Exercises from Day 2 talk by Jeremy Rouse

4. Let $\sigma(n)$ denote the sum of the divisors of n. Compute
$$\sum_{\substack{d<10^{100} \\ d \text{ odd}}} \sigma(d)\sigma(10^{100}-d).$$

Hints:

(a) Define $E_4(z) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n$, where $\sigma_3(n) = \sum_{d|n} d^3$.

(b) Show that $h(z) = \frac{f(z)-f(2z)}{24} = \sum_{n \text{ odd}} \sigma(n)q^n \in M_2(\Gamma_0(4), \chi_4)$.

(c) Show that $h(z)^2 \in M_4(\Gamma_0(4), \chi_4)$. Use that this space is spanned by $E_4(z)$, $E_4(2z)$ and $E_4(4z)$.

5.

(a) Define $E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma(n)q^n$. It's a fact that for any $N \geq 1$, $E_{2,N}(z) = E_2(z) - NE_2(Nz)$ is an Eisenstein series in $M_2(\Gamma_0(N), \chi_1)$.

(b) Let $Q(x,y,z,w) = x^2 + xy + y^2 + 11(z^2 + zw + w^2)$. Express $\theta_Q(z)$ in terms of $E_{2,3}(z)$, $E_{2,11}(z)$ and $E_{2,33}(z)$, $f(z)$, $f(z)|V(3)$, and $g(z)$.

(c) Find a number B so that if $n > B$ is squarefree, then n is represented by Q. What's the minimal such B?