Physics 712
Solutions to Chapter 11 Problems

1. [10] We want to consider the effect of two boosts along the $x$-axis. The following hyperbolic identities will prove useful:

\[
\cosh (\phi_1 \pm \phi_2) = \cosh \phi_1 \cosh \phi_2 \pm \sinh \phi_1 \sinh \phi_2, \quad \text{and} \quad \tanh (\phi_1 \pm \phi_2) = \frac{\tanh \phi_1 \pm \tanh \phi_2}{1 \pm \tanh \phi_1 \tanh \phi_2}.
\]

(a) For two successive boosts with rapidity $\phi_1$ and $\phi_2$ find the equivalent rapidity $\phi_{\text{tot}}$.

We simply combine two boosts using these formulas, which gives us

\[
\Lambda_{\text{tot}} = \Lambda_2 \Lambda_1 = \begin{pmatrix}
\cosh \phi_1 & -\sinh \phi_1 & 0 & 0 \\
-\sinh \phi_1 & \cosh \phi_1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cosh \phi_2 & -\sinh \phi_2 & 0 & 0 \\
-\sinh \phi_2 & \cosh \phi_2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\cosh (\phi_1 + \phi_2) & -\sinh (\phi_1 + \phi_2) & 0 & 0 \\
-\sinh (\phi_1 + \phi_2) & \cosh (\phi_1 + \phi_2) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Obviously, $\phi_{\text{tot}} = \phi_1 + \phi_2$.

(b) For two successive boosts with velocity $v_1$ and $v_2$ find the equivalent velocity $v_{\text{tot}}$.

We know that $v = c \tanh \phi$, so we have

\[
v_{\text{tot}} = c \tanh \phi_{\text{tot}} = c \tanh (\phi_1 + \phi_2) = c \frac{\tanh \phi_1 + \tanh \phi_2}{1 + \tanh \phi_1 \tanh \phi_2} = c \frac{v_1/c + v_2/c}{1 + v_1 v_2/c^2} = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}.
\]