Dynamic Model: Practice Problem Key
Intermediate Macroeconomics
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Question 1

a) A competitive equilibrium in the dynamic, two period classical model with a capital market is a price \( r^* \) and allocations \( (c^*_1, c^*_2, S^*, K^*, Y^S^*) \) such that the following conditions hold:

1) Given the price \( r^* \), the representative household solves

\[
\max_{c_1, c_2, S} U(c_1, c_2) \quad \text{s.t.} \quad c_1 + S = Y_1 \\
c_2 = Y_2 + (1 + r^*)S \\
Y_1, Y_2 \text{ given}
\]

2) Given the price \( r^* \), the representative firm solves

\[
\max_{Y^S, K} Y^S - (1 + r^*)K \quad \text{s.t.} \quad Y^S = f(K)
\]

3) Markets clear

\[ S^* = K^* \]

b) Savings supply is derived from the household’s problem.

Rewriting the representative household’s maximization problem in terms of \( S \), we have the following:

\[
\max_{S} Y_1 - S + \beta(Y_2 + (1 + r)S)
\]

Using the fact that we know \( Y_1 = 25, Y_2 = 0, \beta = \frac{4}{5} \), the above can be written as:
\[
\max_S \ 25 - S + \frac{4}{5}(1 + r)S \\
L = 25 - S + \frac{4}{5}(1 + r)S \\
\frac{\partial L}{\partial S} = -1 + \frac{4}{5}(1 + r) = 0 \\
1 = \frac{4}{5}(1 + r) \\
\Rightarrow r^* = \frac{1}{4}
\]

Notice that since the \( S \) term has disappeared, we will have a perfectly elastic supply of savings.

Investment demand is derived from the firm’s problem:

\[
\max_{Y^S, K} \ Y^S - (1 + r)K \quad s.t. \quad Y^S = 10K^{\frac{1}{2}}
\]

This can be written as the following:

\[
\max_K \ 10K^{\frac{1}{2}} - (1 + r)K \\
L = 10K^{\frac{1}{2}} - (1 + r)K \\
\frac{\partial L}{\partial K} = 5K^{-\frac{1}{2}} - (1 + r) = 0 \\
\Rightarrow K = \frac{25}{(1 + r)^2}
\]

We now can combine the supply of savings and demand for investment to obtain an equilibrium in the capital market.
From the representative household’s problem, we have $r^* = \frac{1}{4}$. We can plug this value into our expression for $K$, $K = \frac{25}{\left(\frac{3}{4}\right)^2}$. Thus, $K^* = 16$.

From the market clearing condition, $S^* = 16$ as well.

To solve for $Y^S$, plug the equilibrium value of capital, $K^*$, into the representative firm’s production function, $Y^S = f(K^*) = 10(16)^{\frac{1}{2}} = 40$.

Using the representative household’s period 1 and period 2 budget constraints, $C_1^* = Y_1 - S^* = 25 - 16 = 9$ and $C_2^* = Y_2 + (1 + r^*)S^* = 0 + (1 + \frac{1}{4})16 = 20$.

**Question 2**

a) The savings supply curve is derived from the household’s problem, which is written as follows:

$$\max_{c_1, c_2, S} ln c_1 + \frac{3}{4} ln c_2 \quad s.t. \quad c_1 + S = 50$$

$$c_2 = (1 + r)S$$

Rewriting the representative household’s maximization problem in terms of $S$ and solving for $S$, we have the following:

$$\max_S \quad ln(50 - S) + \frac{3}{4}ln((1 + r)S)$$

$$\mathcal{L} = ln(50 - S) + \frac{3}{4}ln((1 + r)S)$$

$$\frac{\partial \mathcal{L}}{\partial S} = -\frac{1}{50 - S} + \frac{3}{4} \frac{(1 + r)}{4(1 + r)S} = 0$$

$$\frac{1}{50 - S} = \frac{3}{4S}$$

$$4S = 150 - 3S$$

$$3S = 150$$

$$S = 50$$
\[ \Rightarrow S = \frac{150}{7} = 21.43 \]

Notice we have a perfectly inelastic savings supply curve, so the equilibrium quantity of savings \( S^* \) is immediately determined: \( S^* = 21.43 \).

Investment demand is derived from the firm’s problem:

\[
\max_{Y^s, K} Y^s - (1 + r)K \quad s.t. \quad Y^s = 20K^{\frac{1}{2}}
\]

This can be written as the following:

\[
\max_K 20K^{\frac{1}{2}} - (1 + r)K
\]

\[
\mathcal{L} = 20K^{\frac{1}{2}} - (1 + r)K
\]

\[
\frac{\partial \mathcal{L}}{\partial K} = 10K^{-\frac{1}{2}} - (1 + r) = 0
\]

\[
K^{-\frac{1}{2}} = \frac{(1 + r)}{10}
\]

\[
\Rightarrow K = \frac{100}{(1 + r)^2}
\]

We now can combine the supply of savings and demand for investment to obtain an equilibrium in the capital market. From the market clearing condition, \( S^* = K^* = 21.43 \). Again, using the market clearing condition, we can equate the saving supply curve and the investment demand curve to find the equilibrium interest rate \( r^* \):

\[
\frac{150}{7} = \frac{100}{(1 + r^*)^2}
\]
\[(1 + r^*)^2 = \frac{700}{150}\]

\[\Rightarrow r^* = \left(\frac{700}{150}\right)^{\frac{1}{2}} - 1 = 1.16\]

To solve for \(Y^{S*}\), plug the equilibrium value of capital, \(K^*\), into the representative firm’s production function, \(Y^{S*} = f(K^*) = 20\left(\frac{150}{7}\right)^{\frac{1}{2}} = 92.58\).

Using the representative household’s period 1 and period 2 budget constraints, \(C_1^* = 50 - S^* = 50 - \frac{150}{7} = 28.57\) and \(C_2^* = (1 + r^*)S^* = (1 + 1.16)\frac{150}{7} = 46.29\).