Question 1

a) We want to show that \( f(\lambda K, \lambda L) = \lambda f(K, L) \ \forall \ \lambda > 0: \)

\[
Y(\lambda K, \lambda L) = A(\lambda K)^{\frac{1}{2}}(\lambda L)^{\frac{1}{2}} = A\lambda^{\frac{1}{2}}K^{\frac{1}{2}}\lambda^{\frac{1}{2}}L^{\frac{1}{2}} = \lambda AK^{\frac{1}{2}}L^{\frac{1}{2}} = \lambda Y
\]

b) \[
MPK = \frac{\partial Y}{\partial K} = \frac{1}{3} AK^{-\frac{2}{3}}L^{\frac{2}{3}}
\]

\[
MPL = \frac{\partial Y}{\partial L} = \frac{2}{3} AK^{\frac{1}{3}}L^{-\frac{1}{3}}
\]

c) \[
\frac{\partial MPK}{\partial K} = -\frac{2}{9} AK^{-\frac{5}{3}}L^{\frac{2}{3}} < 0
\]

\[
\frac{\partial MPL}{\partial L} = -\frac{2}{9} AK^{\frac{1}{3}}L^{-\frac{4}{3}} < 0
\]

Question 2

a) We want to show that \( f_t(\lambda K_t, \lambda L_t) = \lambda f_t(K_t, L_t) \ \forall \ \lambda > 0: \)

\[
Y_t(\lambda K_t, \lambda L_t) = \left(A(\lambda K_t)^{\frac{1}{2}} + \frac{3}{2}(\lambda L_t)^{\frac{1}{2}}\right)^2 = \left(A\lambda^{\frac{1}{2}}K_t^{\frac{1}{2}} + \frac{3}{2}\lambda^{\frac{1}{2}}L_t^{\frac{1}{2}}\right)^2
\]

\[
= \left(\lambda^{\frac{1}{2}}(AK_t^{\frac{1}{2}} + \frac{3}{2}L_t^{\frac{1}{2}})\right)^2 = \lambda^2 \left(AK_t^{\frac{1}{2}} + \frac{3}{2}L_t^{\frac{1}{2}}\right)^2 = \lambda Y_t
\]

And, the marginal product of capital and marginal product of labor are given by
\[ MPK = \frac{\partial Y_t}{\partial K_t} = 2 \left( AK_t^{\frac{1}{2}} + \frac{3}{2} L_t^{\frac{1}{2}} \right) \frac{1}{2} AK_t^{-\frac{1}{2}} = \left( AK_t^{\frac{1}{2}} + \frac{3}{2} L_t^{\frac{1}{2}} \right) AK_t^{-\frac{1}{2}} \]

\[ MPL = \frac{\partial Y_t}{\partial L_t} = 2 \left( AK_t^{\frac{1}{2}} + \frac{3}{2} L_t^{\frac{1}{2}} \right) \frac{3}{2} 2 L_t^{-\frac{1}{2}} = \frac{3}{2} \left( AK_t^{\frac{1}{2}} + \frac{3}{2} L_t^{\frac{1}{2}} \right) L_t^{-\frac{1}{2}} \]

b) Dividing both sides of the production function by \( N_t \),

\[ \frac{Y_t}{N_t} = \left( AK_t^{\frac{1}{2}} + \frac{3}{2} L_t^{\frac{1}{2}} \right)^2 \frac{1}{N_t} = \left( AK_t^{\frac{1}{2}} + \frac{3}{2} L_t^{\frac{1}{2}} \right)^2 \frac{1}{(N_t^2)^2} \]

\[ = \left( AK_t^{\frac{1}{2}} + \frac{3}{2} L_t^{\frac{1}{2}} \right) \left( \frac{K_t}{N_t} \right)^\frac{1}{2} + \frac{3}{2} \left( \frac{L_t}{N_t} \right) \left( \frac{1}{N_t^2} \right)^\frac{1}{2} \]

\[ \Rightarrow y_t = \left( AK_t^{\frac{1}{2}} + \frac{3}{2} \right)^2 \]

where the last step follows from A1 and using the per capita variables \( y_t \) and \( k_t \).

c) Using the per capita production function from part b) and the identities provided in the problem,

\[ k_{t+1} = (1 - \delta)k_t + x_t = (1 - \delta)k_t + s_t = (1 - \delta)k_t + s y_t \]

\[ \Rightarrow k_{t+1} = (1 - \delta)k_t + s \left( AK_t^{\frac{1}{2}} + \frac{3}{2} \right)^2 \]

Question 3
a) Plugging the per capita production function into the per capita neo-classical growth equation given in the problem, we have the following:

\[ k_{t+1} = (1 - \delta)k_t + sy_t \]

Assuming a steady state,

\[ k_{ss} = (1 - \delta)k_{ss} + sy_{ss} \]

\[ \delta k_{ss} = sy_{ss} \]

\[ \Rightarrow \frac{k_{ss}}{y_{ss}} = \frac{s}{\delta} \]

b) \[
\frac{\partial k_{ss}}{\partial s} = \frac{1}{\delta} > 0
\]

c) \[
\frac{\partial y_{ss}}{\partial \delta} = -s\delta^{-2} < 0
\]

d) \[
k_{ss} = \left( \frac{sA}{\delta} \right)^{\frac{1}{1-\alpha}} = \left( \frac{.20(5)}{.05} \right)^{\frac{1}{.33}} = 87.4653
\]

\[ y_{ss} = Ak_{ss}^{\alpha} = 5(87.4653)^{.33} = 21.8663 \]

\[ x_{ss} = \delta k_{ss} = .05(87.4653) = 4.3733 \]

\[ \frac{k_{ss}}{y_{ss}} = \frac{s}{\delta} = \frac{.20}{.05} = 4 \]
Question 4

a) Using the per capita endogenous growth equation,

\[ k_{t+1} = (1 - \delta + sA)k_t \]

\[ \frac{k_{t+1}}{k_t} = 1 - \delta + sA \]

\[ 1 + g_{k_t} = 1 - \delta + sA \]

\[ \Rightarrow g_{k_t} = sA - \delta \]

b) Using the per capita social technology function,

\[ y_t = Ak_t \]

\[ \Rightarrow \frac{y_{t+1}}{y_t} = \frac{Ak_{t+1}}{Ak_t} \]

\[ 1 + g_{y_t} = 1 + g_{k_t} \]

\[ \Rightarrow g_{y_t} = g_{k_t} \]

c) The growth rate of per capita capital is ten percent:

\[ g_{k_t} = .20(1) - .10 = .10 \]