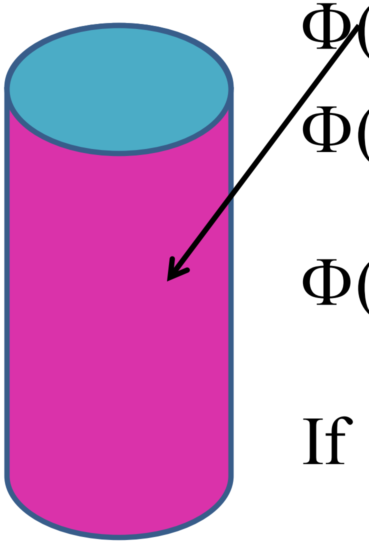
$$\Phi(\rho=a,\phi,z)=V(\phi,z)$$

$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(m\phi+\alpha_{mn}\right)$$



Solutions of Laplace equation inside cylindrical shape Example with non-trivial boundary value at ρ=*a*



$$\Phi(\rho=a,\varphi,z)=V(\varphi,z)$$

$$\Phi(\rho,\varphi,z)=0 \quad \text{on all other boundaries}$$

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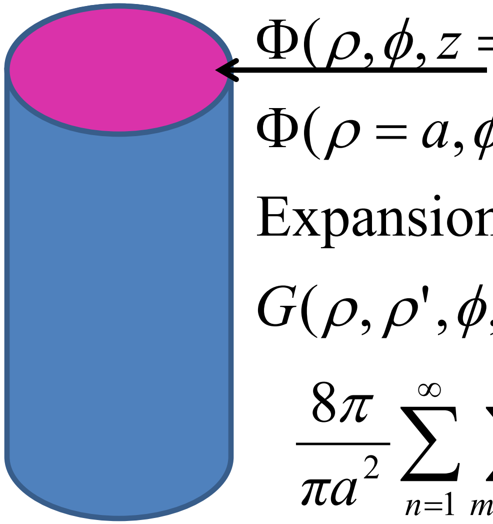
If $V(z,\varphi)$ is an even function of φ so that $\alpha_{mn}=\pi/2$:

$$A_{mn}=\frac{\int_0^{2\pi} d\varphi\cos\left(m\varphi\right)\int_0^{L} dz\sin\left(\frac{n\pi z}{L}\right)V(z,\varphi)}{I_m\left(\frac{n\pi a}{L}\right)\int_0^{2\pi} d\varphi\cos^2\left(m\varphi\right)\int_0^{L} dz\sin^2\left(\frac{n\pi z}{L}\right)}$$

Green's function for Dirchelet boundary value inside cylinder:



$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

$$\Phi(\rho=a,\phi,z)=0\,,\;\;\Phi(\rho,\phi,z=0)=0$$

Expansion in terms of Bessel function zeros : $J_m(k_{mn}a)=0$

$$G(\rho,\rho',\phi,\phi',z,z')=$$

$$\frac{8\pi}{\pi a^2}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{im(\phi-\phi')}J_m(k_{mn}\rho)J_m(k_{mn}\rho')\sinh(k_{mn}z_<)\sinh(k_{mn}(L-z_>))}{k_{mn}(J_{m+1}(k_{mn}a))^2\sinh(k_{mn}L)}$$

$$\Phi(\rho,\phi,z)=\frac{1}{4\pi\varepsilon_0}\int_V d\phi'\,\rho'\,d\rho'\,dz'\,G(\rho,\rho',\phi,\phi',z,z')\rho(\rho',\phi',z')$$

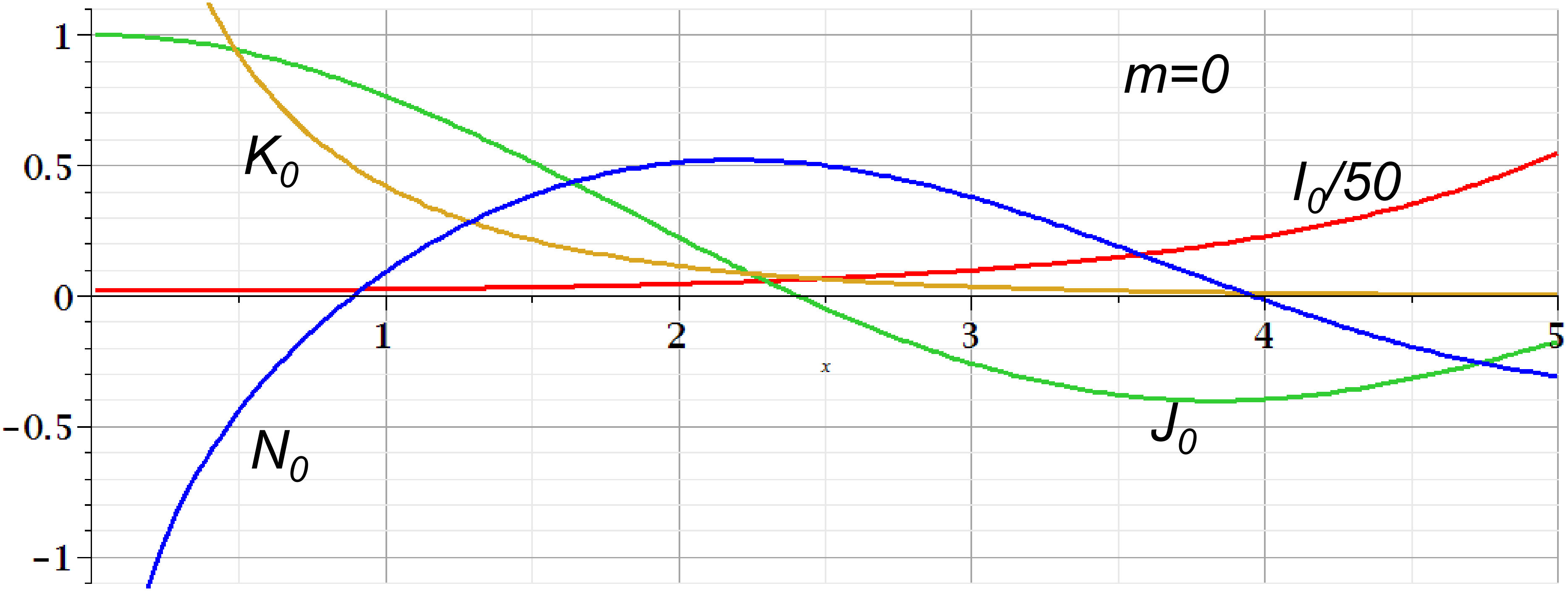
$$+\frac{1}{4\pi}\int_{S;z'=L}d\phi'\,\rho'\,d\rho'\left.\frac{\partial G(\rho,\rho',\phi,\phi',z,z')}{\partial z'}\right|_{z'=L}V(\rho',\phi')$$

Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u), N_m(u), H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u), K_m(u)$$



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