5. [5] For problem 4.1, find the total energy if the cylinder has length $L$. For problem 4.2, find the total energy. In each case, show that the answer is equivalent to $W = \frac{1}{2} Q \Delta \Phi$.

The energy is given for problem 4.1 by

$$W = \frac{1}{2} \int \mathbf{E} \cdot d^3 \mathbf{x} = \frac{1}{2} \int_0^{2\pi} d\phi \int_0^L dz \left[ \int_a^b \varepsilon \left( \frac{\lambda}{2\pi \varepsilon_0 \rho} \right)^2 \rho d\rho + \int_b^c \varepsilon_0 \left( \frac{\lambda}{2\pi \varepsilon_0 \rho} \right)^2 \rho d\rho \right]$$

$$= \frac{2\pi \lambda^2 L}{8\pi^2} \left( \frac{1}{\varepsilon} \ln \rho|_b^b + \frac{1}{\varepsilon_0} \ln \rho|_b^c \right) = \frac{\lambda^2 L}{4\pi} \left[ \frac{1}{\varepsilon} \ln \left( \frac{b}{a} \right) + \frac{1}{\varepsilon_0} \ln \left( \frac{c}{b} \right) \right] = \frac{1}{2} \lambda L \Delta \Phi = \frac{1}{2} Q \Delta \Phi,$$

where at the last step, we interpreted $\lambda L = Q$ as the total charge. For problem 4.2, we have

$$W = \frac{1}{2} \int \mathbf{E} \cdot d^3 \mathbf{x} = \frac{1}{2} \int_0^{2\pi} d\phi \int_a^b \left[ \frac{Q}{\pi (3\varepsilon_0 + \varepsilon) r^2} \right]^2 r^2 dr \left( \int_0^{\pi} \varepsilon \sin \theta d\theta + \int_0^{\pi} \varepsilon_0 \sin \theta d\theta \right)$$

$$= \frac{2\pi Q^2}{2\pi^2 (3\varepsilon_0 + \varepsilon)^2} \left[ -1 \right]^b_a \left( -\varepsilon \cos \theta|_0^{\pi} - \varepsilon_0 \cos \theta|_0^{\pi} \right) = \frac{Q^2}{\pi (3\varepsilon_0 + \varepsilon)^2} \left( \frac{1}{a} - \frac{1}{b} \right) \left( \frac{1}{2} \varepsilon + \frac{3}{2} \varepsilon_0 \right)$$

$$= \frac{Q^2 (b-a)}{2\pi ab (3\varepsilon_0 + \varepsilon)} = \frac{1}{2} Q \Delta \Phi.$$
1. We are trying to trap a charged particle of mass $q > 0$ and mass $m$ by using a combination of magnetic and electric fields given by $\mathbf{B} = B\hat{z}$ and $\mathbf{E} = A(x\hat{x} + y\hat{y} - 2z\hat{z})$.

(a) Obviously, $\nabla \cdot \mathbf{B} = \nabla \times \mathbf{B} = 0$. Check that it also satisfies $\nabla \cdot \mathbf{E} = \nabla \times \mathbf{E} = 0$.

We simply see that $\nabla \cdot \mathbf{E} = A + A - 2A = 0$, and all the terms in $\nabla \times \mathbf{E}$ vanish.

(b) Assume the particle has motion given by $x = R \cos(\omega t)$, $y = R \sin(\omega t)$. Find an equation for $\omega$ in terms of $A$ and $B$.

The velocity and acceleration can be found by simply taking derivatives:

$$\mathbf{v} = \dot{x} = R\omega \left[-\sin(\omega t) \hat{x} + \cos(\omega t) \hat{y}\right], \quad \mathbf{a} = \dot{\mathbf{v}} = R\omega^2 \left[-\cos(\omega t) \hat{x} - \sin(\omega t) \hat{y}\right]$$

We therefore have

$$m\mathbf{a} = \mathbf{F} = (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = qRA \left[\cos(\omega t) \hat{x} + \sin(\omega t) \hat{y}\right] + qBR\omega \left[-\cos(\omega t) \hat{x} + \sin(\omega t) \hat{y}\right] \times \hat{z},$$

$$-mR\omega^2 \left[\cos(\omega t) \hat{x} + \sin(\omega t) \hat{y}\right] = qRA \left[\cos(\omega t) \hat{x} + \sin(\omega t) \hat{y}\right] + qBR\omega \left[\cos(\omega t) \hat{x} + \sin(\omega t) \hat{y}\right],$$

$$-mR\omega^2 = qRA + qBR\omega, \quad m\omega^2 + qB\omega + qA = 0.$$ 

We then solve this using the quadratic equation, so

$$\omega = \frac{-qB \pm \sqrt{q^2B^2 - 4mqA}}{2m}.$$ 

Until I solved this problem myself, I didn’t even realize there were two solutions to this equation.

(c) Argue that there is a maximum value of $A$ for which circular motion is possible. Also argue that for $A > 0$, the particle will not “wander off” in the $z$-direction.

The solution only makes sense if the discriminant is positive, so we must have

$$q^2B^2 \geq 4mqA,$$

or $A \leq \frac{qB^2}{(4m)}$. Although we have not discusses motion in the $z$-direction, it is pretty easy to see that the magnetic field has no influence on it, so the only vertical force is $F_z = E_z q = -2Azq$. Such a linear restoring force will result in simple harmonic motion in the $z$-direction, so it is stable against motion in the $z$-direction.