# Integers represented by positive-definite quadratic forms - the modular approach

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# Notes and exercises

- You can find slides and exercises from my series of talks online at http://users.wfu.edu/rouseja/caaantquafs/.

# Outline

- Overview of modular forms

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- Theta series, Eisenstein series, newforms

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- Theta series, Eisenstein series, newforms
- Determining the integers represented by a quadratic form

# What is a modular form?

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- Also, a modular form has a weight $k$ and level $N$. This means that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$ and $N|c$.

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- We require $f(z)$ to have a Fourier expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a(n) e^{2\pi i n z}$$

where $a(n) = 0$ if $n < 0$ and $|a(n)| \leq C_1 n^{C_2}$ for some $C_1$ and $C_2$.

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*There are five fundamental operations of arithmetic: addition, subtraction, multiplication, division, and modular forms.*

# A picture

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- Dirichlet characters modulo $N$ are in bijection with homomorphisms from $(\mathbb{Z}/N\mathbb{Z})^\times$ to $\mathbb{C}^\times$.

# Modular forms with character

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• This vector space is finite-dimensional!!!

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- Define $q = e^{2\pi iz}$ and

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where $r_Q(n) = \#\{\vec{x} \in \mathbb{Z}^r : Q(\vec{x}) = n\}$.

- Then $\theta_Q(z) \in M_{r/2}(\Gamma_0(N(Q)), \chi)$.

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- If $D$ is a positive integer, define $\chi_D$ to be the unique Dirichlet character with

$$\chi_D(p) = \begin{cases} 0 & \text{if } \gcd(p, D) > 1 \\ 1 & \text{if } \exists x \in \mathbb{Z} \text{ so } x^2 \equiv D \pmod{p} \\ -1 & \text{otherwise.} \end{cases}$$

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- Let $D(Q) = \det(A)$. The character $\chi$ for $\theta_Q$ is $\chi_{D(Q)}$.

# Example (1/3)

- Let $Q = x^2 + y^2 + z^2 + w^2$. We have $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

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- So

$$\theta_Q(z) = 1 + 8q + 24q^2 + 32q^3 + 48q^4 + 96q^5 + \cdots \in M_2(\Gamma_0(4), \chi_4).$$

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- The space $M_2(\Gamma_0(4), \chi_4)$ is 2-dimensional. One basis element is

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- The other basis element is $f(2z)$.
- It's not hard to compute that $\theta_Q(z) = \frac{1}{3} f(z) + \frac{2}{3} f(2z)$.

# Example (3/3)

**Theorem (Jacobi, 1834)**

*The number of ways to write an integer as a sum of four squares is*

$$r_Q(n) = \begin{cases} 8 \sum_{\substack{d \mid n \\ d \text{ odd}}} d & \text{if } n \text{ is odd} \\ 24 \sum_{\substack{d \mid n \\ d \text{ odd}}} d & \text{if } n \text{ is even.} \end{cases}$$

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- The next slide has hints.

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- Show that $h(z) = \frac{f(z)-f(2z)}{24} = \sum_{n \text{ odd}} \sigma(n)q^n \in M_2(\Gamma_0(4), \chi_4)$.

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- Show that $h(z) = \frac{f(z)-f(2z)}{24} = \sum_{n \text{ odd}} \sigma(n) q^n \in M_2(\Gamma_0(4), \chi_4)$.

- Show that $h(z)^2 \in M_4(\Gamma_0(4), \chi_4)$. Use that this space is spanned by $E_4(z)$, $E_4(2z)$ and $E_4(4z)$.

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- The coefficients of $E(z)$ are "large and predictable" and the coefficients of $C(z)$ are "small and mysterious."

- Let $S_2(\Gamma_0(N), \chi)$ denote the subspace of cusp forms.

# Eisenstein series part

- If $f(z) = \theta_Q(z)$, then $E(z) = \sum_{n=0}^{\infty} a_E(n) q^n$ has its coefficients given by

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- There is a constant $C_E$ so that if $n$ is squarefree,

$$a_E(n) \geq C_E n \prod_{\substack{p \mid n \\ \chi(p) = -1}} \frac{p-1}{p+1}.$$

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- If $f(z) \in S_2(\Gamma_0(N), \chi)$, then so is $f|T(p)$.

# Another operator

- If $p \neq q$, then $T(p)T(q) = T(q)T(p)$. We'd like to find a basis for $S_2(\Gamma_0(N), \chi)$ consisting of simultaneous eigenforms for the $T(p)$.

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- If $d|N$ one can define a map $V(d) : S_2(\Gamma_0(N/d), \chi) \to S_2(\Gamma_0(N))$. Let $f(z) = \sum a(n)q^n$, and define

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• In order to diagonalize, we need to isolate that forms that "don't come from lower level."

## The Petersson inner product

- For $f, g \in S_2(\Gamma_0(N), \chi)$, define

$$\langle f, g \rangle = \frac{3}{\pi[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \iint_{\mathbb{H}/\Gamma_0(N)} f(x+iy)\overline{g(x+iy)}\, dx\, dy.$$

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- We define the old subspace $S_2^{\mathrm{old}}(\Gamma_0(N), \chi)$ to be

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- The new subspace $S_2^{\mathrm{new}}(\Gamma_0(N), \chi)$ is the orthogonal complement of the old subspace.

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**Theorem (Eichler-Igusa-Shimura, 1950s)**

*If $f(z)$ is a newform, then $|a(n)| \leq d(n)\sqrt{n}$, where $d(n)$ is the number of divisors of $n$.*

# Example

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$$f(z) = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + \cdots \in S_2(\Gamma_0(11), \chi).$$

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- The third basis element of $S_2(\Gamma_0(33), \chi)$ is a newform of level 33:

$$g(z) = q + q^2 - q^3 - q^4 - 2q^5 - q^6 + 4q^7 + \cdots .$$

# Exercise 5

- Define $E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n$. It's a fact that for any $N \geq 1$, $E_{2,N}(z) = E_2(z) - NE_2(Nz)$ is an Eisenstein series in $M_2(\Gamma_0(N), \chi_1)$.

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- Let $Q(x, y, z, w) = x^2 + xy + y^2 + 11(z^2 + zw + w^2)$. Express $\theta_Q(z)$ in terms of $E_{2,3}(z)$, $E_{2,11}(z)$ and $E_{2,33}(z)$, $f(z)$, $f(z)|V(3)$, and $g(z)$.

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- Find a number $B$ so that if $n > B$ is squarefree, then $n$ is represented by $Q$. What's the minimal such $B$?

# Cusp form coefficients

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- Then, there is a decomposition

$$C(z) = \sum_{M|N} \sum_{i=1}^{s} \sum_{d} c_{M,i,d} g_{M,i}(z) | V(d)$$

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- This gives the bound

$$|a_C(n)| \leq \left( \sum_{M|n} \sum_{i=1}^{s} \frac{|c_{M,i,d}|}{\sqrt{d}} \right) d(n) \sqrt{n}.$$

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- We have

$$E(z) = 1 + \frac{12}{5}\sum_{n=1}^{\infty}(\sigma(n) - 11\sigma(n/11))q^n.$$

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- If $Q = x^2 + y^2 + 3z^2 + 3w^2 + xz + yw$, then

$\theta_Q(z) = 1 + 4q + 4q^2 + 8q^3 + 20q^4 + 16q^5 + \cdots \in M_2(\Gamma_0(11), \chi_1)$.

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- Thus,

$$r_Q(n) \geq \frac{12}{5} \sum_{\substack{d \mid n \\ 11 \nmid d}} d - \frac{8}{5} d(n)\sqrt{n}.$$

# Example (2/2)

- If $n$ is squarefree and $\gcd(n, 11) = 1$, assuming that $r_Q(n) = 0$ yields the inequality

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• The above inequality is true for precisely 110 squarefree integers. The form $Q$ represents all of these. It follows that $Q$ represents all positive integers.

# General setup - Eisenstein

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- We have $E(z) = \sum_{n=0}^{\infty} a_E(n) q^n$ and there is a constant $C_E$ so that

$$a_E(n) \geq C_E n \prod_{\substack{p \mid n \\ \chi(p)=-1}} \frac{p-1}{p+1}.$$

# General setup - cusp forms

- We can decompose

$$C(z) = \sum_{M|N} \sum_{i=1}^{s} \sum_{d} c_{M,i,d} g_{M,i}(z) | V(d).$$

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- Then $|a_C(n)| \leq C_Q d(n) \sqrt{n}$.

# Representability

## Theorem (Hanke, 2004)

*If $n$ is squarefree and not represented by $Q$, then*

$$F_4(n) = \frac{\sqrt{n}}{d(n)} \prod_{\substack{p|n, p\nmid N \\ \chi(p)=-1}} \frac{p-1}{p+1} \leq \frac{C_Q}{C_E}.$$

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• The integers $n$ that satisfy the inequality above can be enumerated efficiently and checked, provided one can compute $C_E$ and $C_Q$.

• Computing $C_E$ is straightforward using formulas for local densities.

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• The dimension of $S_2(\Gamma_0(N), \chi)$ is approximately $\frac{N}{6}$. Computing $C_Q$ can be extremely time-consuming.

# Example (from 451)

- For

$$Q(x, y, z, w) = x^2 - xy + 2y^2 + yz - 2yw + 5z^2 + zw + 29w^2$$

we have $\theta_Q \in M_2(\Gamma_0(4200), \chi_{168})$.

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- It takes almost a day to compute that $C_Q \approx 31.0537$.

- Once this is known, it takes 10 seconds to check that $Q$ represents every odd number.

# Example (from 290)

- The form

$$Q(x, y, z, w) = x^2 - xz - xw + 2y^2 + yz + yw + 5z^2 + 5zw + 29w^2$$

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- We have that $\dim S_2(\Gamma_0(4092), \chi) = 760$, but this space contains a newform $g(z)$ (and its Galois conjugates) with coefficients in a degree 672 number field.

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- We have that $\dim S_2(\Gamma_0(4092), \chi) = 760$, but this space contains a newform $g(z)$ (and its Galois conjugates) with coefficients in a degree 672 number field.

- The modular symbols algorithm requires 46 days to compute $C_Q$.

# Preview of tomorrow

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- The Petersson inner product gives another way to measure how "big" a cusp form is.

- The goal is to find an efficient way to compute $\langle C, C \rangle$ and translate that into a bound on $C_Q$.