2. [15] Consider a particle moving along the x-axis whose 4-velocity is given at proper time \( \tau \) by
\[
U^\mu = c (\cosh \phi, \sinh \phi, 0, 0),
\]
where \( \phi \) is an unknown function of time.

(a) Check that \( U \cdot U = c^2 \). Find the proper acceleration \( a(\tau) \) at time \( \tau \) for an arbitrary function \( \phi(\tau) \).

Trivially, \( U \cdot U = c^2 \cosh^2 \phi - c^2 \sinh^2 \phi = c^2 \). The four-acceleration is given by
\[
A^\mu = \frac{d}{d\tau} U^\mu = \left( c \sinh \phi \frac{d\phi}{d\tau}, c \cosh \phi \frac{d\phi}{d\tau}, 0, 0 \right)
\]
We then have
\[
a = \sqrt{-A \cdot A} = \sqrt{-c^2 \left( \frac{d\phi}{d\tau} \right)^2 \sinh^2 \phi + c^2 \left( \frac{d\phi}{d\tau} \right)^2 \cosh^2 \phi} = c \frac{d\phi}{d\tau}.
\]

(b) Suppose \( a(\tau) = g \), a constant. Assuming the particle starts at the origin at \( \tau = 0 \) and is initially at rest, find \( \phi(\tau) \), \( U(\tau) \) and \( x(\tau) \).

Initially at rest means that at \( \tau = 0 \) we have \( U^\mu = (c, 0, 0, 0) \), or \( \phi = 0 \). Assuming the absolute values is always positive, we now integrate the equation \( c (d\phi/d\tau) = g \), and find
\[
\phi(\tau) = \frac{g}{c} \tau
\]
This can, of course, be substituted into the four velocity to yield
\[
U(\tau) = c \left( \cosh \left( \frac{g\tau}{c} \right), \sinh \left( \frac{g\tau}{c} \right), 0, 0 \right)
\]
Since \( U^\mu = dx^\mu/d\tau \), we can integrate these equations to yield
\[
x(\tau) = c^2 \left( \frac{1}{g} \sinh \left( \frac{g\tau}{c} \right), \cosh \left( \frac{g\tau}{c} \right), -1, 0, 0 \right).
\]
The constant of integration was chosen in each case to make sure \( x(0) = (0, 0, 0, 0) \).

(c) How much proper time (in years) would it take to get to Alpha Centauri (4.3 c·y), the center of our galaxy (2.6×10^4 c·y), or the edge of the visible universe (4.5×10^{10} c·y) if you start at rest and accelerate in a straight line at proper acceleration \( g = 9.8 \text{ m/s}^2 \)?

We simply let \( x \) be the distance to our object and solve for \( \tau \) in each case.
\[ x = \frac{c^2}{g} \left[ \cosh \left( \frac{g\tau}{c} \right) - 1 \right], \]
\[ \cosh \left( \frac{g\tau}{c} \right) = \frac{xg}{c^2} + 1, \]
\[ \tau = \frac{c}{g} \cosh^{-1} \left( \frac{xg}{c^2} + 1 \right) = \frac{2.998 \times 10^8 \text{ m/s}}{9.8 \text{ m/s}^2} \cosh^{-1} \left( \frac{x}{c} \cdot \frac{9.8 \text{ m/s}^2}{2.998 \times 10^8 \text{ m/s}} + 1 \right) \]
\[ = \frac{3.059 \times 10^7 \text{ s}}{3.156 \times 10^7 \text{ s/yr}} \cosh^{-1} \left( \frac{x}{c} \cdot \frac{3.156 \times 10^7 \text{ s/yr}}{3.059 \times 10^7 \text{ s}} + 1 \right) = (0.9693 \text{ yr}) \cosh^{-1} \left( \frac{x}{0.9693 \cdot \text{ yr}} + 1 \right). \]

We now simply substitute each of the distances into the formula to get the final answer:

\[ \alpha \text{ Centauri: } \tau = (0.9693 \text{ yr}) \cosh^{-1} \left( \frac{4.3 \cdot \text{ yr}}{0.9693 \cdot \text{ yr}} + 1 \right) = 2.305 \text{ yr}, \]

Center of Galaxy: \[ \tau = (0.9693 \text{ yr}) \cosh^{-1} \left( \frac{2.6 \times 10^4 \cdot \text{ yr}}{0.9693 \cdot \text{ yr}} + 1 \right) = 10.56 \text{ yr}, \]

Edge of Universe: \[ \tau = (0.9693 \text{ yr}) \cosh^{-1} \left( \frac{4.5 \times 10^6 \cdot \text{ yr}}{0.9693 \cdot \text{ yr}} + 1 \right) = 24.48 \text{ yr}. \]